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Research Article

Geometric Properties and Neighborhoods of Certain Subclass of Analytic Functions Defined by Using Bell Distribution

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Abstract: A differential operator is defined on an open unit disk $\mathbb{D}$ using the innovative Bell Distribution operator. This operator introduces a new perspective in the study of complex functions within the disk. In this research, the established concept of neighborhoods plays a crucial role. By utilizing these neighborhoods, we aim to derive inclusion relations specifically concerning the (t, n)-neighborhoods of the classes defined by this operator. This approach allows for a deeper understanding of how these classes interact and overlap, providing valuable insights into their structural properties and potential applications in geometric function theory. Through this analysis, we hope to uncover new relationships and behaviors that can enhance our comprehension of differential operators in complex analysis.

***Keywords*:** univalent function, bell distribution (BD), starlike functions, convex functions, (t, n)-neighborhood, lnclusion relations

MSC: 30C45

1. Introduction

Geometric function theory is a branch of complex analysis that focuses on the study of analytic functions with geometric properties. It explores how these functions map shapes and regions in the complex plane, often using tools like conformal mappings and Riemann surfaces. This area of mathematics provides insights into understanding the behavior of functions in relation to geometric transformations, examining concepts such as univalence, quasiconformal mappings, and the theory of Teichmüller spaces. Geometric function theory has applications in various fields, including engineering, physics, and computer graphics, where the understanding of complex mappings is crucial.

The Bell distribution, often associated with the Bell curve or normal distribution, is a probability distribution that is symmetric and bell-shaped. It is characterized by its mean, median, and mode being equal and located at the center of the distribution. The spread of the curve is determined by the standard deviation, which measures the dispersion of data points around the mean. This distribution is fundamental in statistics because it describes how data tends to cluster around a central point, making it useful for modeling a wide range of natural and social phenomena. The Bell distribution is applied in fields such as psychology, finance, and the natural sciences, where it helps in making

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inferences about populations based on sample data. Its utility extends beyond numerical domains, impacting the modeling of complex systems and informing predictions based on empirical data in areas such as finance, economics, and psychology. The Bell distribution, introduced by Castellares et al. in 2018 [1], represents an improvement over the Bell numbers [2], showcasing its evolution and refinement.

The probability density function of a discrete random variable X that follows the Bell distribution is expressed in a concise manner, ensuring consistent representation of its statistical characteristics:

$$P(X = s) = \frac{\beta^s e^{e^{(-\beta^2)+1}} \lambda_s}{s!};\ s = 1,\ 2,\ 3,\ \ldots$$

where $\lambda_s = \frac{1}{e}\sum_{x=0}^{\infty} \frac{x^s}{x!}$ is the Bell numbers, $s \geq 2, \beta > 0$.

The first few numbers in the Bell sequence are $\lambda_2 = 2,\ \lambda_3 = 5,\ \lambda_4 = 15$, and $\lambda_5 = 52$.

We will now introduce a fresh power series, utilizing Bell distribution probabilities as coefficients.

$$\mathbb{T}(\beta,\ \varphi) = \varphi + \sum_{s=t+1}^{\infty} \frac{\beta^{s-1} e^{e^{(-\beta^2)+1}} \lambda_s}{(s-1)!} \varphi^s,\ \varphi \in \mathbb{D},\ \text{where } \beta > 0 \tag{1}$$

Coefficients can be viewed as probabilities linked to the Bell distribution.

Using the ratio test, it is possible to establish that the series mentioned earlier converges on the unit disk $\mathbb{D}$. The ratio test is a widely used and proven method for this purpose. This is certainly an achievable goal. Recently, the Bell distribution has been used to address specific problems related to probability in complex analysis. Notable contributions include the works of Aldawish et al. [3], Amourah et al. [4], and Alnajar et al. [5]. The advancement of complex analysis is the driving force behind this line of research.

