Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

New Oscillation Conditions Test for Neutral Differential Equations

Omar Bazighifan[1,2], Anwar Al-Batati[1], Nawa Alshammari[3], Loredana Florentina Iambor[4*]

[1] Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout, Yemen
[2] Jadara Research Center, Jadara University, Irbid 21110, Jordan
[3] Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia
[4] Department of Mathematics and Computer Science, University of Oradea, University Street, 410087 Oradea, Romania
E-mail: iambor.loredana@gmail.com

Received: 1 August 2024; **Revised:** 26 August 2024; **Accepted:** 29 August 2024

Abstract: In this paper, we aim to explore the oscillation of solutions for second-order delay differential equations with multiple delays. Delay equations are characterized by being rich in both practical and theoretical aspects. We propose new criteria to ensure that all obtained solutions are oscillatory. The obtained results can be used to develop and provide theoretical support for and further develop the oscillation study for a class of second-order differential equations. Finally, some illustrated examples are given to demonstrate the effectiveness of our new criteria.

***Keywords*:** oscillation conditions, second-order, delay differential equations

MSC: 34C10, 34K11

1. Introduction

In this work, we focus our attention on the oscillation of the second-order nonlinear delay differential equation with the form

$$\left(f(s)\left(u^{'}(s)\right)^{\mu}\right)^{'}+\sum_{j=0}^{m}q_j(s)u^{\mu}(\tau_j(s))=0,\ s_u\geq s_0. \tag{1}$$

Throughout this paper, we assume that:
(i) μ is quotient of odd positive integers,
(ii) $q_j(s), \in C\left([s_0,\ \infty),\ [0,\ \infty)\right),\ j=1,\ 2,\ \ldots,\ m,$
(iii) $f(s)\in C([s_0,\ \infty),\ (0,\ \infty))$, and

$$\int_{S_0}^{\infty}\frac{1}{f^{\frac{1}{\mu}}(s)}ds<\infty, \tag{2}$$

DOI: https://doi.org/10.37256/cm.5420245428

(iiii) $\tau_j(s) \in C([s_0, \infty), R)$, and there exists $\tau \in C([s_0, \infty), R)$ such that $\tau(s) \leq \tau_j(s)$, $\tau(s) < s$ and $\lim_{s\to\infty} \tau(s) = \infty$, $j = 1, 2, \ldots, m$.

By a solution of (1), we mean a nontrivial function $u \in C^1([s_0, \infty), R)$, $s_u \geq s_0$, which has the properties $f(s)\left(u'(s)\right)^{\mu} \in C^1([s_0, \infty), R)$, and u satisfies (1) on $[s_u, \infty)$, then u is a solution of (1). We focus in our study on the solutions that satisfy $sup\{u(s) : s \geq s_u\} > 0$, for every $s_u \geq s_x$, and we assume that (1) possesses such solutions. Such a solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. An equation is called oscillatory if all of its solutions are oscillatory.

The delay differential equations find a wide range of applications in certain high-tech fields, such as control theory, mechanical engineering, physics, population dynamics, economics, and so on, see [1–7].

The property of oscillation is widespread in many physical, natural, and even social phenomena, so the study of oscillatory properties for solutions of differential equations is an interesting issue not only for its applied importance, but because it also contains many interesting analytical issues, see [7–10].

Thus, we can see that investigating the oscillatory and asymptotic behavior of solutions of neutral differential equations is of great importance. During the past period, many papers appeared on the oscillatory behavior of differential equations of neutral and delay type. Investigations by Zhang et al. [11], Li et al. [12–15], Baculikova et al. [16], and Grace and Lalli [17] have yielded techniques and methodologies aimed at enhancing the oscillatory attributes of these equations. Furthermore, the work of Zhang et al. and Agarwal et al. [6, 18–21] has expanded this inquiry to encompass differential equations of the neutral variety. In recent years, there has also been a significant exploration of oscillation behaviors in fourth-order delay differential equations, as evidenced by studies such as [22, 23].

Balatta et al. [24] studied the oscillatory properties of higher -order delay half linear differential equations with non-canonical operators. They improved new criteria, extended, and greatly simplified the previously established criteria, but they also have the potential to act as a reference point for the theory of delay differential equations of higher order.

