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Research Article

Even-Order Delay Differential Equations with p-Laplacian Like Operator: New Oscillation Criteria

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Abstract: As a pivotal branch within the realm of differential equations, the theory of oscillation holds a crucial position in the exploration of natural sciences and the construction of modern control theory Frameworks. In this manuscript, we obtain some oscillatory properties for the noncanonical even-order differential equation with a p-Laplacian-like operator by using comparison with first-order differential inequality. Some examples are discussed to illustrate the effectiveness of our main results.

Keywords: differential equations, oscillation criteria, p-Laplacian operator, even-order

MSC: 34C10, 34K1

1. Introduction

In recent years, the investigation of problems involving delay equation and Laplacian operators has become very interesting. These types of problems illustrate how delay equations with a p-Laplacian-like operator extend traditional differential equations to be able to model systems with memory effects behaviors which is not captured by the differential operators with integer-order. Also, oscillation theory of DDEs and fractional calculus appear in several applications of many fields, such as biology, finance, physics, engineering, image processing, and the modeling of nonlocal interactions in materials, for these and other applications, we cite for instance the papers see [1–6] and the references therein. Due to their importance, several researchers have concentrated in the development of problems of involving Laplacian differential operators.

In this manuscript, we establish the asymptotic criteria of even-order neutral differential equation with a p-Laplacian-like operator

$$\left(r(t)\left(w^{(n-1)}(t)\right)^{p_1-1}\right)' + q(t)w^{p_2-1}(\tau(t)) = 0,\ t \geq t_0 \tag{1}$$

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where n is an even integer and $p_i > 1,\ i = 1,\ 2$, are real numbers.

Throughout this paper, we assume that

(I_1) $r(t) \in C^1([t_0,\ \infty),\ (0,\ \infty)),\ r(t) > 0,\ r'(t) \geq 0$ verifying

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{p_1-1}}(t)} dt < \infty \tag{2}$$

(I_2) $q(t) \in C([t_0,\ \infty),\ R),\ \tau(t) \in C([t_0,\ \infty),\ R),\ q(t) > 0,\ \tau(t) < t$, and $\lim_{t\to\infty} \tau(t) = \infty$.

A function $w(t) \in C^{n-1}[T_w,\ \infty),\ T_w \geq t_0$, is said to be a solution of (1), that can have the condition $r(t)\left(w^{(n-1)}(t)\right)^{p_1-1} \in C^1[J_w,\ \infty)$ and meets the condition of Equation (1) on $[T_w,\ \infty)$. The solutions $w(t)$ of (1) are only considered when they meet the following condition $\sup\{|w(t)| : t \geq T\} > 0$ for all $T \geq T_w$. A solution (1) is said to oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to non-oscillatory. If all solutions of (1) are oscillatory, then (1) is said to be oscillatory.

Exploring Laplace differential equations yields manifold essential utilities in mechanical systems, electrical circuits, and the regulation of chemical processes. Furthermore, their utility extends to ecological systems, epidemiology, and the modeling of population dynamics, as detailed in references.

Presently, substantial attention is directed towards establishing criteria for the occurrence of oscillations in solutions to diverse categories of differential equations, see [7–11]. Several researchers have directed their efforts towards cutinizing oscillatory behavior, particularly within the realm of fourth-order differential equations encompassing delays and advanced terms, see [12–16]. Investigations by Zhang et al. [17], Li et al. [18, 19], Baculikova et al. [20], and Grace and Lalli [21] have yielded techniques and methodologies aimed at enhancing the oscillatory attributes of these equations. Furthermore, the work of Zhang et al. [4, 14, 22, 23] and Agarwal et al. [5, 24] has expanded this inquiry to encompass differential equations of the neutral variety. In recent years, there has also been a significant exploration of oscillation behaviors in fourth-order delay differential equations, as evidenced by studies such as [25–29].

Numerous investigations have addressed the criteria governing oscillation in solutions to diverse differential equations.

Bazighifan [12] analyzed the oscillation conditions of the fourth-order differential equations with p-Laplacian like operator

$$\left(r(t)\left(w^{(3)}(t)\right)^{p-1}\right)' + q(t)w^{p-1}(\tau(t)) = 0$$

and used the comparison method with a first-order differential.