Over the past few years, there has been a considerable amount of scholarly interest in the distribution operator, particularly in its correlation with univalent functions. Researchers have been exploring various aspects of this relationship, as highlighted in several key studies [6-10]. These studies have contributed to a deeper understanding of how distribution operators can be applied to univalent functions, which are functions that are analytic and injective in a given domain. The findings from this body of work not only advance theoretical knowledge but also have potential implications for practical applications in complex analysis and related fields. As the exploration of this topic continues, it is likely that new insights and methodologies will emerge, further enriching the academic discourse surrounding univalent functions and their associated distribution operators.

Let A represent the collection of all analytic functions, and let f be defined on the open unit disk. The conditions $\mathbb{D} = \{\varphi \in \mathbb{C}:\ |\varphi| < 1\}$, $f(0) = 0$, and $f'(0) - 1 = 0$ are necessary for this assertion to hold true. For every $f \in A$, construct the corresponding Taylor series, given by:

$$f(\varphi) = \varphi + \sum_{s=t+1}^{\infty} a_s \varphi^s,\ \left(\varphi \in \mathbb{D},\ a_s \geq 0;\ t \in N:\ = \{1,\ 2,\ 3,\ \ldots\}\right) \tag{2}$$

The investigation into the relationships of inclusion among analytic functions within specific sets was previously a topic covered in the study of geometric function theory. This was a topic that was once of interest to researchers. A study carried out by Amourah et al. in [11] and Hazha in [12] focused on the neighborhood and inclusion relations of univalent functions as its theme. Amourah et al. [13] conducted research on all of the inclusion characteristics of multivalent functions. In the field of geometric function theory, scholars have been focusing their attention on various subclasses of univalent functions in recent years.

Let us consider the linear operator $\vartheta_\beta f(\varphi): \mathcal{A} \to \mathcal{A}$ as

$$\vartheta_\beta f(\varphi) = \mathbb{T}(\beta,\ \varphi) * f(\varphi) = \varphi + \sum_{s=t+1}^{\infty} \frac{\beta^{s-1} e^{e\left(-\beta^2\right)+1} \lambda_s}{(s-1)!} a_s \varphi^t,\ \ \varphi \in \mathbb{D}$$

For $Q \geq 0$, we define the operator $\vartheta_{\beta,\ Q}^m f(\varphi) : \mathcal{A} \to \mathcal{A}$ as

$$\vartheta_{\beta,\ Q}^0 f(\varphi) = \varphi + \sum_{s=t+1}^{\infty} \frac{\beta^{s-1} e^{e\left(-\beta^2\right)+1} \lambda_s}{(s-1)!} a_s \varphi^s,$$

$$\vartheta_{\beta,\ Q}^1 f(\varphi) = (1-Q)\vartheta_{\beta,\ Q}^0 f(\varphi) + Q\varphi\left(\vartheta_{\beta,\ Q}^0 f(\varphi)\right)'$$

$$= \varphi + \sum_{s=t+1}^{\infty} \frac{\beta^{s-1} e^{e\left(-\beta^2\right)+1} \lambda_s}{(s-1)!} [1 + Q(s-1)] a_s \varphi^s,$$

$$\vartheta_{\beta,\ Q}^2 f(\varphi) = (1-Q)\vartheta_{\beta,\ Q}^1 f(\varphi) + Q\varphi\left(\vartheta_{\beta,\ Q}^1 f(\varphi)\right)$$

$$= \varphi + \sum_{s=t+1}^{\infty} \frac{\beta^{s-1} e^{e\left(-\beta^2\right)+1} \lambda_s}{(s-1)!} [1 + Q(s-1)]^2 a_s \varphi^s,$$

$$\vartheta_{\beta,\ Q}^m f(\varphi) = (1-Q)\vartheta_{\beta,\ Q}^{m-1} f(\varphi) + Q\varphi\left(\vartheta_{\beta,\ Q}^{m-1} f(\varphi)\right)'$$

$$\vdots$$

$$\vartheta_{\beta,\ Q}^m f(\varphi) = \varphi + \sum_{s=t+1}^{\infty} \frac{\beta^{s-1} e^{e\left(-\beta^2\right)+1} \lambda_s}{(s-1)!} [1 + Q(s-1)]^m a_s \varphi^s,$$

