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Research Article

Existence Results and Trajectory Controllability of Conformable Hilfer Fractional Neutral Stochastic Integro-differential Equations

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Abstract: This manuscript commits to analyzing the existence and uniqueness of mild solution and trajectory controllability of conformable Hilfer fractional neutral stochastic integro-differential system with infinite delay through Lipschitz, growth conditions and properties of semigroup theory, controllability theory, stochastic analysis techniques, and Banach fixed point theorem with example. This manuscript mainly focuses on fractional system provide the mathematical foundation to understand and control complex, real-world systems with memory, and randomness.

***Keywords*:** Hilfer fractional derivative, conformable fractional derivative, trajectory controllability, stochastic differential equation

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1. Introduction

Though researchers are familiar with classical and weak solutions of the ordinary or partial differential equations, it is necessary to contemplate, "why generalized functions from Banach space, Hilbert space are needed to study the mild solutions of abstract differential systems?", and "why it is needed to seek a solution of differential equation in an abstract space?". The answer is not straight forward. For the last two decades, the concept of existence of weak solutions is dominated the applied mathematics research field due to its inevitable role in solving ordinary or partial differential equations which cannot be solved by analytical methods. Certain partial differential systems do not have classical solutions such as equations under the hyperbolic conservation law and there are continuous functions satisfying those partial differential equations but still not a solution of the equation due to the non-differentiability. There comes the significant role of abstract theory functional analysis which brings the compactly supported test functions from Banach space or from L^2 space to generate the functions as continuous linear functional in order to have a solution called weak solution, such systems are called abstract differential systems. Since set of all test functions form a linear space, it acts as a linear differential operator in the abstract differential system like abstract Cauchy problem, 1-D diffusion equations and as a kernel under the integral sign like Volterra integral equations. Test functions along with semigroup theory form a new set of notions results bounded linear operators. There comes the role of Banach space and Hilbert space which contain the class of generalized functions satisfying required notions like compactly supported, inner product, norm, and linearity. The main motive of using test function in control theory and dynamical systems is to restrict the impulsive

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behavior and control the dynamics of the system.

Fractional calculus helps identify specific dynamic behaviors of the dynamical system that integer order differential equations are unable to represent because it takes memory effect into consideration. They have drawn more attention and are widely used by scholars to construct and solve models and nonlocal or memory-based Cauchy systems. Additionally, fractional stochastic differential equations are used to solve complex medical biology systems (e.g., chemotherapy, epidemiology).

Readers can refer to [1-3] for a better knowledge of semigroup theory and fractional calculus. Different definitions of fractional derivatives (FDs) exist, such as Caputo, Riemann-Liouville, Grunwald-Letnikov, Hilfer, and others. Hilfer proposed the Hilfer fractional, which combines the Riemann-Liouville and Caputo derivatives [4]. Whether or not there are weak solutions to differential equations with Hilfer fractional derivatives was examined by Gu et al. [5].

Stochastic differential equation plays a crucial role in the area of fractional calculus to model the system where the random change, random growth or random heat conduction occur. For further details refer [6-10]. Ahmed et al. investigated the Hilfer fractional stochastic integro-differential systems with non-local conditions using Sadovskii fixed point theorem in [11]. Jingyun et al. [12] examined the approximate controllability of Hilfer fractional neutral stochastic differential systems through Banach contraction principle. The controllability of nonlinear integro-differential systems in Banach space was explored in [13] and this work was extended in [14] to the third order dispersion system. And in the recent years many authors have made their contribution in the control theory [15, 16]. Mourad K [17] explored the approximate controllability of FDS with fBm. However, these derivatives cannot be used in conjunction with the chain rule, quotient rule, or product rule. In order to solve the problem with nonlocal FDs. Khalil et al. [18-20] recommended a limit-based fractional derivative known as conformable FD. This helps to avoid the challenge of solving the complex systems using the previously described rules. For this reason, conformable FD is a preferable idea when managing complex systems.

The purpose of using Hilfer fractional derivative is that it interpolates between Riemann-Liouville and Caputo fractional derivative. This versatility makes Hilfer fractional derivatives useful for modeling intricate dynamical systems, such as the heat conduction model and infectious disease models.

The fundamental notion behind trajectory controllability is that it directs the solution curve along the predetermined path, allowing for more precise control of the dynamic system. For instance, normal cells may be impacted while are treated in cancer cells in cancer therapy. Trajectory controllability ($\mathfrak{T}$-controllability) will be helpful in reducing such effects and maintaining the normal cells at a safe level. Researchers have been paying attention to trajectory control theory lately because it is a crucial component of control theory and makes a significant contribution in dynamical systems. Since trajectory controllability is independent of both the starting and final states, it is more precise and efficient than other controllability's such as exact controllability, approximate controllability. More specifically, as the majority of dynamical systems are nonlinear, this idea will be useful in solving such systems. Chalishajar et al. [21] the ones who originally proposed the novel idea of $\mathfrak{T}$-controllability and [22, 23] discussed $\mathfrak{T}$-controllability of non-linear integro-differential systems for 1st and 2nd order using by numerical approach. Malik et al. [24] shared their opinion on the $\mathfrak{T}$-controllability of a fractional differential system a few years ago. Then, Dhayal et al. [25] investigated the approximation and Tcontrollability for fractional stochastic differential sytem. Later, [26] explored the $\mathfrak{T}$-controllabilty for Hilfer fractional stochastic equation.

Recently, the existence results and trajectory controllability of conformable fractional stochastic integro-differential equation with infinite delay was discussed in [27]. As accurate as possible, the trajectory controllability of conformable Hilfer fractional neutral stochastic integro-differential has not been discussed. In this manuscript, our motive is to study the conformable Hilfer fractional neutral stochastic integro-differential systems with infinite delay.

First let us consider the conformable Hilfer fractional stochastic integro-differential systems with infinite delay,

$$\begin{cases} \mathrm{D}_{0^+}^{\rho,\mathfrak{e}} u(\mathfrak{r}) = \mathscr{G}u(\mathfrak{r}) + y(\mathfrak{r}) + \mathscr{L}\left(\mathfrak{r},\ u_{\mathfrak{r}},\ \int_0^{\mathfrak{r}} \mathscr{M}\left(\mathfrak{r},\ \theta,\ u_\theta\right) d\theta\right) + \mathscr{N}\left(\mathfrak{r},\ u_{\mathfrak{r}},\ \int_0^{\mathfrak{r}} \tilde{\mathscr{M}}\left(\mathfrak{r},\ \theta,\ u_\theta\right) d\theta\right) \dfrac{d\gamma(\mathfrak{r})}{d\mathfrak{r}}, \\ \mathrm{I}^{(1-\rho)(1-\mathfrak{e})} u(0) = \kappa(\mathfrak{r}),\quad \in L^2\left(\Lambda,\ \mathscr{O}_{\mathbf{H}}\right),\ \mathfrak{r} \in (-\infty,\ 0], \end{cases} \tag{1}$$

where I is Riemann-Liouville fractional integral. $\mathrm{D}_{0^+}^{\rho,\mathfrak{e}}$ is the Hilfer conformable fractional derivative for $\mathfrak{r} \in \mathrm{J}' = (0, \mathrm{b}]$,