Baculikova et al. [13] proved that the even-order equation with a p-Laplacian-like operator

$$\left(r(t)\left(w^{(n-1)}(t)\right)^{p-1}\right)' + q(t)w(\tau(t)) = 0$$

is oscillatory under canonical case

$$\int_{t}^{\infty} \frac{1}{r^{\frac{1}{p-1}}(t)} dt = \infty$$

and obtained new results for the oscillation of this equation using the Riccati technique.

The main motivation for work is to contribute to the development of the oscillation theory for higher-order neutral equations by finding sufficient conditions that guarantee that the solutions of this type of equations are oscillatory.

This work aims to broaden the scope of inquiry by incorporating the higher-order p-Laplacian-like operator under condition (2) into the study. Within this context, the paper introduces innovative criteria for analyzing oscillatory solutions of Equation (1).

The investigation employs the Riccati technique and the comparison method as tools to deduce the sought-after outcomes.

1.1 *Materials and methods*

Materials are raw materials, tools, objects, or important chemicals used in experiments. Basically, it is the important details of what has been used in the research. Provides all details on how the data was measured and used, as well as instrument type, manufacturer and model.

The methods section is how the study was conducted. Describe in this section any steps or procedures taken to achieve your research goals, including experimental design and data analysis. For the statistical analysis, all details related to the statistical tests are important. These details include preliminary analysis, study sample size, the type of data (mean, median, standard deviation, standards error, and confidence intervals), normalization of your data, statistical methods used, and information for the statistical software program (name of the program, company, city and country).

2. Basic lemmas

To obtain our main results, we need the following lemmas:

Lemma 1 [24] Let $f \in C^n\left([t_0, \infty), \mathbb{R}^+\right)$ and suppose that $f^{(n)}$ has a positive (negative) sign and nonzero on an interval in $[t_0, \infty)$, as well there is a $t_1 \geq t_0$ in a way that $f^{(n-1)}(t) f^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} f(t) \neq 0$, then $f \geq \dfrac{\lambda}{(n-1)!} t^{n-1} \left| f^{(n-1)} \right|$, for every $\lambda \in (0, 1)$, holds on $[j_\lambda, \infty)$.

Lemma 2 [15], Lemma 2.1 Let α be a ratio of two odd numbers, $N > 0$ and Q are constants. Then, $Qs - Ns^{\frac{\alpha+1}{\alpha}} \leq \dfrac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \dfrac{Q^{\alpha+1}}{N^\alpha}$, $N > 0$

Lemma 3 [6] Suppose that $w(j)$ is an eventually positive solution of (1), then, there exist two cases

Case 1:

$$w(t) > 0,\ w^{(n-1)}(t) > 0,\ w^{(n)}(t) < 0,\ \left(r(t)\left(w^{(n-1)}(t)\right)^{p_1-1}\right)' < 0.$$

Case 2:

$$w(t) > 0,\ w^{(n-2)}(t) > 0,\ w^{(n-1)}(t) < 0,\ \left(r(t)\left(w^{(n-1)}(t)\right)^{p_1-1}\right)' < 0.$$

3. Main results

In this section, we shall establish some oscillation criteria for (1).

Theorem 1 Let $n \geq 2$ and (2) holds. If the equation

$$y^{'}(t)+q(t)\left(\frac{\lambda_0\tau^{n-1}(t)}{(n-1)!r^{\frac{1}{p_1-1}}(\tau(t))}\right)^{p_2-1} y^{\frac{p_2-1}{p_1-1}}(\tau(t))=0, \tag{3}$$

for $\lambda_0 \in (0,\ 1)$, is oscillatory, and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[M^{p_2-p_1}q(u)\left(\frac{\lambda_1}{(n-2)!}\tau^{n-2}(u)\right)^{p_2-1}\delta^{p_1-1}(u)-\frac{(p_1-1)^{p_1}}{p_1{}^{p_1}}\frac{1}{\delta(u)r^{1/p_1-1}(u)}\right]ds=\infty \tag{4}$$

is satisfied for several constant $\lambda_1 \in (0,\ 1)$ as well as to each constant $M>0$, such that

$$\delta(t):=\int_t^\infty \frac{1}{r^{\frac{1}{p_1}-1}(s)}ds.$$

Subsequently, each solution of (1) either demonstrates oscillatory behavior or converges to zero.

