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Formulation of Problems for Stationary Dispersive Equations of Higher Orders on Bounded Intervals with General Boundary Conditions

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Abstract: Boundary value problems for linear stationary dispersive equations of order $2l + 1$, $l \in \mathbb{N}$ with general linear boundary conditions have been considered on finite intervals $(0, L)$. The existence and uniqueness of regular solutions have been established.

Keywords**:** stationary dispersive equations, general boundary conditions, regular solutions

1. Introduction

This work concerns solvability of boundary-value problems on bounded intervals for linear stationary dispersive equations

$$\lambda u + \sum_{j=1}^{l} (-1)^{j+1} D_x^{2j+1} u = f(x),\ x \in (0, L);\ l \in \mathbb{N}, \tag{1}$$

where λ, L are real positive numbers and f is a given function. This class of stationary equations appears naturally while one wants to solve a corresponding evolution equation

$$u_t + \sum_{j=1}^{l} (-1)^{j+1} D_x^{2j+1} u + u D_x u = 0,\ x \in (0, L);\ t > 0, \tag{2}$$

making use of the semigroup theory (see [1], Theorem 4.1) or a semi-discrete approach [2]. Here, we propose (1) as a stationary analog of (2) because the last equation includes as special cases classical dispersive equations: when $l = 1$, we have the Korteweg-de Vries (KdV) equation [3, 4] and for $l = 2$ the Kawahara equation [5-7]. These equations play an important role in the development of science due to various applications in physics, such as dynamics of fluids and plasma physics [8-10]. There are a number of papers dedicated to initial-boundary value problems for dispersive equations (which included KdV and Kawahara equations) posed on bounded domains [2, 11-17]. Dispersive equations such as KdV and Kawahara equations have been deduced for unbounded regions of wave propagations, however, if one is interested in implementing numerical schemes to calculate solutions in these regions, there arises the issue of cutting off a spatial domain approximating unbounded domains by bounded ones [7, 18]. In this occasion, some boundary conditions are needed to specify the solution. Therefore, precise mathematical analysis of mixed problems in bounded domains for dispersive equations is welcome and attracts attention of specialists in this area [2, 11-13, 15-18].

Last years, publications on stationary and evolution dispersive equations of higher orders appeared [1, 15, 19-22]. Usually, simple boundary conditions at $x = 0$ and $x = L$ such as $D^i u(0) = D^i u(L) = D^l u(L) = 0$, $i = 0, ..., l - 1$ for (1) were imposed, see [1, 21, 22]. Different kinds of boundary conditions for KdV and Kawahara equations were considered in [16, 23-25]. We must mention [26] where general mixed problems for linear multidimensional $(2b+1)$-hyperbolic equations were studied by means of functional analisys methods.

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The goal of our work is to formulate correct general boundary value problems for (1) and to prove the existence and uniqueness of regular solutions. Obviously, boundary conditions for (1) are the same as for (2). Because of that, study of boundary value problems for (1) helps to understand solvability of initial-boundary value problems for (2) and may be considered as the first and a necessary step in study of (2). Therefore, this study is interesting from the purely mathematical point of view because it generalizes some results on dispersive equations. Moreover, these results can be used for constructing of numerical schemes for studying various models of initialboundary value problems for higher-order dispersive equations depending on a choise of l and coefficients. Some techniques proposed in this manuscript are used to solve problems involving dispersive equations [21, 22, 24, 25]. Moreover, they can be found in different problems of mathematical physics [27-29].

Our paper has the following structure: Chapter 1 is introduction. Chapter 2 contains notations and auxiliary facts. In Chapter 3, the formulation of problems to be considered is given. In Chapter 4, the existence and uniqueness of regular solution have been established.

2. Notations and auxiliary facts

Let $x \in (0, L)$, $D^i = D_x^i = \dfrac{\partial^i}{\partial x^i}$, $i \in \mathbb{N}$; $D = D^1$. As in [30] p. 23, we denote for scalar functions $f(x)$ the Banach space $L^p(0, L)$, $1 \le p \le +\infty$ with the norm:

$$\| f\|_{L^p(0,L)}^p = \int_0^L |f(x)|^p\, dx,\ p \in [1,+\infty),\ \ \| f\|_\infty = \operatorname*{ess\,sup}_{x\in(0,L)} |f(x)|.$$

For $p = 2$, $L^2(0, L)$ is a Hilbert space with the scalar product

$$(u,v) = \int_0^L u(x)v(x)dx \ \text{ and the norm }\ \| u\|^2 = \int_0^L |u(x)|^2\, dx.$$

The Sobolev space $W^{m,p}(0, L)$, $m \in \mathbb{N}$ is a Banach space with the norm:

$$\| u \|_{W^{m,p}(0,L)}^p = \sum_{0\le|\alpha|\le m} \| D^\alpha u \|_{L^p(0,L)}^p,\ 1 \le p < +\infty.$$

When $p = 2$, $W^{m,2}(0, L) = H^m(0, L)$ is a Hilbert space with the following scalar product and the norm:

$$((u,v))_{H^m(0,L)} = \sum_{0\le|j|\le m} (D^j u, D^j v),\ \| u \|_{H^m(0,L)}^2 = \sum_{0\le|j|\le m} \| D^j u \|^2.$$

For any space of functions, defined on an interval $(0, L)$, we omit the symbol $(0, L)$, for example, $L^p = L^p(0, L)$, $H^m = H^m(0, L)$, $H_0^m = H_0^m(0, L)$ etc. We will use the following form of the Cauchy inequality:

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon};\ \varepsilon > 0.$$

Lemma 2.1 Let $u \in C^{2j+1}([0, L])$, $j \in \mathbb{N}$. Then

$$(D^{2j+1}u, u) = \sum_{k=1}^{j} (-1)^{k+1} D^{k-1}u D^{(2j+1)-k} u \Big|_0^L + (-1)^j \frac{1}{2}(D^j u)^2 \Big|_0^L, \tag{3}$$

$$(D^{2j}u, u) = \sum_{k=1}^{j} (-1)^{k+1} D^{k-1}u D^{2j-k} u \Big|_0^L + (-1)^j \| D^j u \|^2, \tag{4}$$

$$(D^{2j+1}u, xu) = \sum_{k=1}^{j}(-1)^{k+1}xD^{k-1}uD^{(2j+1)-k}u\Big|_0^L + (-1)^j\frac{x}{2}(D^ju)^2\Big|_0^L$$

$$+\sum_{k=1}^{j}(-1)^k kD^{k-1}uD^{2j-k}\Big|_0^L + (-1)^{j+1}\frac{(2j+1)}{2}\|D^ju\|^2. \tag{5}$$

Proof. Since

$$(D^{2j+1}u, u) = \int_0^L D^{2j+1}u(x)u(x)dx;\ (D^{2j}u, u) = \int_0^L D^{2j}u(x)u(x)dx,$$

the proof of (3), (4) are based on integration by parts and mathematical induction. The proof of (5) is similar taking into account (4).