$$m \in \mathbb{N} \bigcup \{0\},\ Q \geq 0,\ 0 < \beta \leq 1$$

Let $\mathcal{A}(t)$ represent the set of functions $f(\varphi)$ that have the specific form:

$$f(\varphi) = \varphi - \sum_{s=t+1}^{\infty} a_s \varphi^s,\ \left(\varphi \in \mathbb{D},\ a_s \geqq 0;\ t \in N: \ = \{1,\ 2,\ 3,\ \ldots\}\right). \tag{3}$$

Following (4), we define the $(n,\ t)$-neighborhood of a function $f(\varphi) \in \mathcal{A}(t)$ by

$$N_{t,\ \Phi}(f): \ = \left\{ g \in \mathcal{A}(t):\ g(\varphi) = \varphi - \sum_{s=t+1}^{\infty} b_s \varphi^s \text{ and } \sum_{s=t+1}^{\infty} s\left|a_s - b_s\right| \leqq n \right\}. \tag{4}$$

More specifically, about the identity function

$$e(\varphi) = (\varphi)$$

We define as follows:

$$N_{t,\Phi}(e):=\left\{g\in\mathcal{A}(t):\ g(\varphi)=\varphi-\sum_{s=t+1}^{\infty}b_s z^s \text{ and } \sum_{s=t+1}^{\infty}s\left|b_s\right|\leqq n\right\}. \tag{5}$$

The main objective of this work is to investigate the $(t,\ \delta)$-neighborhoods of the following subclasses of $\mathcal{A}(t)$, which consist of normalized analytic functions in $\mathbb{D}$ with negative coefficients.

If the function $f(\varphi)\in\mathcal{A}(t)$ also satisfies the inequality, then it is considered to be starlike of complex order $\gamma(\gamma\in\mathbb{C}\setminus\{0\})$, which means that f belongs to the set $f\in S_t^*(w)$:

$$\operatorname{Re}\left\{1+\frac{1}{w}\left(\frac{\varphi f'(\varphi)}{f(\varphi)}-1\right)\right\}>0,\ (\varphi\in\mathbb{D},\ w\in\mathbb{C}\setminus\{0\})$$

In addition, a function $f(\varphi)$ that is part of the set $\mathcal{A}(t)$ is considered convex with respect to a complex number $w\left(w\in\mathbb{C}\setminus\{0\}\right)$, denoted as $f\in\mathcal{C}_t(w)$, if it satisfies the following condition:

$$\operatorname{Re}\left\{1+\frac{1}{w}\frac{\varphi f''(\varphi)}{f'(\varphi)}\right\}>0,\ (\varphi\in\mathbb{D},\ w\in\mathbb{C}\setminus\{0\})$$

It is worth noting that the classes $\mathcal{S}_t^*(w)$ and $\mathcal{C}_t(w)$ are derived from the classes of starlike and convex functions of complex order. Nasr and Aouf [14] previously studied these classes, and Wiatrowski [15] studied them as well. More information can be found in [16-18].

Definition 1 The subclass of. $A(n)$ comprised of functions $f(z)$ that meet the inequality is denoted as $S_t(w, h, q)$.

$$\left|\frac{1}{w}\left(\frac{z\left(\vartheta_{\beta,\,Q}^{m}f(\varphi)\right)'+h\varphi^2\left(\vartheta_{\beta,\,Q}^{m}f(\varphi)\right)''}{hz\left(\vartheta_{\beta,\,Q}^{m}f(\varphi)\right)'+(1-h)\left(\vartheta_{\beta,\,Q}^{m}f(\varphi)\right)}-1\right)\right|<q, \tag{6}$$

where $\varphi\in\mathbb{D},\ w\in\mathbb{C}\setminus\{0\},\ 0\le h\le 1,\ 0<q\le 1$.