Proof. Let us consider the scenario where Equation (1) possesses a non-oscillatory solution denoted as $w(j)$. Without loss of generality, we can assume that $w(j)$ eventually exhibits positivity. Furthermore, let us suppose that $\lim_{t\to\infty} w(t)\neq 0$, for $t\geq t_1$, t_1 is large enough. □

From Lemma 3, we have two cases. First, if case (1) holds. From Lemma 1, we have

$$w(t)\geq \frac{\lambda j^{n-1}}{(n-1)!r^{1/p_1-1}(t)}\left(r^{1/p_1-1}(t)w^{(n-1)}(t)\right)$$

for every $\lambda\in(0,1)$ and for all sufficiently large j. Hence by Equation (1), we see that $y(t):=r(t)\left(w^{(n-1)}(t)\right)^{p_1-1}$ is a positive solution of the differential inequality

$$y^{'}(t)+q(t)\left(\frac{\lambda\tau^{n-1}(t)}{(n-1)!r^{1/p_1-1}(\tau(t))}\right)^{p_2-1}y^{p_2-1/p_1-1}(\tau(t))\leq 0.$$

Using [28], Corollary 1, one can check that Equation (3) also has a positive solution, which is a contradiction.

Now, if that case (2) holds. Define the function φ by

$$\varphi(t)=\frac{r(t)\left(w^{(n-1)}(t)\right)^{p_1-1}}{\left(w^{(n-2)}(t)\right)^{p_1-1}},\ t\geq t_1. \tag{5}$$

Then $\varphi(t)<0$ for $t\geq t_1$. Noting that $r(t)\left(w^{(n-1)}(t)\right)^{p_1-1}$ is decreasing, we have

$$r^{1/p_1-1}(u)w^{(n-1)}(u)\leq r^{1/p_1-1}(t)w^{(n-1)}(t),\ u\geq t\geq t_1$$

Dividing the above by $r^{1/p_1-1}(u)$ and integrating it from t to k, we find

$$w^{(n-2)}(k) \leq w^{(n-2)}(t)+r^{1/p_1-1}(t)w^{(n-1)}(t)\int_t^k \frac{1}{r^{1/p_1-1}(u)}du$$

Letting $k \to \infty$, we see

$$w^{(n-2)}(t)+r^{1/p_1-1}(t)w^{(n-1)}(t)\delta(t) \geq 0,$$

which yields

$$-\frac{r^{\frac{1}{p_1}-1}(t)w^{(n-1)}(t)}{w^{(n-2)}(t)}\delta(t) \leq 1.$$

Thus, by (5), we obtain

$$-\varphi(t)\delta^{p_1-1}(t) \leq 1. \tag{6}$$

From (5), we find

$$\varphi^{'}(t) = \frac{\left(r(t)\left(w^{(n-1)}(t)\right)^{p_1-1}\right)^{'}}{\left(w^{(n-2)}(t)\right)^{p_1-1}} - (p_1-1)\frac{r(t)\left(w^{(n-1)}(t)\right)^{p_1}}{\left(w^{(n-2)}(t)\right)^{p_1}},$$

which follows from (1) and (5) that

$$\varphi^{'}(t) = -q(t)\frac{w^{p_2-1}\left(\tau_i(t)\right)}{\left(w^{(n-2)}(t)\right)^{p_1-1}} - (p_1-1)\frac{\varphi^{p_1/p_1-1}(t)}{r^{1/p_1-1}(t)}.$$

On the other hand, by Lemma 1, we see

$$w(t) \geq \frac{\lambda}{(n-2)!}t^{n-2}w^{(n-2)}(t)$$

which yields

$$\varphi'(t) = -q(t)\frac{w^{p_2-1}(\tau(t))}{\left(w^{(n-2)}(\tau(t))\right)^{p_2-1}}\left(w^{(n-2)}(\tau(t))\right)^{p_2-p_1}\frac{\left(w^{(n-2)}(\tau(t))\right)^{p_1-1}}{\left(w^{(n-2)}(t)\right)^{p_1-1}}-(p_1-1)\frac{\varphi^{\frac{p_1}{p_1}-1}(t)}{r^{\frac{1}{p_1}-1}(t)},$$

$$\leq -M^{p_1-p_1}q(t)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(t)\right)^{p_2-1}-(p_1-1)\frac{\varphi^{\frac{p_1}{p_1}-1}(t)}{r^{\frac{1}{p_1}-1}(t)}.$$