Lemma 2.2 Let $u \in C^{2l+1}([0, L])$, $l \in \mathbb{N}$. Then

$$\sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u, u) = \sum_{i=0}^{l-1}D^iu\Big(\sum_{k=1}^{l-i}(-1)^{k+1}D^{2k+i}u\Big)\Big|_0^L - \frac{1}{2}\sum_{j=1}^{l}(D^ju)^2\Big|_0^L. \tag{6}$$

Proof. The case $l = 1$ follows by (3). Suppose assertion (6) is valid for some integer $n \geq 1$ and assume $u \in C^{2n+3}([0, L])$. By induction hypothesis and (3), we get

$$\sum_{j=1}^{n+1}(-1)^{j+1}(D^{2j+1}u, u) = \sum_{j=1}^{n}(-1)^{j+1}(D^{2j+1}u, u) + (-1)^n(D^{2n+3}u, u)$$

$$= \sum_{i=0}^{n-1}D^iu\Big(\sum_{k=1}^{n-i}(-1)^{k+1}D^{2k+i}u\Big)\Big|_0^L - \frac{1}{2}\sum_{j=1}^{n}(D^ju)^2\Big|_0^L$$

$$+\sum_{k=1}^{n+1}(-1)^{n+k+1}D^{k-1}uD^{(2n+3)-k}u\Big|_0^L - \frac{1}{2}(D^{n+1}u)^2\Big|_0^L$$

$$= \sum_{i=0}^{n-1}D^iu\Big(\sum_{k=1}^{n-i}(-1)^{k+1}D^{2k+i}u + (-1)^{n-i}D^{2n+2-i}u\Big)\Big|_0^L$$

$$+D^nuD^{n+2}u\Big|_0^L - \frac{1}{2}\sum_{j=1}^{n+1}(D^ju)^2\Big|_0^L$$

$$= \sum_{i=0}^{n}D^iu\Big(\sum_{k=1}^{n+1-i}(-1)^{k+1}D^{2k+i}u\Big)\Big|_0^L - \frac{1}{2}\sum_{j=1}^{n+1}(D^ju)^2\Big|_0^L.$$

This implies (6) for all $l \in \mathbb{N}$.

Lemma 2.3 Let $u \in C^{2l+1}([0, L])$, $l \in \mathbb{N}$. Then

$$\sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u, xu) = \sum_{i=0}^{l-1}xD^iu\Big(\sum_{k=1}^{l-i}(-1)^{k+1}D^{2k+i}u\Big)\Big|_0^L$$

$$+\sum_{i=0}^{l-1}(1+i)D^i u\Big(\sum_{k=1}^{l-i}(-1)^k D^{2k+i-1}u\Big)\Big|_0^L - \frac{x}{2}\sum_{j=1}^{l}(D^j u)^2\Big|_0^L$$

$$+\sum_{j=1}^{l}\frac{(2j+1)}{2}\| D^j u\|^2. \tag{7}$$

Proof. The case $l = 1$ follows by (5). Suppose assertion (7) is valid for some integer $n \geq 1$ and assume $u \in C^{2n+3}([0, L])$. Induction hypothesis and (5) imply

$$\sum_{j=1}^{n+1}(-1)^{j+1}(D^{2j+1}u, xu) = \sum_{j=1}^{n}(-1)^{j+1}(D^{2j+1}u, xu)$$

$$+(-1)^n(D^{2n+3}u, xu) = \sum_{i=0}^{n-1} xD^i u\Big(\sum_{k=1}^{n-i}(-1)^{k+1}D^{2k+i}u\Big)\Big|_0^L$$

$$+\sum_{i=0}^{n-1}(1+i)D^i u\Big(\sum_{k=1}^{l-i}(-1)^k D^{2k+i-1}u\Big)\Big|_0^L - \frac{x}{2}\sum_{j=1}^{n}(D^j u)^2\Big|_0^L$$

$$+\sum_{j=1}^{n}\frac{(2j+1)}{2}\| D^j u\|^2 + \sum_{k=1}^{n+1}(-1)^{n+k+1}xD^{k-1}uD^{(2n+3)-k}u\Big|_0^L$$

$$-\frac{x}{2}(D^{n+1}u)^2\Big|_0^L + \sum_{k=1}^{n+1}(-1)^{n+k}kD^{k-1}uD^{(2n+2)-k}\Big|_0^L + \frac{(2n+3)}{2}\| D^{n+1}u\|^2$$

$$= \sum_{i=0}^{n-1} xD^i u\Big(\sum_{k=1}^{n-i}(-1)^{k+1}D^{2k+i}u + (-1)^{n-i}D^{2n+2-i}\Big)\Big|_0^L + xD^n u D^{n+2}u\Big|_0^L$$

$$+\sum_{i=0}^{n-1}(1+i)D^i u\Big(\sum_{k=1}^{n-i}(-1)^k D^{2k+i-1}u + (-1)^{n-i+1}D^{2n+1-i}\Big)\Big|_0^L$$

$$-(1+n)D^n u D^{n+1}u\Big|_0^L - \frac{x}{2}\sum_{j=1}^{n+1}(D^j u)^2\Big|_0^L + \sum_{j=1}^{n+1}\frac{(2j+1)}{2}\| D^j u\|^2$$

$$= \sum_{i=0}^{n} xD^i u\Big(\sum_{k=1}^{n+1-i}(-1)^{k+1}D^{2k+i}u\Big)\Big|_0^L$$

$$+\sum_{i=0}^{n}(1+i)D^i u\Big(\sum_{k=1}^{n+1-i}(-1)^k D^{2k+i-1}u\Big)\Big|_0^L - \frac{x}{2}\sum_{j=1}^{n+1}(D^j u)^2\Big|_0^L$$

$$+\sum_{j=1}^{n+1}\frac{(2j+1)}{2}\| D^j u\|^2.$$

This proves (7) for all $l \in \mathbb{N}$.