$0 < \rho \leq 1$ and $\frac{1}{2} < \mathfrak{e} \leq 1$. $\mathscr{O}_H$-phase space. $\mathscr{G} : D(\mathscr{G}) \subset \mathscr{V} \to \mathscr{V}$ is a linear operator which generates a strongly continuous semigroup $\{T(\mathfrak{r})\}$, $\mathfrak{r} \geq 0$ on a Hilbert space $\mathscr{V}$ with inner product and norm defined in $\mathscr{V}$. $y(\mathfrak{r})$ is a control function which is measurable and square integrable and assumes values on J = [0, b] in the reflexive Hilbert space K. The function $u_\mathfrak{r} : (-\infty, 0] \to \mathscr{V}$ is described by $u_\mathfrak{r}(\theta) = u(\mathfrak{r} + \theta)$ in $\mathscr{O}_H$. The functions $\mathscr{L}$, $\mathscr{N}$, $\mathscr{M}$ and $\tilde{\mathscr{M}}$ are defined as follows $\mathscr{L} : J \times \mathscr{O}_H \times \mathscr{V} \to \mathscr{V}$, $\mathscr{N} : J \times \mathscr{O}_H \times \mathscr{V} \to L^2$ and $\mathscr{M}$, $\tilde{\mathscr{M}} : J \times J \times \mathscr{O}_H \to \mathscr{V}$ are bounded and continuouson J' = (0, b]. Assume that z is a standard $\mathfrak{Q}$-Wiener process on a separable Hilbert space U with $<\cdot, \cdot>_u$ and $\|\cdot\|_u$ as itsnorm. $\{\gamma(\mathfrak{r}), \mathfrak{r} \geq 0\}$ is a U-valued Brownian motion with finite-trace nuclear covariance operator $\mathfrak{Q} \geq 0$. $\kappa = \{\kappa(\mathfrak{r}) : \mathfrak{r} \in (-\infty, 0\}$ with the finite second moment is $\mathscr{O}_H$-valued random variable, measurable and independent of Wiener process $\{\gamma(\mathfrak{r})\}$.

The main contribution of this paper is to exhibit the existence results of complex system like conformable Hilfer fractional derivative with stochastic process and establishing the controllability. Conformable fractional derivative is an efficient tool since it preserves the chain and quotient rule for derivatives. This manuscript is organized as follows: preliminaries, existence and uniqueness of mild solution of (1) without neutral term, with neutral term and trajectory controllability of (1). An illustration is given to understand the theoretical approach.

2. Preliminaries

Consider the followings. Set of all bounded linear operators $S(\mathfrak{r})_{\mathfrak{r} \geq 0}$ generated by $\mathscr{G}$ forms an analytic semi-group. $\rho(\mathscr{G})$ is the resolvent set of $\mathscr{G}$ which contains zero. Then $\forall \zeta \in (0, 1]$, $\mathscr{G}^\beta$ is a closed linear operator on $D(\mathscr{G}^\beta)$. The complete probability space with a normal filtration $\{\mathscr{K}_\mathfrak{r}\}_{\mathfrak{r} \in [0,b]}$ is denoted by $(\Lambda, \mathscr{K}, P)$. Z: $J \times \Lambda \to H$ is a $\mathfrak{Q}$-Wiener process on $(\Lambda, \mathscr{K}, P)$ with the linear bounded covariance operator $\mathfrak{Q}$ such that $Tr(\mathfrak{Q}) < \infty$, which is adapted to normal filtration $\{\mathscr{K}_\mathfrak{r}\}_{\mathfrak{r} \in [0, b]}$. The complete orthonormal set $\{h_n\}_{n \geq 1}$ in H, with sequence of nonnegative real numbers $\{w_n\}_{n \in N}$ such that $\mathfrak{Q} h_n = w_n h_n$, $w_n \geq 0$, $n = 1, 2, \cdots$ and $\langle Z(\mathfrak{r}), w\rangle = \sum_{n=1}^{\infty} \sqrt{w_n} \langle h_n, w\rangle \gamma_n(\mathfrak{r})$, $w \in U$, $\mathfrak{r} \in [0, b]$ where $\gamma_n(\mathfrak{r})$ is a sequence of independent Brownian motions.

Consider the Hilbert spaces $L^2(\Lambda, \mathscr{V})$, L_2^0, $L^2_{\mathscr{K}}(J, U)$, where

$$L^2(\Lambda, \mathscr{V}) := \{\mathscr{L} \mid \mathscr{L} \text{ is an } \mathscr{K}\text{-measurable and square integrable random variables with values in } \mathscr{V}\},$$

$$L_2^0 := \left\{ \mathscr{L} \mid \mathscr{L} \text{ is a Hilbert-Schmidt operator from } \mathfrak{Q}^{\frac{1}{2}}(U) \text{ to } \mathscr{V} \right\},$$

$$L^2_{\mathscr{K}}(J, U) := \left\{ c \mid c : J \times \Lambda \to \mathscr{V} \text{ is a square integrable } \mathscr{K}_\mathfrak{r}\text{-adapted process with values in } \mathscr{V} \right\}.$$

Let $1 - p = (1 - \rho)(1 - \mathfrak{e}) \in (0, 1)$. Let $C(J, L^2(\Lambda, \mathscr{V})) := \{c : J \to L^2(\Lambda, \mathscr{V}) \mid c$ be an $\mathscr{K}_\mathfrak{r}$-a continuous mapping and adapted stochastic process such that $\sup_{\mathfrak{r} \in J} \mathscr{E} \|u(\mathfrak{r})\|^2 < \infty\}$ and a complete normed linear space with $\|u\|_{C(J, L^2(\Lambda, \mathscr{V}))} = \left(\sup_{\mathfrak{r} \in J} \mathscr{E} \| u(\mathfrak{r})\|^2\right)^{\frac{1}{2}}$. Assume $C_{1-p}\left(J, L^2(\Lambda, \mathscr{V})\right) := \left\{ c \in C\left(J', L^2(\Lambda, \mathscr{V})\right) \middle| \mathfrak{r}^{1-p} c(\mathfrak{r}) \in C\left(J, L^2(\Lambda, \mathscr{V})\right)\right\}$. Let $L^2(\Lambda, \mathscr{V})$ is a Banach space equipped with the norm $\|u\|_{C_{1-p}(J, L^2(\Lambda, \mathscr{V}))} = \left(\sup_{\mathfrak{r} \in J} \mathscr{E} \left\|\mathfrak{r}^{1-p} u(\mathfrak{r})\right\|^2\right)^{\frac{1}{2}}$ which can be described as the set of all Lebesgue integrable $\mathscr{K}$-valued random values in $\mathscr{V}$.