Integrating (10) from t_1 to t, we have

$$\delta^{p_1-1}(t)\varphi(t)-\delta^{p_1-1}(t_1)\varphi(t_1)+\int_{t_1}^{t} M^{p_2-p_1}q(u)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(u)\right)^{p_2-1}\delta^{p_1-1}(u)du$$

$$+(p_1-1)\int_{t_1}^{t} r^{-\dfrac{1}{p_1-1}}(u)\delta^{p_1-2}(u)\varphi(u)du+\int_{t_1}^{t}(p_1-1)\frac{\varphi^{\frac{p_1}{p_1-1}}(u)}{r^{\frac{1}{p_1-1}}(u)}\delta^{p_1-1}(u)du\leq 0.$$

From this Equation and Lemma 2, we have

$$\int_{t_1}^{t}\left[M^{p_2-p_1}q(u)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(u)\right)^{p_2-1}\delta^{p_1-1}(u)-\frac{(p_1-1)^{p_1}}{p_1{}^{p_1}}\frac{1}{\delta(u)r^{1/p_1-1}(u)}\right]du$$

$$\leq\delta^{p_1-1}(t_1)\varphi(t_1)+1,$$

due to (6), which contradicts (4). This completes the proof.

Corollary 1 Let $n\geq 2$. Suppose that (2) holds. Further, assume that $p_1=p_2$,

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} q(u)\frac{\left(\tau^{n-1}(u)\right)^{p_1-1}}{r(\tau(u))}du>\frac{((n-1)!)^{p_1-1}}{e}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[q(u)\left(\frac{\lambda_1}{(n-2)!}\tau^{n-2}(u)\right)^{p_1-1}\delta^{p_1-1}(u)-\frac{(p_1-1)^{p_1}}{p_1{}^{p_1}}\frac{1}{\delta(u)r^{1/p_1-1}(u)}\right]du=\infty$$

Then every solution of (1) is oscillatory or tends to zero.

Corollary 2 Let $n\geq 2$. Assume that (2) holds. Moreover, suppose that$p_1>p_2$, τ is a strictly increasing function,

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} q(u)\frac{\left(\tau^{n-1}(u)\right)^{p_{2-1}}}{r^{p_2-1/p_1-1}(\tau(u))}du>0.$$

If inequality (3) is satisfied with a specific constant value λ_1 within the interval $(0,\ 1)$ and for all constants $M > 0$, then it can be concluded that each solution of Equation (1) either exhibits oscillatory behavior or converges to zero.

4. Numerical examples

To provide a practical demonstration of our findings, we present the following examples.

Example 1 Let us consider the differential equation:

$$\left(j^5 w'''(t)\right)' + \beta t w(t) = 0,\ t \geq 1. \tag{7}$$

Let $p_1 = p_2 = 2,\ n = 4,\ r(t) = t^5,\ q(t) = \beta t;\ \beta > \dfrac{8}{\lambda_1},\ \tau(t) = \dfrac{t}{2.}$

Thus, it is easy to verify that

$$\begin{aligned} \limsup_{t\to\infty} \int_{\tau(t)}^{t} q(u) \frac{\left(\tau^{n-1}(u)\right)^{p_1-1}}{r(\tau(u))} du &= \limsup_{t\to\infty} \int_{\frac{t}{2}}^{t} \beta u \frac{\left(\frac{u}{2}\right)^3}{\left(\frac{u}{2}\right)^5} du, \\ &= \limsup_{t\to\infty} \int_{\frac{t}{2}}^{t} \frac{4\beta}{u} du = 4\beta \ln(2) > \left(\frac{6}{e} = \frac{(n-1)!^{p_1-1}}{e}\right). \end{aligned} \tag{8}$$