Definition 2 The subclass of. $A(t)$ comprised of functions $f(\varphi)$ that meet the inequality is denoted as $R_n(w, h, q)$.

$$\left|\frac{1}{W}\left(\left(\vartheta_{\beta,\,Q}^{m}f(\varphi)\right)'+h\varphi\left(\vartheta_{\beta,\,Q}^{m}f(\varphi)\right)''-1\right)\right|<q, \tag{7}$$

where $\varphi\in\mathbb{D},\ w\in\mathbb{C}\setminus\{0\},\ 0\le h\le 1,\ 0<q\le 1$.

Clearly, we have

$$\mathcal{S}_t(w,\ 0,\ 1)\subset\mathcal{S}_t^*(w) \text{ and } \mathcal{R}_t(w,\ 0,\ 1)\subset\mathcal{C}_t(w)$$

$$(t\in\mathbb{N};\ w\in\mathbb{C}\setminus\{0\})$$

2. A collection of inclusion relations that involve the variables $N_{t,\Phi}(e)$

Definition 1 and 2 are going to be necessary for our analysis of the inclusion relations involving $N_{t,\Phi}(e)$, which will be discussed further below.

Theorem 1 If we assume that the function $f(\varphi) \in \mathcal{A}(t)$ is defined by the equation (3), then it follows that $f(\varphi)$ belongs to the class $S_n(w, h, q)$ if and only if

$$\sum_{s=t+1}^{\infty} (h(s-1)+1)(s+q\,|\,w\,|-1)\frac{\beta^{s-1}e^{e\left(-\beta^2\right)+1}\lambda_s}{(s-1)!}[1+Q(s-1)]^m a_s \le q\,|\,w\,| \tag{8}$$

Proof. Initially, we assume that $f(\varphi) \in \mathcal{S}_{\mathrm{n}}(w,\ h,\ q)$. Next, by utilizing condition (6), we can easily obtain

$$\mathrm{Re}\left\{\frac{\varphi\left(\vartheta^m_{\beta,\,Q} f(\varphi)\right)' + h\varphi^2\left(\vartheta^m_{\beta,\,Q} f(\varphi)\right)''}{h\varphi\left(\vartheta^m_{\beta,\,Q} f(\varphi)\right)' + (1-h)\left(\vartheta^m_{\beta,\,Q} f(\varphi)\right)} - 1\right\} > -q\,|\,w\,|,\ (\varphi \in \mathbb{D})$$

or equivalently

$$\mathrm{Re}\left\{\frac{-\sum_{s=t+1}^{\infty}(h(s-1)+1)(s-1)\frac{\beta^{s-1}e^{e\left(-\beta^2\right)+1}\lambda_s}{(s-1)!}[1+Q(s-1)]^m a_s\varphi^s}{1-\sum_{s=t+1}^{\infty}(h(s-1)+1)\frac{\beta^{s-1}e^{e}\left(-\beta^2\right)+1}{(s-1)!}\lambda_s[1+Q(s-1)]^m a_s\varphi^s}\right\} > -q\,|\,w\,|. \tag{9}$$

This is the point at which we have utilized equation (3).

The next step is to select values of φ along the real axis, and then allow $\varphi \to 1-$ up through the real values. In that case, inequality (9) immediately results in the condition given by (8).

Alternatively, by putting hypothesis (8) into practice and allowing $|\varphi|$ to equal 1, we discover that

$$\left|\frac{\varphi\left(\vartheta^m_{\beta,\,Q} f(z)\right)' + h\varphi^2\left(\vartheta^m_{\beta,\,Q} f(\varphi)\right)''}{h\varphi\left(\vartheta^m_{\beta,\,Q} f(\varphi)\right)' + (1-h)\left(\vartheta^m_{\beta,\,Q} f(\varphi)\right)} - 1\right|$$

$$= \left|\frac{\sum_{s=t+1}^{\infty}(h(s-1)+1)(s-1)\frac{\beta^{s-1}e^{e\left(-\beta^2\right)+1}\lambda_s}{(s-1)!}[1+Q(s-1)]^m a_s\varphi^s}{1-\sum_{s=t+1}^{\infty}(h(s-1)+1)\frac{\beta^{s-1}e^{\left(-\beta^2\right)+1}\lambda_s}{(s-1)!}[1+Q(s-1)]^m a_s\varphi^s}\right|$$