Lemma 2.4 (See [31], p.125). Suppose u and $D^m u$, $m \in \mathbb{N}$ belong to $L^2(0, L)$. Then for the derivatives $D^i u$, $0 \le i < m$, the following inequality holds:

$$\| D^i u \| \le C_1 \| D^m u \|^{\frac{i}{m}} \| u \|^{1-\frac{i}{m}} + C_2 \| u \|, \tag{8}$$

where C_1, C_2 are constants depending only on L, m, i.

3. Formulation of the problem

Let L, λ be real positive numbers and $l \in \mathbb{N}$. Consider the higher-order stationary dispersive equation

$$\lambda u + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} u = f(x),\ x \in (0, L), \tag{9}$$

subject to a correct set of boundary conditions (l conditions at $x = 0$ and $l + 1$ conditions at $x = L$, see [32])
l = 1:

$$u(0) = u(L) = Du(L) = 0; \tag{10}$$

l ≥ 2:

$$u(0) = u(L) = 0, \tag{11}$$

$$D^i u(0) = \sum_{j=1}^{l} a_{ij} D^j u(0),\ i = l+1, \dots, 2l-1, \tag{12}$$

$$D^i u(L) = \sum_{j=1}^{l-1} b_{ij} D^j u(L),\ i = l, \dots, 2l-1, \tag{13}$$

where a_{ij}, b_{ij} are real constants and $f \in L^2(0, L)$ is a given function. Assumptions on the coefficients imply estimate in L^2 -norm. In other words, multiplying (9) by u and integrating over $(0, L)$, we get

$$\lambda \| u \|^2 + \sum_{j=1}^{l} (-1)^{j+1} (D^{2j+1} u, u) \le \| f \| \, \| u \|.$$

A natural way to obtain $\| u \| \le \frac{1}{\lambda} \| f \|$ is to choose a_{ij}, b_{ij} such that $I = \sum_{j=1}^{l} (-1)^{j+1} (D^{2j+1} u, u) \ge 0$. When $l = 2$, (11)-(13) become

$$u(0) = u(L) = 0,\ D^3 u(0) = a_{31} Du(0) + a_{32} D^2 u(0),$$

$$D^2 u(L) = b_{21} Du(L),\ D^3 u(L) = b_{31} Du(L). \tag{14}$$

Substituting (14) into (6), we obtain

$$I = \Big(b_{31} - \frac{1}{2} - \frac{b_{21}^2}{2}\Big)(Du(L))^2 + \Big(-a_{31} + \frac{1}{2}\Big)(Du(0))^2$$

$$-a_{32}Du(0)D^2u(0)+\frac{1}{2}(D^2u(0))^2.$$

By the Cauchy inequality with $\varepsilon = 1$, we get

$$I \geq \left(b_{31}-\frac{1}{2}-\frac{b_{21}^2}{2}\right)(Du(L))^2+\left(-a_{31}+\frac{1}{2}-a_{32}^2\right)(Du(0))^2+\left(\frac{1}{2}-\frac{1}{4}\right)(D^2u(0))^2.$$

In order to obtain $I \geq 0$, we must have

$$B_1 = b_{31}-\frac{1}{2}-\frac{b_{21}^2}{2} > 0,\ A_1 = -a_{31}+\frac{1}{2}-a_{32}^2 > 0,\ A_2 = \frac{1}{4}. \tag{15}$$

This implies that $b_{31} > \frac{1}{2}$, $a_{31} < \frac{1}{2}$, and $|a_{32}|$, $|b_{21}|$ should be sufficiently small or zero. For $l = 3$, (11)-(13) become

$$\begin{aligned}
&u(0) = u(L) = 0,\\
&D^4u(0) = a_{41}Du(0) + a_{42}D^2u(0) + a_{43}D^3u(0),\\
&D^5u(0) = a_{51}Du(0) + a_{52}D^2u(0) + a_{53}D^3u(0),\\
&D^3u(L) = b_{31}Du(L) + b_{32}D^2u(L),\\
&D^4u(L) = b_{41}Du(L) + b_{42}D^2u(L),\\
&D^5u(L) = b_{51}Du(L) + b_{52}D^2u(L).
\end{aligned} \tag{16}$$

Substituting (16) into (6), we obtain

$$I = \left(b_{31}-b_{51}-\frac{1}{2}-\frac{b_{31}^2}{2}\right)(Du(L))^2+\left(b_{42}-\frac{1}{2}-\frac{b_{32}^2}{2}\right)(D^2u(L))^2$$

$$+\left(b_{32}-b_{52}+b_{41}-b_{31}b_{32}\right)Du(L)D^2u(L)+\left(a_{51}+\frac{1}{2}\right)(Du(0))^2$$

$$+\left(-a_{42}+\frac{1}{2}\right)(D^2u(0))^2+\frac{1}{2}(D^3u(0))^2+(a_{52}-a_{41})Du(0)D^2u(0)$$

$$+(-1+a_{53})Du(0)D^3u(0)-a_{43}D^2u(0)D^3u(0).$$

By the Cauchy inequality with $\varepsilon = 1$, it follows that

$$I \geq \left(b_{31}-b_{51}-\frac{1}{2}-b_{31}^2-\frac{1}{2}(|b_{32}|+|b_{52}|+|b_{41}|)\right)(Du(L))^2$$

$$+\left(b_{42}-\frac{1}{2}-b_{32}^2-\frac{1}{2}(|b_{32}|+|b_{52}|+|b_{41}|)\right)(D^2u(L))^2$$

$$+\left(a_{51}-\frac{1}{2}-\frac{1}{2}(|a_{52}|+|a_{41}|+|a_{53}|)\right)(Du(0))^2$$

$$+\left(-a_{42}+\frac{1}{2}-\frac{1}{2}(|a_{52}|+|a_{41}|+|a_{43}|)\right)(D^2u(0))^2$$

$$+\left(\frac{1}{4}-\frac{1}{2}(|a_{53}|+|a_{43}|)\right)(D^3u(0))^2.$$

To have $I \geq 0$, the coefficients must satisfy the following inequalies:

$$B_1 = b_{31}-b_{51}-\frac{1}{2}-b_{31}^2-\frac{1}{2}(|b_{32}|+|b_{52}|+|b_{41}|)>0,$$

$$B_2 = b_{42}-\frac{1}{2}-b_{32}^2-\frac{1}{2}(|b_{32}|+|b_{52}|+|b_{41}|)>0,$$

$$A_1 = a_{51}-\frac{1}{2}-\frac{1}{2}(|a_{52}|+|a_{41}|+|a_{53}|)>0, \tag{17}$$

$$A_2 = -a_{42}+\frac{1}{2}-\frac{1}{2}(|a_{52}|+|a_{41}|+|a_{43}|)>0,$$

$$A_3 = \frac{1}{4}-\frac{1}{2}(|a_{53}|+|a_{43}|)>0.$$

This implies that $b_{51} < -\frac{1}{2}$, $b_{42} > \frac{1}{2}$, $a_{51} > \frac{1}{2}$, $a_{42} < \frac{1}{2}$ and the remaining coefficients should be sufficiently small or zero. Let $l \geq 4$. By (6),

$$I = \sum_{i=0}^{l-1} D^i u(L)\left(\sum_{k=1}^{l-i}(-1)^{k+1}D^{2k+i}u(L)\right)-\frac{1}{2}\sum_{i=0}^{l-1}(D^{i+1}u(L))^2 \tag{18}$$

$$+\sum_{i=0}^{l-1} D^i u(0)\left(\sum_{k=1}^{l-i}(-1)^{k}D^{2k+i}u(0)\right)+\frac{1}{2}\sum_{i=0}^{l-1}(D^{i+1}u(0))^2. \tag{19}$$

Conditions at x = L: Substituting (11), (13) into (18), we find

$$I_L = \sum_{i=0}^{l-1} D^i u(L)\left(\sum_{k=1}^{l-i}(-1)^{k+1}D^{2k+i}u(L)\right)-\frac{1}{2}\sum_{i=0}^{l-1}(D^{i+1}u(L))^2$$

$$=\sum_{i=1}^{l-1}\Bigg[\sum_{\substack{k=1\\2k+i\leq l-1}}(-1)^{k+1}D^iu(L)D^{2k+i}u(L)+\sum_{\substack{k=1\\2k+i\geq l}}^{l-i}(-1)^{k+1}D^iu(L)D^{2k+i}u(L)\Bigg]$$

$$-\frac{1}{2}\sum_{i=0}^{l-2}(D^{i+1}u(L))^2-\frac{1}{2}(D^lu(L))^2=\sum_{i=1}^{l-3}\sum_{\substack{k=1\\2k+i\leq l-1}}(-1)^{k+1}D^iu(L)D^{2k+i}u(L)$$

$$+\sum_{i=1}^{l-1}\sum_{\substack{k=1\\2k+i\geq l}}^{l-i}\sum_{j=1}^{l-1}(-1)^{k+1}b_{2k+i,j}D^iu(L)D^ju(L)-\frac{1}{2}\sum_{i=0}^{l-2}(D^{i+1}u(L))^2$$

$$-\frac{1}{2}\Big(\sum_{j=1}^{l-1} b_{lj}(D^j u(L))\Big)^2 = \sum_{i=1}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\geq l}}^{l-i}(-1)^{k+1}b_{2k+i,i} - \frac{1}{2} - \frac{b_{li}^2}{2}\Big)(D^i u(L))^2$$

$$+\sum_{i=1}^{l-3}\sum_{\substack{k=1\\2k+i\leq l-1}}(-1)^{k+1}D^i u(L)D^{2k+i}u(L)$$

$$+\sum_{\substack{i,j=1\\i\neq j}}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\geq l}}^{l-i}(-1)^{k+1}b_{2k+i,j}\Big)D^i u(L)D^j u(L) - \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{l-1} b_{li}b_{lj}D^i u(L)D^j u(L).$$

We deduce

$$I_1 = \sum_{i=1}^{l-3}\sum_{\substack{k=1\\2k+i\leq l-1}}(-1)^{k+1}D^i u(L)D^{2k+i}u(L) \geq \frac{3-l}{2}\sum_{i=1}^{l-1}(D^i u(L))^2. \tag{20}$$

The proof is an induction on l. For $l = 4$, we have

$$Du(L)D^3u(L) \geq -\frac{1}{2}\sum_{i=1}^{3}(D^i u(L))^2 = \frac{3-4}{2}\sum_{i=1}^{4-1}(D^i u(L))^2.$$

Assume assertion (20) is valid for some integer $m \geq 4$. Then

$$\sum_{i=1}^{m-2}\sum_{\substack{k=1\\2k+i\leq m}}(-1)^{k+1}D^i u(L)D^{2k+i}u(L) = \sum_{i=1}^{m-3}\sum_{\substack{k=1\\2k+i\leq m}}(-1)^{k+1}D^i u(L)D^{2k+i}u(L)$$

$$+D^{m-2}u(L)D^m u(L) = \sum_{i=1}^{m-3}\sum_{\substack{k=1\\2k+i\leq m-1}}(-1)^{k+1}D^i u(L)D^{2k+i}u(L)$$

$$+\sum_{i=1}^{m-3}\sum_{\substack{k=1\\2k+i=m}}(-1)^{k+1}D^i u(L)D^m u(L) + D^{m-2}u(L)D^m u(L)$$

$$\geq \frac{3-m}{2}\sum_{i=1}^{m-1}(D^i u(L))^2 - \frac{1}{2}\sum_{i=1}^{m-3}(D^i u(L))^2 + \frac{3-m}{2}(D^m u(L))^2$$

$$-\frac{1}{2}\sum_{i=m-2}^{m}(D^i u(L))^2 \geq \Big(\frac{3-m}{2} - \frac{1}{2}\Big)\sum_{i=1}^{m-3}(D^i u(L))^2$$

$$+\Big(\frac{3-m}{2} - \frac{1}{2}\Big)\sum_{i=m-2}^{m-1}(D^i u(L))^2 + \Big(\frac{3-m}{2} - \frac{1}{2}\Big)(D^m u(L))^2$$

$$= \frac{2-m}{2}\sum_{i=1}^{m}(D^i u(L))^2.$$

This proves (20) for all $l \ge 4$.
For i, j fixed, by the Cauchy inequality, we obtain

$$\Big(\sum_{\substack{k=1\\2k+i\ge l}}^{l-i}(-1)^{k+1}b_{2k+i,j}\Big)D^i u(L)D^j u(L)$$

$$\ge -\frac{1}{2}\Big(\sum_{\substack{k=1\\2k+i\ge l}}^{l-i}\left|b_{2k+i,j}\right|\Big)^2(D^i u(L))^2 - \frac{1}{2}(D^j u(L))^2.$$