Definition 2.1 [2] The Riemann-Liouville fractional integral of order ρ for $\mathfrak{F} : [0, \infty) \to R$ shall be expressed as

$$I^{\rho}_{0^+} \mathfrak{F}(\mathfrak{r}) = \frac{1}{\Gamma(\rho)} \int_0^{\mathfrak{r}} \frac{\mathfrak{F}(\theta)}{(\mathfrak{r} - \theta)^{1-\rho}} d\theta, \ \mathfrak{r} > 0; \ \rho > 0.$$

Definition 2.2 [2] Riemann-Liouville's fractional derivative of order ρ for $\mathfrak{F} : [0, \infty) \to R$ shall be expressed as

$$ {}^{L}D_{0^+}^{\rho}\mathfrak{F}(\mathfrak{r}) = \frac{1}{\Gamma(n-\rho)}\frac{d^n}{d\mathfrak{r}^n}\int_0^{\mathfrak{r}}\frac{\mathfrak{F}(\theta)}{(\mathfrak{r}-\theta)^{\rho+1-n}}d\theta,\ \mathfrak{r}>0;\ n=[\rho]+1. $$

Definition 2.3 [2] Caputo derivative of order ρ for a function $\mathfrak{F}:[0,\infty)\to R$ is defined as

$$ {}^{C}D_{0^+}^{\rho}\mathfrak{F}(\mathfrak{r}) = \frac{1}{\Gamma(n-\rho)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\theta)^{n-\rho-1}\mathfrak{F}^{n}(\theta)d\theta,\ \mathfrak{r}>0,\ n=[\rho]+1. $$

Definition 2.4 [4] The Hilfer fractional derivative of order $\rho\in[0,1]$ and $\mathfrak{e}\in(0,1)$ with the lower limit 0 is defined as

$$ D_{0^+}^{\rho,\mathfrak{e}}\mathfrak{F}(\mathfrak{r}) = I_{0^+}^{\rho(1-\mathfrak{e})}\frac{d}{d\mathfrak{r}}I_{0^+}^{(1-\rho)(1-\mathfrak{e})}\mathfrak{F}(\mathfrak{r}). $$

Note

- If $\mathfrak{e}=0$, then $D_{0^+}^{\rho,0}$ is a Riemann-Liouville fractional derivative:

$$ D_{0^+}^{\rho,0}\mathfrak{F}(\mathfrak{r}) = \frac{d}{d\mathfrak{r}}I_{0^+}^{1-\rho}\mathfrak{F}(\mathfrak{r}) = {}^{L}D_{0^+}^{\rho}\mathfrak{F}(\mathfrak{r}). $$

- If $\mathfrak{e}=1$, then $D_{0^+}^{\rho,1}$ is a Caputo fractional derivative:

$$ D_{0^+}^{\rho,1}\mathfrak{F}(\mathfrak{r}) = I_{0^+}^{1-\rho}\frac{d}{d\mathfrak{r}}\mathfrak{F}(\mathfrak{r}) = {}^{C}D_{0^+}^{\rho}\mathfrak{F}(\mathfrak{r}). $$

Definition 2.5 [5] Wright function $M_{\mathfrak{e}}$, is described as,

$$ M_{\mathfrak{e}}(\mathfrak{r}) = \sum_{n=1}^{\infty}\frac{(-\mathfrak{r})^{n-1}}{(n-1)!\Gamma(1-n\mathfrak{e})},\ \mathfrak{e}\in(0,\ 1),\ \mathfrak{r}\in\mathbb{C}, $$

and satisfies

$$ \int_0^{\infty}\mathfrak{r}^{p}M_{\mathfrak{e}}(\mathfrak{r})d\mathfrak{r} = \frac{\Gamma(1+p)}{\Gamma(1+\mathfrak{e}p)},\ \mathfrak{r}\geq 0. $$

In addition to the above, we shall construct few more important results, lemmas and properties to support the main results.

Lemma 2.1 [5, 11] Let the semigroup $T(\mathfrak{r})$ be generated by the infinitesimal generator $\mathscr{G}:\mathscr{V}\to\mathscr{V}$ and there exists $M>0$ such that $\|T(\mathfrak{r})\|\leq M\ \forall\mathfrak{r}\in$ J. Then

1. $T_{\mathfrak{e}}(\mathfrak{r})$, $P_{\mathfrak{e}}(\mathfrak{r})$ and $S_{\rho,\mathfrak{e}}(\mathfrak{r})$ are bounded linear operators, i.e. $\forall\ \mathfrak{r}>0$, $u\in\mathscr{V}$, $\mathrm{p}=\rho+\mathfrak{e}-\rho\mathfrak{e}$, we have

$$ \|T_{\mathfrak{e}}(\mathfrak{r})u\|\leq\frac{M\|u\|}{\Gamma(\mathfrak{e})},\ \ \|P_{\mathfrak{e}}(\mathfrak{r})u\|\leq\frac{Mt^{\mathfrak{e}-1}\|u\|}{\Gamma(\mathfrak{e})}\ \text{and}\ \|S_{\rho,\mathfrak{e}}(\mathfrak{r})u\|\leq\frac{M\mathfrak{r}^{\mathrm{p}-1}\|u\|}{\Gamma(\mathrm{p})}. $$

2. $T_{\mathfrak{e}}(\mathfrak{r})$, $P_{\mathfrak{e}}(\mathfrak{r})$ and $S_{\rho,\mathfrak{e}}(\mathfrak{r})$ are strongly continuous.

Lemma 2.2 [11] For $\forall u\in\mathscr{V}$, $\gamma\in(0,1)$ and $\beta\in(0,1]$, we have

$$\mathscr{G}T_{\mathfrak{e}}(\mathfrak{r})u = \mathscr{G}^{1-\gamma}T_{\mathfrak{e}}(\mathfrak{r})\mathscr{G}^{\gamma}u,\ \mathfrak{r} \in \mathrm{J},\ \ \| \mathscr{G}^{\beta}T_{\mathfrak{e}}(\mathfrak{t})u \| \leq \frac{\mathfrak{e}C_{\beta}\Gamma(2-\beta)\| u \|}{\mathfrak{r}^{\beta}\Gamma(1+\mathfrak{e}(1-\beta))},\ \mathfrak{r} \in \mathrm{J}'.$$

Lemma 2.3 [12] For arbitrary L_2^0-valued stochastic process $\Upsilon(\mathfrak{r})$, $\mathfrak{r} \in [\tau_1, \tau_2]$, satisfying

$$\mathscr{E}\left(\int_{\tau_1}^{\tau_2} \| \Upsilon(\theta) \|_{L_2^0}^2 \, d\theta\right) < \infty,\ 0 \leq \tau_1 < \tau_2 \leq b,$$

thus

$$\mathscr{E} \| \int_{\tau_1}^{\tau_2} \Upsilon(\theta) d\gamma(\theta) \|^2 \leq \mathrm{Tr}(\mathfrak{Q}) \int_{\tau_1}^{\tau_2} \mathscr{E} \| \Upsilon(\theta) \|_{L_2^0}^2 \, d\theta,$$

where

$$\mathrm{Tr}(\mathfrak{Q}) < \infty.$$

Definition 2.6 [30] The abstract phase space $\mathscr{O}_{\mathrm{H}}$ is defined as follows,

$$\mathscr{O}_{\mathrm{H}} = \left\{\kappa : (-\infty,\ 0] \to \mathscr{V},\ \text{such that } \forall \text{ with } \int_{-\infty}^{0} H(s) \sup_{\theta \leq \zeta \leq 0} \left(\mathscr{E}\| \kappa(\zeta) \|^2\right)^{1/2} d\theta < +\infty,\right.$$

where the continuous function H: $(-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^{0} \mathrm{H}(\mathfrak{r}) d\mathfrak{r} < \infty$ and

$$\| \kappa \|_{\mathscr{O}_{\mathrm{H}}} = \int_{-\infty}^{0} H(\theta) \sup_{\theta \leq \zeta \leq 0} \left(\mathscr{E} \| \kappa(\zeta) \|^2\right)^{1/2} d\theta,\ \forall\ \kappa \in \mathscr{O}_{\mathrm{H}}.$$

Obviously $\left(\mathscr{O}_{\mathrm{H}},\ \|\cdot\|_{\mathscr{O}_{\mathrm{H}}}\right)$ is a Banach space.