Also,

$$\begin{aligned} &\limsup_{t\to\infty} \int_{t_0}^{t} \left[q(u) \left(\frac{\lambda_1}{(n-2)!} \tau^{n-2}(u) \right)^{p_1-1} \delta^{p_1-1}(u) - \frac{(p_1-1)^{p_1}}{p_1{}^{p_1}} \frac{1}{\delta(u) r^{\frac{1}{p_1}-1}(u)} \right] du, \\ &= \limsup_{t\to\infty} \int_{t_0}^{t} \left[\beta u \frac{\lambda_1 \left(\frac{u^2}{4}\right)}{2} \left(\frac{1}{4u^4}\right) - \frac{1}{2^2} \frac{1}{\left(\frac{1}{4u^4}\right) u^5} \right] du \\ &= \limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\beta\lambda_1}{8u} - \frac{1}{u} \right] du = \infty. \end{aligned} \tag{9}$$

From (8), (9) and by using Corollary 1, we obtain every solution of (8) is oscillatory or tends to zero.

Example 2 Let us consider the fourth-order differential equation:

$$\left(t^6 \left(w'''(t)\right)^3\right)' + \left(\frac{\gamma}{t}\left(t^2+t+1\right)(t-1) + \frac{\gamma}{t}\right) w(t) = 0,\ \gamma > 0 \tag{10}$$

we note that $p_1 = 4,\ p_2 = 2,\ n = 4,\ r(t) = t^6,\ \tau(t) = \dfrac{t}{2}$ and

$$q(t)=\left(\frac{\gamma}{t}\left(t^2+t+1\right)(t-1)+\frac{\gamma}{t}\right)$$

Thus, it is easy to verify that

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} q(u)\frac{\left(\tau^{n-1}(u)\right)^{p_2-1}}{r^{p_2-1/p_1-1}\left(\tau(u)\right)}du,$$

$$=\mathit{lim\,sup}_{t\to\infty}\int_{t-1}^{t}\frac{\left(\frac{\gamma}{u}\left(u^2+u+1\right)(u-1)+\frac{\gamma}{u}\right)\left(\frac{t}{2}\right)^3}{16\left(t^6\right)^{\frac{1}{2}}}du,$$

$$=\infty>0.$$

By using Corollary 2, we obtain every solution of (10) is oscillatory or tends to zero.

5. Conclusions

The aim of this paper is to investigate the oscillatory characteristics inherent in higher-order differential equations featuring a p-Laplacian-like operator. This investigation is conducted through the application of Riccati transformations and a comparison analysis with first-order equations, ultimately leading to the derivation of oscillation criteria. The study culminates in the establishment of a central theorem pertaining to the oscillation behavior of higher-order differential equations. Additionally, some illustrative examples are presented to elucidate the findings.

In future work, we will study some the oscillatory characteristics inherent in fourth-order differential equations with a p-Laplacian-like operator of the form

$$\left(r(t)\left(w'''(t)\right)^{p_1-1}\right)'+q(t)w^{p_2-1}\left(\tau(t)\right)=0,\ t\geq t_0$$

Progress is already underway in investigating these particular equation types.

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Conflict of interest

The authors declare no competing financial interest.

References

[1] Agarwal RP, Grace SR, O'Regan D. *Oscillation Theory for Difference and Functional Differential Equations*. Dordrecht, The Netherlands: Marcel Dekker; 2000.

[2] Ladde GS, Lakshmikantham V, Zhang BG. *Oscillation Theory of Differential Equations with Deviating Arguments*. New York: Marcel Dekker; 1987.

[3] Bliss GA, Schoenberg IJ. On separation, comparison and oscillation theorems for self-adjoint systems of linear second-order differential equations. *American Journal of Mathematics*. 1931; 53(4): 781-800.

[4] Zhang C, Agarwal RP, Bohner M, Li T. Oscillation of fourth-order delay dynamic equations. *Science China Mathematics*. 2015; 58(1): 143-160.

[5] Graef JR, Grace SR, Tunç E. Oscillatory behavior of even-order nonlinear differential equations with a sublinear neutral term. *Opuscula Mathematica*. 2019; 39(1): 39-47.

[6] Graef JR, Grace SR, Jadlovská I, Tunç E. Some new oscillation results for higher-order nonlinear differential equations with a nonlinear neutral term. *Mathematics*. 2022; 10(16): 2997.

[7] Karpuz B, Ocalan O, Ozturk S. Comparison theorems on the oscillation and asymptotic behavior of higher-order neutral differential equations. *Glasgow Mathematical Journal*. 2010; 52(01): 107-114.