Summing over $i, j = 1,\cdots, l-1$ with $i \ne j$, we get

$$I_2 = \sum_{\substack{i,j=1\\i\ne j}}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\ge l}}^{l-i}(-1)^{k+1}b_{2k+i,j}\Big)D^i u(L)D^j u(L)$$

$$\ge -\frac{1}{2}\sum_{i=1}^{l-1}\Big[\sum_{\substack{j=1\\j\ne i}}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\ge l}}^{l-i}\left|b_{2k+i,j}\right|\Big)^2 + l-2\Big](D^i u(L))^2. \tag{21}$$

It is easy to see that

$$I_3 = -\frac{1}{2}\sum_{\substack{i,j=1\\i\ne j}}^{l-1}b_{li}b_{lj}D^i u(L)D^j u(L) \ge \frac{2-l}{2}\sum_{i=1}^{l-1}b_{li}^2(D^i u(L))^2.$$

Substituting $I_1 + I_2 + I_3$ into I_L, we conclude

$$I_L \ge \sum_{i=1}^{l-1}\Big[\sum_{\substack{k=1\\2k+i\ge l}}^{l-i}(-1)^{k+1}b_{2k+i,i} + (2-l)$$

$$+\frac{(1-l)}{2}b_{li}^2 - \frac{1}{2}\sum_{\substack{j=1\\j\ne i}}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\ge l}}^{l-i}\left|b_{2k+i,j}\right|\Big)^2\Big](D^i u(L))^2.$$

Hence, for $I_L \ge 0$, the coefficients b_{ij} must satisfy

$$B_i = \sum_{\substack{k=1\\2k+i\ge l}}^{l-i}(-1)^{k+1}b_{2k+i,i} + (2-l) + \frac{(1-l)}{2}b_{li}^2$$

$$-\frac{1}{2}\sum_{\substack{j=1\\j\ne i}}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\ge l}}^{l-i}\left|b_{2k+i,j}\right|\Big)^2 > 0,\ \ i = 1,\ldots, l-1. \tag{22}$$

This implies

$$
\begin{aligned}
&b_{l+1,l-1} > l-2,\\
&b_{l+j,l-j} > \frac{1}{2}\Big(\sum_{m=1}^{\frac{j-1}{2}} |b_{l+2m-1,l-2m+1}|\Big)^2 + l - 2, \quad \underbrace{j = 3,\ldots,\ l-1}_{(j\ \text{odd})},\\
&b_{l+2,l-2} < 2-l,\\
&b_{l+j,l-j} < -\frac{1}{2}\Big(\sum_{m=1}^{\frac{j}{2}-1} |b_{l+2m,l-2m}|\Big)^2 + 2 - l, \quad \underbrace{j = 4,\ldots,l-1}_{(j\ \text{even})},
\end{aligned}
\tag{23}
$$

and the remaining coefficients of (22) should be sufficiently small or zero.

Conditions at x = 0: Substituting (11)-(12) into (19), we get

$$
I_0 = \sum_{i=0}^{l-1} D^i u(0)\Big(\sum_{k=1}^{l-i}(-1)^k D^{2k+i}u(0)\Big) + \frac{1}{2}\sum_{i=0}^{l-1}(D^{i+1}u(0))^2
$$

$$
= \sum_{i=1}^{l-1}\Big[\sum_{\substack{k=1\\2k+i\le l}}(-1)^k D^i u(0) D^{2k+i}u(0) + \sum_{\substack{k=1\\2k+i\ge l+1}}^{l-i}(-1)^k D^i u(0) D^{2k+i}u(0)\Big]
$$

$$
+\frac{1}{2}\sum_{i=0}^{l-1}(D^{i+1}u(0))^2 = \sum_{i=1}^{l-2}\sum_{\substack{k=1\\2k+i\le l}}(-1)^k D^i u(0) D^{2k+i}u(0)
$$

$$
+\sum_{i=1}^{l-1}\sum_{\substack{k=1\\2k+i\ge l+1}}^{l-i}\sum_{j=1}^{l}(-1)^k a_{2k+i,j} D^i u(0) D^j u(0) + \frac{1}{2}\sum_{i=0}^{l-1}(D^{i+1}u(0))^2
$$

$$
= \sum_{i=1}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\ge l+1}}^{l-i}(-1)^k a_{2k+i,i} + \frac{1}{2}\Big)(D^i u(0))^2 + \frac{1}{2}(D^l u(0))^2
$$

$$
+\sum_{i=1}^{l-2}\sum_{\substack{k=1\\2k+i\le l}}(-1)^k D^i u(0) D^{2k+i}u(0) + \sum_{\substack{i,j=1\\i\ne j}}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\ge l+1}}^{l-i}(-1)^k a_{2k+i,j}\Big) D^i u(0) D^j u(0)
$$

$$
+\sum_{i=1}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\ge l+1}}^{l-i}(-1)^k a_{2k+i,l}\Big) D^i u(0) D^l u(0).
$$

Making use of (20) with 0 instead L, and the Cauchy inequality with an arbitrary $\varepsilon > 0$, we obtain

$$
I_1 = \sum_{i=1}^{l-2}\sum_{\substack{k=1\\2k+i\le l}}(-1)^k D^i u(0) D^{2k+i}u(0) = \sum_{i=1}^{l-3}\sum_{\substack{k=1\\2k+i\le l-1}}(-1)^k D^i u(0) D^{2k+i}u(0)
$$

$$+\sum_{i=1}^{l-3}\sum_{\substack{k=1\\2k+i=l}}(-1)^k D^i u(0)D^l u(0)-D^{l-2}u(0)D^l u(0)\geq\frac{3-l}{2}\sum_{i=1}^{l-1}(D^i u(0))^2$$

$$-\frac{\varepsilon}{2}\sum_{i=1}^{l-3}(D^i u(0))^2+\frac{3-l}{2\varepsilon}(D^l u(0))^2-\frac{\varepsilon}{2}(D^{l-2}u(0)+D^{l-1}u(0))$$

$$-\frac{1}{2\varepsilon}(D^l u(0))^2=\Big(\frac{3-l-\varepsilon}{2}\Big)\sum_{i=1}^{l-1}(D^i u(0))^2+\frac{2-l}{2\varepsilon}(D^l u(0))^2.$$