Definition 2.7 [30] The space containing all continuous $\mathscr{V}$-valued stochastic processes $\{h(\mathfrak{r})$: $\mathfrak{r} \in (-\infty, \rho]\}$ is denoted by $\mathscr{C}((-\infty, v], \mathscr{V})$. Furthermore,

$$\mathscr{O}_{\mathrm{H}}' = \{u : u \in \mathscr{C}((-\infty,\ \rho],\ \mathscr{V})\}$$

and

$$\|u\|_{\mathfrak{e}} = \|\kappa\|_{\mathscr{O}_H} + \sup_{\theta \in [0, \mathfrak{b}]} \left(\mathscr{E} \| u(\theta) \|^2\right)^{1/2},\ u \in \mathscr{O}_{\mathrm{H}}'.$$

Lemma 2.4 [30] If $u_0 = \kappa \in \mathscr{O}_{\mathrm{H}}$, $u \in \mathscr{O}_{\mathrm{H}}'$, then for $\mathfrak{r} \in \mathscr{E}$, $u_{\mathfrak{r}} \in \mathscr{O}_{\mathrm{H}}$. Furthermore,

$$l\left(\mathscr{E}\|u(\mathfrak{r})\|^2\right)^{1/2} \leq \|u_{\mathfrak{r}}\|_{\mathscr{O}_{\mathrm{H}}} \leq \|u_0\|_{\mathscr{O}_{\mathrm{H}}} + l \sup_{\theta \in [0, \mathfrak{r}]} \left(\mathscr{E}\|u(\theta)\|^2\right)^{1/2},$$

where $l = \int_{-\infty}^{0} H(s) d\theta < \infty$.

Definition 2.8 [18] The conformable FD of a function $\mathfrak{F}(\cdot)$ of order ρ with $\mathfrak{r} > 0$ is defined as follows:

$$\frac{d^{\rho}\mathfrak{F}(\mathfrak{r})}{d\mathfrak{r}^{\rho}} = \lim_{\rho \to 0} \frac{\mathfrak{F}\left(\mathfrak{r} + \rho \mathfrak{r}^{1-\rho}\right) - \mathfrak{F}(\mathfrak{r})}{\rho},\ 0 < \rho < 1.$$

For $\mathfrak{r} = 0$, the definition becomes,

$$\frac{d^{\rho}\mathfrak{F}(0)}{d\mathfrak{r}^{\rho}} = \lim_{\mathfrak{r}\to 0^+} \frac{d^{\rho}\mathfrak{F}(\mathfrak{r})}{d\mathfrak{r}^{\rho}}.$$

Moreover,

$$\mathscr{I}^{\rho}(\mathfrak{F})(\mathfrak{r}) = \int_0^{\mathfrak{r}} \mathfrak{g}^{\rho-1}\mathfrak{F}(\mathfrak{g})d\mathfrak{g}.$$

Definition 2.9 (Trajectory controllable) [27] The system (1) is $\mathfrak{T}$-controllable on $[0, \mathfrak{T}]$ if for any $w \in \mathscr{T}$, ($w(\cdot)$- a set of all continuous functions (trajectories) defined on $[0, \mathfrak{T}]$ such that $w(0) = u_0$ and $w(\mathfrak{T}) = u_1$), and if fractional derivative $D_0^{\rho} w$ exists almost everywhere, then there exists a control $y \in L^2([0, \mathfrak{T}])$ such that the mild solution of (1) satisfies $u(\mathfrak{r}) = w(\mathfrak{r})$ on $[0, \mathfrak{T}]$.

Lemma 2.5 (Generalized Gronwall's inequality) [31] Suppose $l(\mathfrak{r})$ and $m(\mathfrak{r})$ are non-negative locally integrable functions on $0 \le \mathfrak{r} < \infty$ and $s(\mathfrak{r})$ is a non-negative, non-decreasing function on $0 \le \mathfrak{r} < \infty$ with $s(\mathfrak{r}) \le C$ where C is a constant.

If

$$m(\mathfrak{r}) \le l(\mathfrak{r}) + s(\mathfrak{r})\int_0^{\mathfrak{r}} (\mathfrak{r}-\theta)^{\alpha-1} m(\theta)d\theta$$

for any $\alpha > 0$, then

$$m(\mathfrak{r}) \le l(\mathfrak{r}) + \int_0^{\mathfrak{r}} \sum_{n=1}^{\infty} \frac{(s(\mathfrak{r})\Gamma(\alpha))^n}{\Gamma(n\alpha)} (\mathfrak{r}-\theta)^{\alpha-1} m(\theta)d\theta,\ 0 \le \mathfrak{r} < \infty.$$

Furthermore, if $l(\mathfrak{r}) = 0$, then $m(\mathfrak{r}) = 0\ \forall\ 0 \le \mathfrak{r} < \infty$. From the motivation from [5, 27], we shall construct the mild solution (1) as follows,

Definition 2.10 The $\mathscr{K}_{\mathfrak{r}}$-adapted stochastic process u: $(-\infty, b] \to \mathscr{V}$ with $\kappa \in \mathscr{L}^2(\Lambda, \mathscr{O}_H)$ on $(-\infty, 0]$, $u_0 \in \mathscr{L}_2^0(\Lambda, \mathscr{V})$ is the mild solution of (1) is defined as

$$u(\mathfrak{r}) = I^{\rho(\mathfrak{e}-1)}(\mathfrak{r})\mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\kappa(0) + \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right) y(\theta)d\theta$$

$$+\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{L}\left(\theta, u_\theta, \int_0^{\theta}\mathscr{M}\left(\theta, \rho, u_\rho\right)d\rho\right)d\theta$$

$$+\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{N}\left(\theta, u_\theta, \int_0^{\theta}\tilde{\mathscr{M}}\left(\theta, \rho, u_\rho\right)d\rho\right)d\gamma(\theta), \text{ here } \int_{\theta}^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1}d\mathfrak{g} = \frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}. \tag{2}$$

3. Existence and uniqueness of mild solution

Since prior motive is to prove the uniqueness of mild solution of the Hilfer fractional differential system, assume that all the appropriate functions used in the proposed system satisfies the Lipschitz condition through semigroup theory and stochastic process as follows.