[8] Zafer A. Oscillation criteria for even order neutral differential equations. *Applied Mathematics Letters*. 1998; 11(3): 21-25.

[9] Cesarano C, Pinelas S, Al-Showaikh F, Bazighifan O. Asymptotic properties of solutions of fourth-order delay differential equations. *Symmetry*. 2019; 11(5): 628.

[10] Bazighifan O, Dassios I. Riccati technique and asymptotic behavior of fourth-order advanced differential equations. *Mathematics*. 2020; 8(4): 590.

[11] Grace S, Agarwal R, Graef J. Oscillation theorems for fourth order functional differential equations. *Journal of Applied Mathematics and Computation*. 2009; 30: 75-88. Available from: https://doi.org/10.1007/s12190-008-0158-9.

[12] Bazighifan O, Abdeljawad T. Improved approach for studying oscillatory properties of fourth-order advanced differential equations with p-Laplacian like operator. *Mathematics*. 2020; 8(5): 656.

[13] Li T, Baculikova B, Dzurina J, Zhang C. Oscillation of fourth-order neutral differential equations with p-Laplacian like operators. *Boundary Value Problems*. 2014; 56: 41-58. Available from: https://doi.org/10.1186/1687-2770-2014-56.

[14] Zhang Q, Liu S, Gao L. Oscillation criteria for even-order half-linear functional differential equations with damping. *Applied Mathematics Letters*. 2011; 24(10): 1709-1715.

[15] Agarwal RP, Zhang Ch, Li T. Some remarks on oscillation of second order neutral differential equations. *Applied Mathematics and Computation*. 2016; 274: 178-181. Available from: https://doi.org/10.1016/j.amc.2015.10.089.

[16] Chatzarakis GE, Grace SR, Jadlovska I, Li T, Tunc E. Oscillation criteria for third-order Emden-Fowler differential equations with unbounded neutral coefficients. *Complexity*. 2019; 2019(8): 1-7.

[17] Zhang C, Agarwal RP, Li T. Oscillation and asymptotic behavior of higher-order delay differential equations with p-Laplacian like operators. *Journal of Mathematical Analysis and Applications*. 2014; 409(2): 1093-1106.

[18] Li T, Rogovchenko YV. Oscillation criteria for even-order neutral differential equations. *Applied Mathematics Letters*. 2016; 61: 35-41. Available from: https://doi.org/10.1016/j.aml.2016.04.012.

[19] Bazighifan O, Alotaibi H, Mousa AAA. Neutral delay differential equations: Oscillation conditions for the solutions. *Symmetry*. 2021; 13(1): 101.

[20] Baculikova B, Dzurina J, Graef JR. On the oscillation of higher-order delay differential equations. *Journal of Mathematical Sciences*. 2012; 187(4): 387-400.

[21] Baculíková B. Oscillation of even order linear functional differential equations with mixed deviating arguments. *Opuscula Mathematica*. 2022; 42(4): 549-560.

[22] Zhang C, Li T, Agarwal RP, Bohner M. Oscillation results for fourth-order nonlinear dynamic equations. *Applied Mathematics Letters*. 2012; 25(12): 2058-2065.
[23] Zhang C, Agarwal RP, Bohner M, Li T. New results for oscillatory behavior of even-order half-linear delay differential equations. *Applied Mathematics Letters*. 2013; 26(2): 179-183.
[24] Agarwal R, Grace S, O'Regan D. *Oscillation Theory for Difference and Functional Differential Equations*. Dordrecht, The Netherlands: Kluwer Academic Publishers; 2000.
[25] Bazighifan O, Kumam P. Oscillation theorems for advanced differential equations with p-Laplacian like operators. *Mathematics*. 2020; 8(5): 821.
[26] Bazighifan O. Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations. *Advances in Difference Equations*. 2020; 2020(1): 201.
[27] Bazighifan O. An approach for studying asymptotic properties of solutions of neutral differential equations. *Symmetry*. 2020; 12(4): 555.
[28] Bazighifan O, Cesarano C. A Philos-type oscillation criteria for fourth-order neutral differential equations. *Symmetry*. 2020; 12(3): 379.
[29] Bazighifan O. Fourth-order differential equations with p-Laplacian like operator. *Applied Mathematics and Computation*. 2020; 386: 1-8.