$$\le \frac{q\,|\,w\,|\left(1-\sum_{s=t+1}^{\infty}(h(s-1)+1)\frac{\beta^{s-1}e^{e\left(-\beta^2\right)+1}\lambda_s}{(s-1)!}[1+Q(s-1)]^m a_s\varphi^s\right)}{1-\sum_{s=t+1}^{\infty}(h(s-1)+1)\frac{\beta^{s-1}e^{\left(-\beta^2\right)+1}\lambda_s}{(s-1)!}[1+Q(s-1)]^m a_s\Big)} \#$$

$$\le q\,|\,w\,|. \tag{10}$$

According to the maximum modulus theorem, this means that we have

$$f \in \mathcal{S}_{\mathrm{n}}(w,\ h,\ q)$$

and therefore, conclusively completing the evidence of Theorem 1, it would appear. Similarly, we can illustrate the following.

Theorem 2 If we assume that the function $f(\varphi) \in \mathcal{A}(t)$ is defined by the equation (3), then it follows that $f(\varphi)$ belongs to the class $\mathcal{R}_n(w,\ h,\ q)$ if and only if

$$\sum_{s=t+1}^{\infty} s(h(s-1)+1)\frac{\beta^{s-1} e^{e\left(-\beta^2\right)+1}\lambda_s}{(s-1)!}[1+Q(s-1)]^m a_s \leq q\,|\,w\,| \tag{11}$$

Remark 1 A special case of the Theorem 1 is when

$$w = 1 \text{ and } q = 1-\alpha,\ (0 \leq \alpha < 1). \tag{12}$$

The following is the expression that characterizes our initial inclusion relation that involves $N_{t,\,\Phi}(e)$.

Theorem 3 Let

then

$$\mathcal{S}_{\mathrm{n}}(w,\ h,\ q) \subset N_{t,\,\Phi}(e) \tag{13}$$

Proof. In the case where f belongs to the set $f \in \mathcal{S}_t(w,\ h,\ q)$, Theorem 1 gevien

$$(ht+1)(t+q\,|\,w\,|)\frac{\beta^t e^{e^{\left(-\beta^2\right)+1}}\lambda_s}{(t)!}[1+Qt]^m \sum_{s=t+1}^{\infty} a_s \leq q\,|\,w\,|$$

so that

On the other hand, we notice that (8) and (14) both come to the conclusion that

$$(ht+1)\frac{\beta^t e^{e\left(-\beta^2\right)+1}\lambda_s}{(t)!}[1+Qt]^m \sum_{s=t+1}^{\infty} sa_s$$

$$\leq q\,|\,w\,| + (1-q\,|\,w\,|)(ht+1)\frac{\beta^t e^{e^{\left(-\beta^2\right)+1}\lambda_s}}{(t)!}[1+Qt]^m \sum_{s=t+1}^{\infty} a_s$$

$$\leq q\,|\,w\,| + (1-q\,|\,w\,|)(ht+1)\frac{\beta^t e^{e\left(-\beta^2\right)+1}\lambda_s}{(t)!}[1+Qt]^m \frac{q\,|\,w\,|}{(ht+1)(t+q\,|\,w\,|)\frac{\beta^t e^{ee^{\left(-\beta^2\right)+1}\lambda_s}(t)!}{(t)!}[1+Qt]^m}$$

$$\leq \frac{(t+1)q\,|\,w\,|}{t+q\,|\,w\,|},\ (|\,w\,|<1)$$

that is, given that (5), and thus Theorem 3 as required.

Through a similar process, we can demonstrate the following by applying Theorem 1 rather than Theorem 2.