Taking $\varepsilon = 2(l-2)$, we conclude

$$I_1\geq\frac{7-3l}{2}\sum_{i=1}^{l-1}(D^i u(0))^2-\frac{1}{4}(D^l u(0))^2.$$

Acting as by the proof of (21), we obtain

$$I_2=\sum_{\substack{i,j=1\\i\neq j}}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\geq l+1}}^{l-i}(-1)^k a_{2k+i,j}\Big)D^i u(0)D^j u(0)$$

$$\geq-\frac{1}{2}\sum_{i=1}^{l-1}\Big[\sum_{\substack{j=1\\j\neq i}}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\geq l+1}}^{l-i}\left|a_{2k+i,j}\right|\Big)^2+l-2\Big](D^i u(0))^2.$$

Applying the Cauchy inequality for i fixed, we get

$$\Big(\sum_{\substack{k=1\\2k+i\geq l+1}}^{l-i}(-1)^k a_{2k+i,l}\Big)D^i u(0)D^l u(0)$$

$$\geq-\frac{1}{2}\sum_{\substack{k=1\\2k+i\geq l+1}}^{l-i}\left|a_{2k+i,l}\right|(D^i u(0))^2-\frac{1}{2}\sum_{\substack{k=1\\2k+i\geq l+1}}^{l-i}\left|a_{2k+i,l}\right|(D^l u(0))^2.$$

Summing over $i = 1,\dots,l-1$, we find

$$I_3=\sum_{i=1}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\geq l+1}}^{l-i}(-1)^k a_{2k+i,l}\Big)D^i u(0)D^l u(0)$$

$$\geq-\frac{1}{2}\sum_{i=1}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\geq l+1}}^{l-i}\left|a_{2k+i,l}\right|\Big)(D^i u(0))^2-\frac{1}{2}\Big[\sum_{i=1}^{l-1}\Big(\sum_{\substack{k=1\\2k+i\geq l+1}}^{l-i}\left|a_{2k+i,l}\right|\Big)\Big](D^l u(0))^2.$$

Substituting $I_1+I_2+I_3$ into I_0, we conclude

$$I_0 \geq \sum_{i=1}^{l-1}\Big[\sum_{\substack{k=1 \\ 2k+i\geq l+1}}^{l-i} (-1)^k a_{2k+i,i} + (5-2l)$$

$$-\frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{l-1}\Big(\sum_{\substack{k=1 \\ 2k+i\geq l+1}}^{l-i} |a_{2k+i,j}|\Big)^2 - \frac{1}{2}\sum_{\substack{k=1 \\ 2k+i\geq l+1}}^{l-i} |a_{2k+i,l}|\Big](D^i u(0))^2$$

$$+\Big[\frac{1}{4} - \frac{1}{2}\sum_{i=1}^{l-1}\Big(\sum_{\substack{k=1 \\ 2k+i\geq l+1}}^{l-i} |a_{2k+i,l}|\Big)\Big](D^l u(0))^2.$$

Obviously, $I_0 \geq 0$ if the coefficients a_{ij} satisfy the following conditions:

$$A_i = \sum_{\substack{k=1 \\ 2k+i\geq l+1}}^{l-i} (-1)^k a_{2k+i,i} + (5-2l) - \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{l-1}\Big(\sum_{\substack{k=1 \\ 2k+i\geq l+1}}^{l-i} |a_{2k+i,j}|\Big)^2$$

$$-\frac{1}{2}\sum_{\substack{k=1 \\ 2k+i\geq l+1}}^{l-i} |a_{2k+i,l}| > 0, \quad i = 1, \ldots, l-1, \tag{24}$$

$$A_l = \frac{1}{4} - \frac{1}{2}\sum_{i=1}^{l-1}\Big(\sum_{\substack{k=1 \\ 2k+i\geq l+1}}^{l-i} |a_{2k+i,l}|\Big) > 0. \tag{25}$$

This implies

$$a_{l+1,\, l-1} < 5 - 2l,$$

$$a_{l+j,l-j} < -\frac{1}{2}\Big(\sum_{m=1}^{\frac{j-1}{2}} |a_{l+2m-1,l-2m+1}|\Big)^2 + 5 - 2l, \quad \underbrace{j = 3, \ldots, l-1}_{(j \text{ odd})},$$

$$a_{l+2,\, l-2} > 2l - 5, \tag{26}$$

$$a_{l+j,l-j} > \frac{1}{2}\Big(\sum_{m=1}^{\frac{j}{2}-1} |a_{l+2m,l-2m}|\Big)^2 + 2l - 5, \quad \underbrace{j = 4, \ldots, l-1}_{(j \text{ even})},$$

and the remaining coefficients of (24)-(25) should be sufficiently small or zero.

Assuming $b_{31} > \frac{1}{2}$, $a_{31} < \frac{1}{2}$ for $l = 2$; $b_{51} < -\frac{1}{2}$, $b_{42} > \frac{1}{2}$, $a_{51} > \frac{1}{2}$, $a_{42} < \frac{1}{2}$ for $l = 3$ and (23), (26) for $l \geq 4$ and the remaining coefficients in (15), (17), (22), (24), (25) equals to zero, we get the following boundary conditions for all $l \in \mathbb{N}$:

$$u(0) = u(L) = D^l u(L) = 0,$$

$$D^{l+j}u(0) = a_{l+j,l-j} D^{l-j} u(0), \quad j = 1, \ldots, l-1, \tag{27}$$

$$D^{l+j}u(L) = b_{l+j,l-j} D^{l-j} u(L), \quad j = 1, \ldots, l-1.$$

Remark 1 We call (12)-(13) general boundary conditions because they follow from a more general form:

$$\sum_{i=1}^{2l-1} \alpha_{ki} D^i u(0) = 0, \ k = 1, \ldots, l-1, \tag{28}$$

$$\sum_{i=1}^{2l-1} \beta_{ki} D^i u(L) = 0, \ k = 1, \ldots, l, \tag{29}$$

where α_{ki}, β_{ki} are real numbers. Write (28)-(29) as

$$\sum_{i=l+1}^{2l-1} \alpha_{ki} D^i u(0) = -\sum_{j=1}^{l} \alpha_{kj} D^j u(0), \ k = 1, \ldots, l-1,$$

$$\sum_{i=l}^{2l-1} \beta_{ki} D^i u(L) = -\sum_{j=1}^{l-1} \beta_{kj} D^j u(L), \ k = 1, \ldots, l.$$