(C1) The linear operator $\mathscr{G}: \mathscr{V} \to \mathscr{V}$ generates C_0-semigroup $T(\mathfrak{r})$. Thus $\exists\ M > 0$ such that $\|T(\mathfrak{r})\| \le M\ \forall \mathfrak{r} \in J$.

(C2) The function $\mathscr{L}$: $J \times \mathscr{O}_H \times \mathscr{V} \to \mathscr{V}$ is continuous for $\mathfrak{r} \in J$. $w_1, \tilde{w}_1 \in \mathscr{O}_H$, $w_2, \tilde{w}_2 \in \mathscr{V}$ and there exist

$Q_{\mathscr{L}}, \hat{Q}_{\mathscr{L}} > 0$ such that

$$\mathscr{E} \| \mathscr{L}(\mathfrak{r}, w_1, w_2) - \mathscr{L}(\mathfrak{r}, \tilde{w}_1, \tilde{w}_2) \|^2 \leq Q_{\mathscr{L}}^2 \left(\mathfrak{r}^{2(1-\mathfrak{p})} \| w_1 - \tilde{w}_1 \|_{\mathscr{C}_{\mathrm{H}}}^2 + \mathscr{E} \| w_2 - \tilde{w}_2 \|^2 \right),$$

$$\mathscr{E} \| \mathscr{L}(\mathfrak{r}, w_1, w_2) \|^2 \leq \hat{Q}_{\mathscr{L}}^2 \left(1 + \mathfrak{r}^{2(1-\mathfrak{p})} \| w_1 \|_{\mathscr{C}_{\mathrm{H}}}^2 + \mathscr{E} \| w_2 \|^2 \right).$$

(C3) The function $\mathscr{N}$: J × $\mathscr{C}_{\mathrm{H}}$ × $\mathscr{V} \rightarrow L_2^0$ is continuous for $\mathfrak{r} \in$ J and for $w_1, \tilde{w}_1 \in \mathscr{C}_{\mathrm{H}}, w_2, \tilde{w}_2 \in \mathscr{V}$ there exist positive constants $Q_{\mathscr{N}}, \hat{Q}_{\mathscr{N}}$ such that

$$\mathscr{E} \| \mathscr{N}(\mathfrak{r}, w_1, w_2) - \mathscr{N}(\mathfrak{r}, \tilde{w}_1, \tilde{w}_2) \|^2 \leq Q_{\mathscr{N}}^2 \left(\mathfrak{r}^{2(1-\mathfrak{p})} \| w_1 - \tilde{w}_1 \|_{\mathscr{C}_{\mathrm{H}}}^2 + \mathscr{E} \| w_2 - \tilde{w}_2 \|^2 \right),$$

$$\mathscr{E} \| \mathscr{N}(\mathfrak{r}, w_1, w_2 \|)^2 \leq \hat{Q}_{\mathscr{N}}^2 \left(1 + \mathfrak{r}^{2(1-\mathfrak{p})} \| w_1 \|_{\mathscr{C}_{\mathrm{H}}}^2 + \mathscr{E} \| w_2 \|^2 \right).$$

(C4) The appropriate functions $\mathscr{M}, \tilde{\mathscr{M}} : \mathrm{J} \times \mathrm{J} \times \mathscr{C}_{\mathrm{H}} \rightarrow \mathscr{V}$ are continuous $\forall(\mathfrak{r}, \theta) \in$ J × J. For all $w, \tilde{w} \in \mathscr{C}_{\mathrm{H}}$, there exist positive constants $n_1, n_2, \tilde{n}_1, \tilde{n}_2$ such that

$$\mathscr{E} \| \mathscr{M}(\mathfrak{r}, \theta, w) - \mathscr{M}(\mathfrak{r}, \theta, \tilde{w}) \|^2 \leq n_1^2 \mathfrak{r}^{2(1-\mathfrak{p})} \| w - \tilde{w} \|_{\mathscr{C}_{\mathrm{H}}}^2,$$

$$\mathscr{E} \| \tilde{\mathscr{M}}(\mathfrak{r}, \theta, w) - \tilde{\mathscr{M}}(\mathfrak{r}, \theta, \tilde{w}) \|^2 \leq n_2^2 \mathfrak{r}^{2(1-\mathfrak{p})} \| w - \tilde{w} \|_{\mathscr{C}_{\mathrm{H}}}^2,$$

$$\mathscr{E} \| \mathscr{M}(\mathfrak{r}, \theta, w) \|^2 \leq \tilde{n}_1^2 \left(1 + \mathfrak{r}^{2(1-\mathfrak{p})} \| w \|_{\mathscr{C}_{\mathrm{H}}}^2 \right),$$

$$\mathscr{E} \| \tilde{\mathscr{M}}(\mathfrak{r}, \theta, w) \|^2 \leq \tilde{n}_2^2 \left(1 + \mathfrak{r}^{2(1-\mathfrak{p})} \| w \|_{\mathscr{C}_{\mathrm{H}}}^2 \right).$$

Theorem 3.1 If the suppositions (C1)-(C4), with $c > 0$ are satisfied, then $\tilde{\Upsilon}(\mathrm{B}_c) \subset \mathrm{B}_c$ and

$$\frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}_{\mathscr{L}}^2 \left(1 + \tilde{n}_1^2 \mathrm{b}^2 \right) + \mathrm{Tr}(\mathfrak{Q}) \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}_{\mathscr{N}}^2 \left(1 + \tilde{n}_2^2 \mathrm{b}^2 \right) < 1. \tag{3}$$

Proof. Define the operator $\psi : \mathscr{C}'_{\mathrm{H}} \rightarrow \mathscr{C}'_{\mathrm{H}}$ be

$$\psi u(\mathfrak{r}) = \begin{cases} \mathrm{I}^{\rho(\mathfrak{e}-1)} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}} \right) \kappa(0), \ \mathfrak{r} \in (-\infty, 0], \\ \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}} \right) y(\theta) d\theta + \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}} \right) \mathscr{L}\left(\theta, u_\theta, \int_0^\theta \mathscr{M}(\theta, \rho, u_\rho) d\rho \right) d\theta \\ + \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}}}{\mathfrak{e}} - \frac{s^{\mathfrak{e}}}{\mathfrak{e}} \right) \mathscr{N}\left(\theta, u_\theta, \int_0^\theta \tilde{\mathscr{M}}(\theta, \rho, u_\rho) d\rho \right) d\gamma(\theta), \ \mathfrak{r} \in \mathrm{J}, \end{cases}$$

$$\overline{\kappa}(\mathfrak{r}) = \begin{cases} \mathscr{I}^{(1-\rho)(1-\mathfrak{e})} u(0) & ; \ \mathfrak{r} \in (-\infty, 0], \\ \kappa(0) & ; \ \mathfrak{r} \in \mathrm{J}, \end{cases}$$

then $\overline{\kappa} \in \mathscr{C}'_{\mathrm{H}}$. Let $u(\mathfrak{r}) = \mathfrak{z}(\mathfrak{r}) + \overline{\kappa}(\mathfrak{r}), \ -\infty < \mathfrak{r} \leq \mathrm{b}$. Clearly this assumption makes $\mathfrak{z}_0 = 0$. Then $\mathfrak{z}(\mathfrak{r})$ becomes

$$\mathfrak{z}(\mathfrak{r}) = \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) y(\theta) d\theta$$

$$+\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \mathscr{L}\left(\theta,\ \mathfrak{z}_\theta + \bar{\kappa}_\theta,\ \int_0^\theta \mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) d\theta$$

$$+\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \mathscr{N}\left(\theta,\ \mathfrak{z}_\theta + \bar{\kappa}_\theta,\ \int_0^\theta \tilde{\mathscr{M}}\left(\theta,\ \rho,\ \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) d\gamma(\theta).$$