Numerous investigations have addressed the criteria governing oscillation in solutions to diverse differential equations. As an illustration, Zhang et al. [25] introduced novel characteristics concerning solutions to delay differential equations

$$\left(f(s)\left(u'(s)\right)^{\mu}\right)' + q(s)u^{\mu}(\tau(s)) = 0,$$

in the canonical case

$$\int_{S_0}^{\infty} \frac{1}{f^{\frac{1}{\mu}}(s)} ds = \infty.$$

Agarwal et al. [26] analyzed the oscillatory behavior for differential equations of second-order

$$\left(f(s)\left(u^{\mu}(s)\right)'\right)' + q(s)u^{r}(\tau(s) = 0.$$

The main purpose of this work is to establish new criteria for oscillation of (1) under condition (2). New criteria ensure that all solutions are oscillatory, which is an extension and expansion of previous results. Some examples were provided to illustrate the results.

The following notation will be used in the remaining sections of this work: $\tau(s) = min\left\{\tau_j(s),\ j = 1, 2, \ldots, m\right\}$.

2. Main results

In this section, we set out to investigate the monotonic properties and oscillatory behavior of solutions to (1).

Lemma 1 [20] Let $f \in C^n\left([j_0,\ \infty),\ \mathbb{R}^+\right)$ and $f^{(n)}$ is not identically zero on a subray of $[j_0,\ \infty)$, and is of fixed sign, such that $f^{(n-1)}(j)f^{(n)}(j) \leq 0$. If $\lim_{t\to\infty} f(j) \neq 0$, then

$$f \geq \frac{\lambda}{(n-1)!} j^{n-1} \left| f^{(n-1)} \right|,\ \lambda \in (0,\ 1),$$

holds on $[j_\lambda,\ \infty)$.

Lemma 2 [26] Let $N > 0$ and Q are constants, and α be a ratio of two odd numbers. Then

$$Qs - Ns^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{Q^{\alpha+1}}{N^\alpha},\ N > 0.$$

Lemma 3 [25] Let u is an positive solution of (1). Then, there exist two cases:
Case 1: $u^{'}(s) < 0$.
Case 2: $u^{'}(s) > 0$.

Lemma 4 [16] If u is a positive solution of (1). Then

$$\left(\frac{u(s)}{\varphi(s,\ \infty)}\right)^{'} \geq 0.$$

and there exists a $s_1 \geq s_0$ such that $\left(\frac{u(s)}{\varphi(s_1,\ \infty)}\right)^{'} \leq 0.$

Theorem 1 If

$$\limsup_{s\to\infty} \int_{s_1}^{s} \frac{1}{f^{\frac{1}{\mu}}(x)} \left(\int_{s_0}^{x} \varphi^\mu(y,\ \infty) \sum_{j=0}^{m} q_j(y)dy\right)^{\frac{1}{\mu}} dx = \infty.$$

Then (1) is oscillatory.

Proof. Suppose the contrary that u is a positive solution of (1), so we have

$$u(s) > 0,\ \tau_j(s) > 0 \text{ and } \left(f(s)\left(u^{'}(s)\right)^\mu\right)^{'} \leq 0,\ \text{for } s \geq s_1 \geq s_0.$$

By Lemma 3, we find two cases.

First, if case (1) holds. From (1), we have

$$\left(f(s)\left(u^{'}(s)\right)^{\mu}\right)^{'}=-\sum_{j=0}^{m}q_{j}(s)u^{\mu}(\tau_{j}(s)),\ \text{for } s\ \geq s_{1}\geq s_{0},$$

$$\leq -u^{\mu}(s)\sum_{j=0}^{m}q_{j}(s). \tag{3}$$

Since $s\geq s_1$, and so,

$$f(s)\left(u^{'}(s)\right)^{\mu}\leq f(s_{1})\left(u^{'}(s_{1})\right)^{\mu},$$

since $u^{'}(s)<0$, we obtain

$$u(s)\geq -\varphi(s,\,\infty)f^{\frac{1}{\mu}}(s)u^{'}(s),$$

$$\geq -\varphi(s,\,\infty)f^{\frac{1}{\mu}}(s_{1})\,u^{'}(s_{1})\,,$$

we put $\sigma=-f(s)\left(u^{'}(s)\right)^{\mu}$, and so,

$$u^{\mu}(s)\geq -\varphi^{\mu}(s,\,\infty)f(s_{1})(u^{'}(s_{1}))^{\mu}=\varphi^{\mu}(s,\,\infty)\sigma\ \text{for } s\geq s_{1}. \tag{4}$$

From (3) and (4), we have

$$\left(f(s)\left(u^{'}(s)\right)^{\mu}\right)^{'}\leq \sigma\varphi^{\mu}(s,\,\infty)\sum_{j=0}^{m}q_{j}(s)\ \ \text{for } s\ \geq s_{1}. \tag{5}$$