If $\det(\alpha_{ki}) \neq 0$, then $D^i u(0) = \dfrac{\det(\widehat{\alpha_{ki}})}{\det(\alpha_{ki})}$, $i = l + 1, ..., 2l - 1$, where $\widehat{(\alpha_{ki})}$ is the matrix formed by replacing the i-th column of (α_{ki}) by $-\sum_{j=1}^{l} \alpha_{kj} D^j u(0)$.

After simple calculations, we arrive to (12). Similarly, if $\det(\beta_{ki}) \neq 0$, then $D^i u(L) = \dfrac{\det(\widehat{\beta_{ki}})}{\det(\beta_{ki})}$, $i = l, ..., 2l - 1$, where $\widehat{(\beta_{ki})}$ is the matrix formed by replacing the i-th column of (β_{ki}) by $-\sum_{j=1}^{l-1} \beta_{kj} D^j u(L)$ and we come to (13).

Remark 2 All results established in this paper are proven for the case $l = 1$, see [21]. From here on, we will consider $l \geq 2$.

4. Existence and uniqueness of regular solutions

For a real $\lambda > 0$ and a given function f, consider the equation

$$\lambda u + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} u = f(x), \ x \in (0, L), \tag{30}$$

subject to boundary conditions (27) with the coefficients satisfying: $b_{31} > \frac{1}{2}$, $a_{31} < \frac{1}{2}$ for $l = 2$; $b_{51} < -\frac{1}{2}$, $b_{42} > \frac{1}{2}$, $a_{51} > \frac{1}{2}$, $a_{42} < \frac{1}{2}$ for $l = 3$ and (23), (26) for $l \geq 4$.

Theorem 4.1 Let $f \in L^2(0, L)$. Then for all $\lambda > 0$, there exists a unique regular solution $u = u(x) \in H^{2l+1}(0, L)$ for the problem (30), (27), such that

$$\| u \|_{H^{2l+1}} \leq C \| f \|, \tag{31}$$

where C is a constant depending on L, l, λ, $a_{l+j,l-j}$, $b_{l+j,l-j}$, $j = 1, ..., l - 1$.

Proof. Suppose initially $f \in C([0, L])$ and consider the homogeneous equation

$$\lambda u + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} u = 0 \ \text{ in } \ (0, L), \tag{32}$$

subject to boundary conditions (27). It is known, see [33], that (30), (27) has a unique classical solution $u \in C^{2l+1}([0, L])$ if and only if (32), (27) has only the trivial solution. Let $u \in C^{2l+1}([0, L])$ be a nontrivial solution of (32), (27), then multiplying (32) by u and integrating over $(0, L)$, we obtain

$$\lambda \| u \|^2 + \sum_{j=1}^{l} (-1)^{j+1} (D^{2j+1}u, u) = 0.$$

Making use of (6) and boundary conditions (27) with the coefficients satisfying: $b_{31} > \frac{1}{2}$, $a_{31} < \frac{1}{2}$ for $l = 2$; $b_{51} < -\frac{1}{2}$, $b_{42} > \frac{1}{2}$, $a_{51} > \frac{1}{2}$, $a_{42} < \frac{1}{2}$ for $l = 3$ and (23), (26) for $l \geq 4$, we get

$$\sum_{j=1}^{l} (-1)^{j+1} (D^{2j+1}u, u) \geq \sum_{i=1}^{l-1} B_i (D^i u(L))^2 + \sum_{i=1}^{l} A_i (D^i u(0))^2 \geq 0, \tag{33}$$

for all $l \geq 2$, which implies $\lambda \| u \|^2 \leq 0$. Since $\lambda > 0$, it follows that $u \equiv 0$ and (30), (27) has a unique classical solution $u = u(x) \in C^{2l+1}([0, L])$.

Estimate 1 Multiply (30) by u and integrate over $(0, L)$ to obtain

$$\lambda \| u \|^2 + \sum_{j=1}^{l} (-1)^{j+1} (D^{2j+1}u, u) = (f, u).$$

Taking $M_1 = \min_{i \in \{1, \ldots, l-1\}} \{B_i, A_i, A_l\}$ in (33) and making use of the Cauchy-Schwarz inequality, we get

$$\lambda \| u \|^2 + M_1 \Big(\sum_{i=1}^{l-1} \left[(D^i u(L))^2 + (D^i u(0))^2 \right] + (D^l u(0))^2 \Big) \leq \| f \| \, \| u \|, \tag{34}$$

which implies

$$\| u \| \leq \frac{1}{\lambda} \| f \|. \tag{35}$$

Substituting (35) into (34), we find

$$\sum_{i=1}^{l-1} \left[(D^i u(L))^2 + (D^i u(0))^2 \right] + (D^l u(0))^2 \leq \frac{1}{\lambda M_1} \| f \|^2. \tag{36}$$

Estimate 2 Multiply (30) by $(1 + x)u$ and integrate over $(0, L)$ to obtain

$$\lambda (1 + x, u^2) + \sum_{j=1}^{l} (-1)^{j+1} (D^{2j+1}u, (1 + x)u) = (f, (1 + x)u). \tag{37}$$

By the Cauchy inequality with an arbitrary $\varepsilon > 0$, we estimate

$$(f, (1 + x)u) \leq \frac{\varepsilon}{2} (1 + x, u^2) + \frac{1}{2\varepsilon} (1 + x, f^2). \tag{38}$$

Substituting (38) into (37) and taking $\varepsilon = \lambda$, we get

$$\frac{\lambda}{2} \| u \|^2 + \sum_{j=1}^{l} (-1)^{j+1} (D^{2j+1}u, (1 + x)u) \leq \frac{1 + L}{2\lambda} \| f \|^2. \tag{39}$$