Let $\mathscr{O}_{\mathrm{H}}'' = \left\{\mathfrak{z} \in \mathscr{O}_{\mathrm{H}}';\ \mathfrak{z}_0 = 0 \in \mathscr{O}_{\mathrm{H}}\right\}$. For any $\mathfrak{z} \in \mathscr{O}_{\mathrm{H}}''$.

$$\|\mathfrak{z}\|_{\mathfrak{b}} = \|\mathfrak{z}_0\|_{\mathscr{O}_{\mathrm{H}}} + \sup_{0 \le \theta \le \mathfrak{b}} \left(\mathscr{E} \|\ \mathfrak{z}(\theta)\ \|^2\right)^{1/2} = \sup_{0 \le \theta \le \mathfrak{b}} \left(\mathscr{E} \|\ \mathfrak{z}(\theta)\ \|^2\right)^{1/2}.$$

This exhibits, $\left(\mathscr{O}_{\mathrm{H}}'',\ \|\cdot\|_{\mathfrak{b}}\right)$ is a Banach space. We consider $\mathrm{B}_c = \left\{\mathfrak{z} \in \mathscr{O}_{\mathrm{H}}'':\ \|\mathfrak{z}\|_{\mathfrak{b}}^2 \le c\right\};\ c > 0$, such that $\mathrm{B}_c \subseteq \mathscr{O}_{\mathrm{H}}''$ uniformly bounded, $\forall$c which leads us to make an assumption with the help of Lemma 2.4,

$$\left\|\mathfrak{z}_{\mathfrak{r}} + \bar{\kappa}_{\mathfrak{r}}\right\|_{\mathscr{O}_{\mathrm{H}}}^2 \le c' \text{ where } c' > 0.$$

Construct $\tilde{\Upsilon} : \mathscr{O}_{\mathrm{H}}'' \to \mathscr{O}_{\mathrm{H}}''$ as follows,

$$\tilde{\Upsilon}\mathfrak{z}(\mathfrak{r}) = \begin{cases} 0,\ \mathfrak{r} \in (-\infty,\ 0) \\ \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) y(\theta) d\theta \\ +\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \mathscr{L}\left(\theta,\ \mathfrak{z}_\theta + \bar{\kappa}_\theta,\ \int_0^\theta \mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) d\theta \\ +\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \mathscr{N}\left(\theta,\ \mathfrak{z}_\theta + \bar{\kappa}_\theta,\ \int_0^\theta \tilde{\mathscr{M}}\left(\theta,\ \rho,\ \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) d\gamma(\theta),\ \mathfrak{r} \in \mathrm{J}. \end{cases}$$

Step 1

To prove $\tilde{\Upsilon}$ has a fixed point, first it is necessary to show that $\tilde{\Upsilon}$ maps into itself. By the method of contradiction, we consider that for any $c > 0 \exists \mathfrak{r}^{1-\mathfrak{p}} \mathfrak{z}^c(\mathfrak{r}) \in \mathrm{B}_c$ and $\tilde{\Upsilon}\left(\mathfrak{r}^{1-\mathfrak{p}} \mathfrak{z}^c\right) \notin \mathrm{B}_c$, (i.e), $\mathscr{E}\left\|\mathfrak{r}^{1-\mathfrak{p}}\left(\tilde{\Upsilon}\mathfrak{z}^c\right)(\mathfrak{r})\right\|^2 > c$ for $\mathfrak{r} \in \mathscr{E}$.

The suppositions (C1)-(C4) with Lemma 2.3 produces

$$c \le \mathscr{E} \|\ \mathfrak{r}^{1-\mathfrak{p}}\left(\tilde{\Upsilon}\mathfrak{z}^c\right)(\mathfrak{r})\ \|^2$$

$$\le 3\left[\mathscr{E}\left\|\mathfrak{r}^{1-\mathfrak{p}} \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) y(\theta) d\theta\right\|^2 + \mathscr{E}\left\|\mathfrak{r}^{1-\mathfrak{p}} \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \mathscr{L}\left(\theta,\ \mathfrak{z}_\theta^c + \bar{\kappa}_\theta,\ \int_0^\theta \mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_\rho^c + \bar{\kappa}_\rho\right) d\rho\right) d\theta\right\|^2\right.$$

$$\left.+\mathscr{E}\left\|\mathfrak{r}^{1-\mathfrak{p}} \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \mathscr{N}\left(\theta,\ \mathfrak{z}_\theta^c + \bar{\kappa}_\theta,\ \int_0^\theta \tilde{\mathscr{M}}\left(\theta,\ \rho,\ \mathfrak{z}_\rho^c + \bar{\kappa}_\rho\right) d\rho\right) d\gamma(\theta)\right\|^2\right]$$

where

$$\mathbf{S}_1 \leq 3\mathscr{E} \| \mathfrak{r}^{1-\mathfrak{p}} \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) y(\theta) d\theta \|^2$$

$$\leq \frac{3M^2}{\Gamma^2(\mathfrak{e})} \mathfrak{r}^{2(1-\mathfrak{p})} \int_0^{\mathfrak{r}} \mathfrak{g}^{2(\mathfrak{e}-1)} \mathscr{E} \| y(\theta) \|^2 \, d\theta$$

$$\leq \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{r}^{2\mathfrak{e}-1+2-2\mathfrak{p}}}{2\mathfrak{e}-1} \| y \|^2$$

$$\leq \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \| y \|^2; \ \mathfrak{r} \in (0, \ \mathrm{b}].$$

$$\mathbf{S}_2 \leq 3\mathscr{E} \| \mathfrak{r}^{1-\mathfrak{p}} \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \mathscr{L}\left(\theta, \ \mathfrak{z}_\theta^c + \overline{\kappa}_\theta, \ \int_0^\theta \mathscr{M}\left(\theta, \ \rho, \ \mathfrak{z}_\rho^c + \overline{\kappa}_\rho\right) d\rho\right) d\theta \|^2$$

$$\leq \frac{3M^2}{\Gamma^2(\mathfrak{e})} \mathfrak{r}^{2(1-\mathfrak{p})} \int_0^{\mathfrak{r}} \mathfrak{g}^{2(\mathfrak{e}-1)} \hat{Q}_{\mathscr{L}}^2 \left(1 + \| \mathfrak{z}_\theta^c + \overline{\kappa}_\theta \|^2 + \mathscr{E} \| \int_0^\theta \mathscr{M}\left(\theta, \ \rho, \ \mathfrak{z}_\rho^c + \overline{\kappa}_\rho\right) d\rho \|^2\right) d\theta$$