Integrating (5) from s_1 to s, we obtain

$$f(s)\left(u^{'}(s)\right)^{\mu}-f(s_{1})\left(u^{'}(s_{1})\right)^{\mu}\leq -\sigma\int_{s_{1}}^{s}\varphi^{\mu}(t,\,\infty)\sum_{j=0}^{m}q_{j}(t)\,dt. \tag{6}$$

From (6), we get

$$u^{'}(s)\leq \frac{-\sigma^{\frac{1}{\mu}}\left(\int_{s_{1}}^{s}\varphi^{\mu}(t,\,\infty)\sum_{j=0}^{m}q_{j}(t)\,dt\ \right)^{\frac{1}{\mu}}}{f(s)^{\frac{1}{\mu}}}. \tag{7}$$

Integrating (7) from s_1 to s, we find

$$u(s) - u(s_1) \leq -\sigma^{\frac{1}{\mu}} \int_{s_1}^{s} \left(\frac{\left(\int_{s_1}^{y} \varphi^{\mu}(t,\ \infty) \sum_{j=0}^{m} q_j(t)\ dt \right)^{\frac{1}{\mu}}}{f(y)^{\frac{1}{\mu}}} \right) dy.$$

when $s \to \infty$, we have arrive at a contradiction with (2).

Now, if that case (2) holds. From (2) and I2, we find

$$\lim_{s \to \infty} \int_{s_1}^{s} \sum_{j=0}^{m} q_j(t)\ dt \ = \infty, \tag{8}$$

and so,

$$\left(f(s) \left(u^{'}(s) \right)^{\mu} \right)^{'} \leq -u^{\mu} \left(\tau(s) \right) \sum_{j=0}^{m} q_j(s) \ \text{ for } s \geq s_1. \tag{9}$$

Integrating (9) from s_2 to s, we have

$$f(s) \left(u^{'}(s) \right)^{\mu} - f(s_2) \left(u^{'}(s_2) \right)^{\mu} \leq - \int_{s_2}^{s} u^{\mu}(\tau(y)) \sum_{j=0}^{m} q_j(y)\ dy$$

$$\leq -u^{\mu} \left(\tau(s_2) \right) \int_{s_2}^{s} \sum_{j=0}^{m} q_j(y)\ dy.$$

Take $s \to \infty$, we arrive at a contradiction with (8).

The proof is complete.

Corollary 1 Let the conditions $I_1 - I_4$ holds. If

$$\underset{s \to \infty}{lim\ sup} \varphi^{\mu}(s,\ \infty) \int_{s_1}^{s} \sum_{j=0}^{m} q_j(v)\ dv > 1.$$

Then (1) is oscillatory.

Corollary 2 Let the conditions $I_1 - I_4$ holds. If $f^{'}(s) > 0$, $\exists \gamma,\ \delta\ \in C^1([s_0,\ \infty),\ (0,\ \infty))$ such that

$$\underset{s \to \infty}{lim\ sup} \frac{\varphi^{\mu}(s,\ \infty)}{\gamma(s)} \int_{s_1}^{s} \left(\gamma(s) \sum_{j=0}^{m} q_j(s) - \frac{f(s) \left(\gamma^{'}(s) \right)^{\mu+1}}{(\mu+1)^{\mu+1}\ \gamma^{\mu}(s)} \right) ds > 1,$$

and

$$\limsup_{s\to\infty}\int_{s_1}^{s}\left(\delta(s)\sum_{j=0}^{m}q_j(s)-\frac{f(s)\left(\delta'(s)\right)^{\mu+1}}{(\mu+1)^{\mu+1}\,\delta^{\mu}(s)(\tau'(s))^{\mu}}\right)ds=\infty.$$

Then (1) is oscillatory.

3. Numerical example

Examples are provided to demonstrate the significance of our results.
Example 1 Consider the second-order equation:

$$(s^2\,u'(s))'+a_0 s\left(u\left(\frac{s}{3}\right)+u\left(\frac{s}{2}\right)\right)=0,\ j\geq 1. \tag{10}$$

Let $\mu=1,\ f(j)=s^2,\ q_1(s)=q_2(s)=a_0 s,\ \tau_1(s)=\frac{s}{3},\ \tau_2(s)=\frac{s}{2}$, and

$$\int_{s_0}^{\infty}\frac{1}{x^2}dx=\frac{1}{s_0}.$$

It is easy to verify the conditions (I_1-I_4).
Now, from Theorem 1, we see that

$$\limsup_{s\to\infty}\int_{s_1}^{s}\frac{1}{f^{\frac{1}{\mu}}(x)}\left(\int_{s_0}^{x}\varphi^{\mu}(y,\,\infty)\sum_{j=0}^{m}q_j(y)dy\right)^{\frac{1}{\mu}}dx,$$

$$=\limsup_{s\to\infty}\int_{s_0}^{s}\frac{1}{y^2}\left(\int_{s_0}^{y}\left(\int_{v}^{\infty}\frac{1}{t^2}dt\right)2a_0 v\,dv\right)dy=\infty.$$