Making use of (6), (7) and boundary conditions (27) with the coefficients satisfying: $b_{31} > \frac{1}{2}$, $a_{31} < \frac{1}{2}$ for $l = 2$; $b_{51} < -\frac{1}{2}$, $b_{42} > \frac{1}{2}$, $a_{51} > \frac{1}{2}$, $a_{42} < \frac{1}{2}$ for $l = 3$ and (23), (26) for $l \geq 4$, we find

$$I = \sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u,(1+x)u) = \sum_{i=0}^{l-1}(1+x)D^iu\Big(\sum_{k=1}^{l-i}(-1)^{k+1}D^{2k+i}u\Big)\Big|_0^L$$

$$+\sum_{i=0}^{l-1}(1+i)D^iu\Big(\sum_{k=1}^{l-i}(-1)^k D^{2k+i-1}u\Big)\Big|_0^L - \frac{(1+x)}{2}\sum_{j=1}^{l}(D^ju)^2\Big|_0^L$$

$$+\sum_{j=1}^{l}\frac{(2j+1)}{2}\| D^ju\|^2 \geq (1+L)\sum_{i=1}^{l-1}B_i(D^iu(L))^2 + \sum_{i=1}^{l}A_i(D^iu(0))^2$$

$$+\sum_{i=1}^{l-1}(1+i)D^iu\Big(\sum_{k=1}^{l-i}(-1)^k D^{2k+i-1}u\Big)\Big|_0^L + \sum_{j=1}^{l}\frac{(2j+1)}{2}\| D^ju\|^2.$$

Substituting I into (39), we obtain

$$\frac{\lambda}{2}\| u\|^2 + \sum_{j=1}^{l}\frac{(2j+1)}{2}\| D^ju\|^2 + (1+L)\sum_{i=1}^{l-1}B_i(D^iu(L))^2 + \sum_{i=1}^{l}A_i(D^iu(0))^2$$

$$\leq \frac{1+L}{2\lambda}\| f\|^2 - \sum_{i=1}^{l-1}(1+i)D^iu\Big(\sum_{k=1}^{l-i}(-1)^k D^{2k+i-1}u\Big)\Big|_0^L. \tag{40}$$

Making use of (27) and applying the Cauchy inequality, we find

$$-\sum_{i=1}^{l-1}(1+i)D^iu\Big(\sum_{k=1}^{l-i}(-1)^k D^{2k+i-1}u\Big)\Big|_0^L \leq \sum_{i=1}^{l-1}(1+i)\,|D^iu(L)|$$

$$\times\Big(\sum_{k=1}^{l-i}|D^{2k+i-1}u(L)|\Big) + \sum_{i=1}^{l-1}(1+i)\,|D^iu(0)|\Big(\sum_{k=1}^{l-i}|D^{2k+i-1}u(0)|\Big)$$

$$\leq M_2\Big(\sum_{i=1}^{l-1}\big[(D^iu(L))^2 + (D^iu(0))^2\big] + (D^lu(0))^2\Big), \tag{41}$$

where M_2 is the maximum among all the coefficients of the derivatives $(D^lu(0))^2$, $(D^iu(0))^2$, $(D^iu(\mathrm{L}))^2$, $i = 1,\ldots, l-1$ in (41). Substituting (41) into (40) and taking into account (36), we get

$$\frac{\lambda}{2}\| u\|^2 + \sum_{j=1}^{l}\frac{(2j+1)}{2}\| D^ju\|^2 \leq \Big(\frac{1+L}{2\lambda} + \frac{M_2}{\lambda M_1}\Big)\| f\|^2.$$

Therefore

$$\| u\|_{H^l} \leq C\| f\|, \tag{42}$$

where C is a constant depending only on L, l, λ, $a_{l+j,l-j}$, $b_{l+j,l-j}$, $j = 1,\ldots, l-1$.

Finally, returning to (30) and making use of (8), we conclude that

$$\| u \|_{H^{2l+1}} \leq C\| f \|,$$

with a constant C depending only on L, l, λ, $a_{l+j,l-j}$, $b_{l+j,l-j}$, (see details in [21], p.4-5). Uniqueness of u follows from (35). In fact, such calculations must be performed for smooth solutions and the general case can be obtained using density arguments.

Remark 3 The problem (9)-(13) in Chapter 3 can be formulated under the following boundary conditions:

$$D^i u(0) = \sum_{j=0}^{l} a_{ij} D^j u(0),\ i = l+1,\ldots, 2l, \tag{43}$$

$$D^i u(L) = \sum_{j=0}^{l-1} b_{ij} D^j u(L),\ i = l,\ldots, 2l, \tag{44}$$

instead of (10)-(13). In fact, boundary conditions (10)-(13) are derived from (43)-(44) while one wants to study the nonlinear equation:

$$\lambda u + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} u + uDu = f(x),\ x \in (0, L). \tag{45}$$

Multiplying (45) by u and integrating over $(0, L)$, we get

$$\lambda \| u \|^2 (t) + \sum_{j=1}^{l} (-1)^{j+1} (D^{2j+1} u, u) + \frac{2}{3} u^3 (x) \Big|_0^L = (f, u).$$

So a natural way to obtain $\| u \| \leq \frac{1}{\lambda} \| f \|$ is suppose $u(0) = u(L) = 0$ and to choose a_{ij}, b_{ij} such that $\Sigma_{j=1}^{l} (-1)^{j+1} (D^{2j+1} u, u) \geq 0$. Note that, assuming $u(0) = u(L) = 0$, (6) gives us $(-1)^{l+1} u(x) D^{2l} u(x)|_0^L = 0$. This allows us to eliminate conditions at (43)-(44) when $i = 2l$, getting a correct set of boundary conditions (l conditions at $x = 0$ and $l + 1$ conditions at $x = L$): When $l = 1$, (43)-(44) become $u(0) = u(L) = Du(L) = 0$ and when $l \geq 2$, we get (11)-(13). We call (43)-(44) general boundary conditions because they follow from a more general form: (see Remark 1)

$$\sum_{i=0}^{2l} \alpha_{ki} D^i u(0) = 0,\ k = 1,\ldots, l,$$

$$\sum_{i=0}^{2l} \beta_{ki} D^i u(L) = 0,\ k = 1,\ldots, l+1,$$

where α_{ki}, β_{ki} are real numbers.

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Conflict of interest

The authors declare that there are no conflict of interest regarding the publication of this paper.

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