$$\leq \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}-1+2-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}_{\mathscr{L}}^2 \left(1 + c' + \tilde{n}_1^2 \mathrm{b}^2 \left(1 + c'\right)\right)$$

$$\leq \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}_{\mathscr{L}}^2 \left(c + \tilde{n}_1^2 \mathrm{b}^2 c\right); \ \text{where } c = 1 + c'.$$

$$\mathbf{S}_3 \leq 3\mathscr{E} \| \mathfrak{r}^{1-\mathfrak{p}} \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \mathscr{N}\left(\theta, \ \mathfrak{z}_\theta^c + \overline{\kappa}_\theta, \ \int_0^\theta \tilde{\mathscr{M}}\left(\theta, \ \rho, \ \mathfrak{z}_\rho^c + \overline{\kappa}_\rho\right) d\rho\right) d\gamma(\theta) \|^2$$

$$\leq \frac{3M^2}{\Gamma^2(\mathfrak{e})} \mathfrak{r}^{2(1-\mathfrak{p})} \mathrm{Tr}(\mathfrak{Q}) \int_0^{\mathfrak{r}} \mathfrak{g}^{2(\mathfrak{e}-1)} \hat{Q}_{\mathscr{N}}^2 \left(1 + \| \mathfrak{z}_\theta^c + \overline{\kappa}_\theta \|^2 + \mathscr{E} \| \int_0^\theta \tilde{\mathscr{M}}\left(\theta, \ \rho, \ \mathfrak{z}_\rho^c + \overline{\kappa}_\rho\right) d\rho \|^2\right) d\theta$$

$$\leq 3\mathrm{Tr}(\mathfrak{Q}) \frac{M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{r}^{2\mathfrak{e}-1+2-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}_{\mathscr{N}}^2 \left(1 + c' + \tilde{n}_2^2 \mathrm{b}^2 \left(1 + c'\right)\right)$$

$$\leq 3\mathrm{Tr}(\mathfrak{Q}) \frac{M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}_{\mathscr{N}}^2 \left(c + \tilde{n}_2^2 \mathrm{b}^2 c\right).$$

Substituting the above values in the original equation, it yields

$$c \leq \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \|y\|^2 + \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}-1+2-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}_{\mathscr{L}}^2 \left(c + \tilde{n}_1^2 \mathrm{b}^2 c\right) + \mathrm{Tr}(\mathfrak{Q}) \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathrm{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}_{\mathscr{N}}^2 \left(c + \tilde{n}_2^2 \mathrm{b}^2 c\right),$$

Divide the above inequality by c and as $c \to \infty$, we obtain

$$1 \le \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}^2_{\mathscr{L}} \left(1+\tilde{n}_1^2 \mathfrak{b}^2\right) + \mathrm{Tr}(\mathfrak{Q}) \frac{3M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} \hat{Q}^2_{\mathscr{N}} \left(1+\tilde{n}_2^2 \mathfrak{b}^2\right),$$

which contradicts the assumption. So, there must be some $c > 0$, such that $\tilde{\Upsilon}(\mathrm{B}_c) \subset \mathrm{B}_c$.

Step 2 To prove the map $\tilde{\Upsilon}$ is a contraction map, we shall assume that $\mathfrak{z}, \hat{\mathfrak{z}} \in \mathrm{B}_c$ and the expectation set be defined as,

$$\begin{aligned}
&\mathscr{E} \| \tilde{\Upsilon}\mathfrak{z}(\mathfrak{r}) - \tilde{\Upsilon}\hat{\mathfrak{z}}(\mathfrak{r}) \|^2 \\
&\le 2\Bigg[\mathfrak{r}^{2(1-\mathfrak{p})} \mathscr{E} \Big\| \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \Big[\mathscr{L}\left(\theta,\, \mathfrak{z}_\theta + \bar{\kappa}_\theta,\, \int_0^\theta \mathscr{M}\left(\theta,\, \rho,\, \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) \\
&\quad - \mathscr{L}\left(\theta,\, \hat{\mathfrak{z}}_\theta + \bar{\kappa}_\theta,\, \int_0^\theta \mathscr{M}\left(\theta,\, \rho,\, \hat{\mathfrak{z}}_\rho + \bar{\kappa}_\rho\right) d\rho\right) d\theta \Big] \Big\|^2 + \mathfrak{r}^{2(1-\mathfrak{p})} \mathscr{E} \left\| \int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) \right. \\
&\quad \times \left. \Big[\mathscr{N}\left(\theta,\, \mathfrak{z}_\theta + \bar{\kappa}_\theta,\, \int_0^\theta \tilde{\mathscr{M}}\left(\theta,\, \rho,\, \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) - \mathscr{N}\left(\theta,\, \hat{\mathfrak{z}}_\theta + \bar{\kappa}_\theta,\, \int_0^\theta \tilde{\mathscr{M}}\left(\theta,\, \rho,\, \hat{\mathfrak{z}}_\rho + \bar{\kappa}_\rho\right) d\rho\right) \Big] d\gamma(\theta) \right\|^2 \Bigg] \\
&\le \frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{r}^{2\mathfrak{e}-1+2-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{L}} \left(\| \mathfrak{z}_\theta - \hat{\mathfrak{z}}_\theta \|^2_{\mathscr{C}_{\mathrm{H}}} + n_1 \mathfrak{b}^2 \| \mathfrak{z}_\rho - \hat{\mathfrak{z}}_\rho \|^2 \right) \\
&\quad + 2\,\mathrm{Tr}(\mathfrak{Q}) \frac{M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{r}^{2\mathfrak{e}-1+2-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{N}} \left(\| \mathfrak{z}_\theta - \hat{\mathfrak{z}}_\theta \|^2_{\mathscr{C}_{\mathrm{H}}} + n_2 \mathfrak{b}^2 \| \mathfrak{z}_\theta - \hat{\mathfrak{z}}_\theta \|^2 \right) \\
&\le \left[\frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}-1+2-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{L}} \left(1+n_1^2 \mathfrak{b}^2\right) + \mathrm{Tr}(\mathfrak{Q}) \frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{N}} \left(1+n_2^2 \mathfrak{b}^2\right) \right] \| \mathfrak{z}_\theta - \hat{\mathfrak{z}}_\theta \|^2_{\mathscr{C}_{\mathrm{H}}} \\
&\le \left[\frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{L}} \left(1+n_1^2 \mathfrak{b}^2\right) + \mathrm{Tr}(\mathfrak{Q}) \frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{N}} \left(1+n_2^2 \mathfrak{b}^2\right) \right] \\
&\quad \times \left(l^2 \sup_{\theta \in \mathrm{J}} \mathscr{E} \| \mathfrak{z}(\theta) - \hat{\mathfrak{z}}(\theta) \|^2 + \| \mathfrak{z}_0 \|^2_{\mathscr{C}_{\mathrm{H}}} + \| \hat{\mathfrak{z}}_0 \|^2_{\mathscr{C}_{\mathrm{H}}} \right) \\
&\le l^2 \left[\frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{L}} \left(1+n_1^2 \mathfrak{b}^2\right) + \mathrm{Tr}(\mathfrak{Q}) \frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{N}} \left(1+n_2^2 \mathfrak{b}^2\right) \right] \sup_{\theta \in \mathrm{J}} \mathscr{E} \| \mathfrak{z}(\theta) - \hat{\mathfrak{z}}(\theta) \|^2 \\
&\le \mathrm{c}^* \sup_{\theta \in \mathrm{J}} \mathscr{E} \| \mathfrak{z}(\theta) - \hat{\mathfrak{z}}(\theta) \|^2,
\end{aligned}$$

here

$$\mathrm{c}^* = l^2 \left[\frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{L}} \left(1+n_1^2 \mathfrak{b}^2\right) + \mathrm{Tr}(\mathfrak{Q}) \frac{2M^2}{\Gamma^2(\mathfrak{e})} \frac{\mathfrak{b}^{2\mathfrak{e}+1-2\mathfrak{p}}}{2\mathfrak{e}-1} Q^2_{\mathscr{N}} \left(1+n_2^2 \mathfrak{b}^2\right) \right],$$