Thus, Equation (10) is oscillatory.
Example 2 Consider the equation:

$$(s^2\,u'(s))'+\left(s+s^2\right)u\left(\frac{s}{2}\right)=0,\ j\geq 1. \tag{11}$$

Let $\mu=1,\ f(j)=s^2,\ q_1(s)=s,\ q_2(s)=s^2,\ \tau_1(s)=\ \tau_2(s)=\frac{s}{2}$.

$$\int_{s_0}^{\infty}\frac{1}{x^2}dx=\frac{1}{s_0}.$$

Now, from Corollary 1, we get

$$\lim_{s\to\infty} \sup \varphi^{\mu}(s,\infty) \int_{s_1}^{s} \sum_{j=0}^{m} q_j(t)dt$$

$$= \lim_{s\to\infty} \sup \frac{1}{s} \int_{s_1}^{s} (t+t^2)dt = \lim_{s\to\infty} \sup \frac{1}{s} \left(\left(\frac{s^3}{3} + \frac{s^2}{2} \right) - \left(\frac{s_1^3}{3} + \frac{s_1^2}{2} \right) \right)$$

$$= \infty > 1.$$

Thus, Equation (11) is oscillatory.

Example 3 Consider the second-order equation:

$$(s^3 u'(s))' + s^2 \left(u\left(\frac{s}{5}\right) + u\left(\frac{s}{3}\right) \right) = 0. \tag{12}$$

Let $\mu = 1,\ f(j) = s^3,\ q_1(s) = q_2(s) = s^3,\ \tau_1(s) = \frac{s}{5},\ \tau_2(s) = \frac{s}{3,}\ \gamma(s) = \frac{1}{s^3} = \delta(s)$ and

$$\int_{s_0}^{\infty} \frac{1}{x^3} dx = \frac{1}{2{s_0}^2}.$$

it is easy to verify the conditions $(I_1 - I_4)$.

Now, by Corollary 2, we have

$$\lim_{s\to\infty} \sup \frac{\varphi^{\mu}(s,\infty)}{\gamma(s)} \int_{s_1}^{s} \left(\gamma(t) \sum_{j=0}^{m} q_j(t) - \frac{f(t)(\gamma'(t))^{\mu+1}}{(\mu+1)^{\mu+1}\gamma^{\mu}(t)} \right) dt,$$

$$= \lim_{s\to\infty} \sup \frac{1}{2} s \int_{s_1}^{s} \left(2 - \frac{9}{4s^2} \right) dt = \infty > 1.$$

Also,

$$\lim_{s\to\infty} \sup \int_{s_1}^{s} \left(\delta(t) \sum_{j=0}^{m} q_j(t) - \frac{f(t)(\delta'(t))^{\mu+1}}{(\mu+1)^{\mu+1}\delta^{\mu}(t)(\tau'(t))^{\mu}} \right) dt$$

$$= \lim_{s\to\infty} \sup \int_{s_1}^{s} \left(2 - \frac{27}{4s^2} \right) dt = \infty.$$

Thus, Equation (12) is oscillatory.

4. Conclusions

This article presents an interesting outcome by studying a class of second-order delay differential equations with multiple delays. By explicitly taking advantage of proposing new criteria, we ensure that all solutions are oscillatory. The obtained results can provide theoretical support and empower the oscillation study for a class of second-order neutral differential equations. An illustrated example is presented to verify our results. For researchers interested in this field, and as part of our future research, it would be interesting to extend this improvement to higher-order differential equations in the non-canonical case.

Acknowledgment

The authors would like to thank the anonymous reviewers for their work and constructive comments that contributed to improve the manuscript. Also, they present their appreciation to University of Oradea for funding the publication of this research.

This research was funded by the University of Oradea.

Conflict of interest

The authors have no conflict of interest either wholly or partially in the content of the article.

References

[1] Agarwal RP, Grace SR, O'Regan D. *Oscillation Theory for Difference and Functional Differential Equations*. Dordrecht: Marcel Dekker, Kluwer Academic; 2000.

[2] Ladde GS, Lakshmikantham V, Zhang BG. *Oscillation Theory of Differential Equations with Deviating Arguments*. New York, NY, USA: Marcel Dekker; 1987.