Theorem 4 Let

then

$$\mathcal{R}_t(w,\ h,\ q) \subset N_{t,\ \delta}(e)$$

Remark 2 A special case of the Theorem 3, is when

$$w = 1 - a,\ (0 \le \alpha < 1),\ h = 0,\ \text{and } q = 1$$

3. The neighborhood conditions for classes $S_t^{(\alpha)}(w,\ h,\ q)$ and $R_t^{(\alpha)}(w,\ h,\ q)$

The neighborhood for each of the classes is determined in this portion of the paper.

$$\mathcal{S}_t^{(\alpha)}(w,\ h,\ q) \text{ and } \mathcal{R}_t^{(\alpha)}(w,\ h,\ q)$$

as we will explain in the following with respect to the class $\mathcal{S}_t^{(\alpha)}(w,\ h,\ q)$, A function $f \in \mathcal{A}(t)$ is considered to be in the class if there exists a function $g \in \mathcal{S}_t(w,\ h,\ q)$ such that

$$\left|\frac{f(\varphi)}{g(\varphi)} - 1\right| < 1 - \alpha,\ (\varphi \in \mathbb{D},\ 0 \le \alpha < 1) \tag{15}$$

Similarly, a function f that belongs to the class $\mathcal{A}(t)$ is considered to be in the class $\mathcal{R}_t^{(\alpha)}(w,\ h,\ q)$ if there is a function g that belongs to the class $\mathcal{R}_t^{(\alpha)}(w,\ h,\ q)$ that proves the inequality (15) to be true.

Theorem 5 Let $g \in \mathcal{S}_t(w,\ h,\ q)$ and

then

$$N_{t,\ \Phi}(g) \subset \delta_t^{(\alpha)}(w,\ h,\ q)$$

Proof. Let us assume that f belongs to the set $N_{t,\ \Phi}(g)$. Then, based on the equation (4), we discover that

$$\sum_{s=t+1}^{\infty} s\left|a_s - b_s\right| \le \Phi$$

This is a direct consequence of the coefficient inequality

$$\sum_{s=t+1}^{\infty} \left|a_s - b_s\right| \le \frac{\Phi}{t+1},\ (t \in \mathbb{N})$$

Next, given that g belongs to the set $\mathcal{S}_s(w,\ h,\ q)$, we may observe that [refer to equation (14)]

so that,

$$\left|\frac{f(\varphi)}{g(\varphi)} - 1\right| < \frac{\sum_{s=t+1}^{\infty}\left|a_s - b_s\right|}{1 - \sum_{s=t+1}^{\infty} b_s} \le \frac{\Phi}{t+1} \cdot \frac{(ht+1)(t+q\,|\,w\,|)\dfrac{\beta^t e^{e^{\left(-\beta^2\right)+1}}\lambda_s}{(n)!}[1+Qt]^m}{t(h(t+q\,|\,w\,|)+1)\dfrac{\beta^t e^{e^{\left(-\beta^2\right)+1}}\lambda_s}{(n)!}[1+Qt]^m} = 1 - \alpha, \tag{16}$$

assuming that the value of is accurately determined by the equation (16). Therefore, according to the definition, f belongs to the set $\mathcal{S}_t^{(\alpha)}(w, h, q)$ for the value of that is provided by (16), which concludes our demonstration of Theorem 5.

We show below can also that Theorem 6 is substantially similar to Theorem 5.

Theorem 6 If $g \in \mathcal{R}_t(w, h, q)$ and then

$$N_{t, \Phi}(g) \subset \mathcal{R}_t^{(\alpha)}(w, h, q)$$

4. Conclusions

Using the Bell distribution of the differential operator, we investigate the neighborhood of several subclasses of functions with complex order. We define a specific subclass of analytic univalent functions through this new operator and examine some of the geometric properties associated with it. Notably, this subclass encompasses several new classes alongside well-known ones, such as the classes of convex and starlike functions. This characteristic is a significant highlight of the class. Moreover, this work enhances our understanding by providing an introduction to the subject and exploring its applications in GFT. Future research should focus on investigating the Hankel determinants of orders two and three within the aforementioned subclasses, which could lead to new avenues for exploration.

Conflict of interest

The authors declare no competing financial interest.

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