[3] Az-Zo'bi EA, Al-Khaled K, Darweesh A. Numeric-analytic solutions for nonlinear oscillators via the modified multi-stage decomposition method. *Mathematics*. 2019; 7: 550.

[4] Thandpani E, Selvaranegam T. Improved approach for studying oscillatory properties of fourth-order advanced differential equations with p-Laplacian like operator. *Mathematics*. 2020; 8: 656.

[5] Li T, Baculikova B, Dzurina J, Zhang C. Oscillation of fourth order neutral differential equations with p-Laplacian like operators. *Boundary Value Problems*. 2014; 56: 41-58.

[6] Zhang C, Agarwal RP, Bohner M, Li T. Oscillation of fourth-order delay dynamic equations. *Science China Mathematics*. 2015; 58: 143-160.

[7] Bazighifan O, Ruggieri M, Scapellato A. An improved criterion for the oscillation of fourth-order differential equations. *Mathematics*. 2020; 8: 1-10.

[8] Karpuz B, Ocalan O, Ozturk S. Comparison theorems on the oscillation and asymptotic behavior of higher-order neutral differential equations. *Glasgow Mathematical Journal*. 2010; 52: 107-114.

[9] Zafer A. Oscillation criteria for even order neutral differential equations. *Applied Mathematics Letters*. 1998; 11: 21-25.

[10] Bazighifan O, Ramos H. On the asymptotic and oscillatory behavior of the solutions of a class of higher-order differential equations with middle term. *Applied Mathematics Letters*. 2020; 107: 106431.

[11] Zhang C, Agarwal RP, Li T. Oscillation and asymptotic behavior of higher-order delay differential equations with p-Laplacian like operators. *Journal of Mathematical Analysis and Applications*. 2014; 409: 1093-1106.

[12] Li T, Rogovchenko YV. Oscillation criteria for even-order neutral differential equations. *Applied Mathematics Letters*. 2017; 67: 53-59.

[13] Az-Zo'bi EA, Al Dawoud K, Marashdeh M. Numeric-analytic solutions of mixed-type systems of balance laws. *Applied Mathematics and Computation*. 2015; 265: 133-143.

[14] Li T, Rogovchenko YV. Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations. *Monthly Journal of Mathematics.* 2017; 184: 489-500.

[15] Li T, Rogovchenko YV. On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations. *Applied Mathematics Letters.* 2020; 105: 1-7.

[16] Baculikova B, Dzurina J. Oscillation theorems for second-order nonlinear neutral differential equations. *Computers and Mathematics with Applications.* 2011; 62: 4472-4478.

[17] Grace SR, Lalli BS. Oscillation theorems for nth-order differential equations with deviating arguments. *Journal of Mathematical Analysis and Applications.* 1984; 101(1): 268-296.

[18] Zhang C, Li T, Agarwal RP, Bohner M. Oscillation results for fourth-order nonlinear dynamic equations. *Applied Mathematics Letters.* 2012; 25: 2058-2065.

[19] Zhang C, Agarwal RP, Bohner M, Li T. New results for oscillatory behavior of even-order half-linear delay differential equations. *Applied Mathematics Letters.* 2013; 26: 179-183.

[20] Agarwal R, Grace S, O'Regan D. *Oscillation Theory for Difference and Functional Differential Equations*. Dordrecht: Kluwer Academic Publishers; 2000.

[21] Agarwal R, Shieh SL, Yeh CC. Oscillation criteria for second order retard differential equations. *Mathematics and Computers in Simulation.* 1997; 26: 1-11.

[22] Balatta SA, Hashim I, Bataineh AS, Ismail ES. Oscillation criteria of fourth-order differential equations with delay terms. *Journal of Function Spaces.* 2022; 2022(1): 9527666.

[23] Ladde GS, Lakshmikantham V, Zhang BG. *Oscillation Theory of Differential Equations with Deviating Arguments*. New York, NY, USA: Marcel Dekker; 1987.

[24] Balatta SA, Ismail ES, Hashim I, Bataineh AS, Momani S. Oscillatory behavior of higher-order differential equations with delay terms. *European Journal of Pure and Applied Mathematics.* 2023; 16(4): 2234-2246.

[25] Zhang Q, Liu S, Gao L. Oscillation criteria for even-order half-linear functional differential equations with damping. *Applied Mathematics Letters.* 2011; 24: 1709-1715.

[26] Agarwal RP, Zhang Ch, Li T. Some remarks on oscillation of second order neutral differential equations. *Applied Mathematics and Computation.* 2016; 274: 178-181.