$c^* \in [0, 1)$. Since $c^* = 0$ is a trivial case, that is $\mathscr{E} \| \mathfrak{z}(\mathfrak{r}) - \hat{\mathfrak{z}}(\mathfrak{r}) \|^2 = 0$ is nothing to do with the uniqueness of mild solution. Therefore $c^* \in (0, 1)$. From the above result, taking supremum over $\mathfrak{r}$, $\| \tilde{\Upsilon}\mathfrak{z}(\mathfrak{r}) - \tilde{\Upsilon}\hat{\mathfrak{z}}(\mathfrak{r}) \|^2 = c^* \| \mathfrak{z}(\mathfrak{r}) - \hat{\mathfrak{z}}(\mathfrak{r}) \|^2$. concluding that $\tilde{\Upsilon}$ is a contraction map having unique fixed point $\mathfrak{z}(.) \in B_c$, which is the mild solution of (1). Hence proved. □

4. Mild solution of conformable Hilfer fractional neutral stochastic integro-differential equation

Neutral fractional equations with dependent delay, infinite delay, or without delay assist partial differential systems in many fields like physics, Biological sciences and control theory, as the importance of neutral differential equations in practical mathematics has grown over the years. Consider the neutral stochastic integro-differential equation infinite-delay of the form:

$$\mathscr{D}_{0^+}^{\rho,\mathfrak{e}}\left[u(\mathfrak{r}) - \mathfrak{k}\left(\mathfrak{r}, u_\mathfrak{r}\right)\right] = \mathscr{G}[u(\mathfrak{r})] + y(\mathfrak{r}) + \mathscr{L}\left(\mathfrak{r}, u_\mathfrak{r}, \int_0^\mathfrak{r} \mathscr{M}\left(\mathfrak{r}, \theta, u_\theta\right) d\theta\right)$$

$$+\mathscr{N}\left(\mathfrak{r}, u_\mathfrak{r}, \int_0^\mathfrak{r} \tilde{\mathscr{M}}\left(\mathfrak{r}, \theta, u_\theta\right) d\theta\right) \frac{d\gamma(\mathfrak{r})}{d\mathfrak{r}}$$

$$\mathscr{I}^{(1-\rho)(1-\mathfrak{e})} u(0) = \kappa(\mathfrak{r}) \in \mathscr{L}^2\left(\Lambda, \mathscr{O}_H\right), \ \mathfrak{r} \in (-\infty, 0]. \tag{4}$$

Consider the following hypothesis:
(C5) $\mathfrak{k} : [0, b] \times \mathscr{O}_H \rightarrow \mathscr{V}$ is continuous such that

$$\mathscr{E} \| \mathscr{G}^\beta (\mathfrak{k}(\mathfrak{r}, w) - \mathfrak{k}(\mathfrak{r}, \hat{w})) \|^2 \le Q_\mathfrak{k}^2 \mathfrak{r}^{2(1-p)} \| w - \hat{w} \|_{\mathscr{O}_H}^2, \ w, \hat{w} \in \mathscr{O}_H, \ \mathfrak{r} \in \mathscr{O}_H,$$

$$\mathscr{E} \| \mathscr{G}^\beta \mathfrak{k}(\mathfrak{r}, w) \|^2 \le \hat{Q}_\mathfrak{k}^2 \left(1 + \mathfrak{r}^{2(1-p)} \| w \|_{\mathscr{O}_H}^2\right), \ w \in \mathscr{O}_H, \ \mathfrak{r} \in \mathscr{O}_H.$$

Definition 4.1 A function $u \in \mathscr{O}_H$ is a $\mathscr{K}_\mathfrak{r}$-adapted stochastic process which is mild solution of (4) is defined by $u : (-\infty, b] \rightarrow \mathscr{V}$ and $\kappa \in \mathscr{L}^2(\Lambda, \mathscr{O}_H)$ on $(-\infty, 0]$, $u_0 \in \mathscr{L}_2^0(\Lambda, \mathscr{V})$ and it satisfies the following

$$u(\mathfrak{r}) = I^{\rho(\mathfrak{e}-1)} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^\mathfrak{e} - \theta^\mathfrak{e}}{\mathfrak{e}}\right)[\kappa(0) - \mathfrak{k}(0, \kappa)] + \mathfrak{k}\left(\mathfrak{r}, u_\mathfrak{r}\right) + \int_0^\mathfrak{r} \mathscr{G}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^\mathfrak{e} - \theta^\mathfrak{e}}{\mathfrak{e}}\right) \mathfrak{k}(\theta, u(\theta)) d\theta$$

$$+\int_0^\mathfrak{r} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^\mathfrak{e} - \theta^\mathfrak{e}}{\mathfrak{e}}\right) y(\theta) d\theta + \int_0^\mathfrak{r} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^\mathfrak{e} - \theta^\mathfrak{e}}{\mathfrak{e}}\right) \mathscr{L}\left(\theta, u_\theta, \int_0^\theta \mathscr{M}\left(\theta, \rho, u_\rho\right) d\rho\right) d\theta$$

$$+\int_0^\mathfrak{r} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^\mathfrak{e} - \theta^\mathfrak{e}}{\mathfrak{e}}\right) \mathscr{N}\left(\theta, u_\theta, \int_0^\theta \tilde{\mathscr{M}}\left(\theta, \rho, u_\rho\right) d\rho\right) d\gamma(\theta), \tag{5}$$

Note The linear operator $\mathscr{G}$ generates $\{\mathscr{O}(\mathfrak{r})\}_{\mathfrak{r} \ge 0}$ on $\mathscr{V}$.
Theorem 4.1 With the suppositions (C1)-(C5), there exists $c > 0$ such that $\tilde{\Upsilon}\left(B_c\right) \subset B_c$ and

$$6\mathfrak{r}^{2(1-\mathfrak{p})}M_0^2\hat{Q}_{\mathfrak{k}}^2+\frac{6\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{b^{2\mathfrak{e}\beta+2(1-\mathfrak{p})}}{2\mathfrak{e}\beta-1}\hat{Q}_{\mathfrak{k}}^2+\frac{6M^2}{\Gamma^2\mathfrak{e}}\frac{\mathrm{b}^{2\mathfrak{e}+2(1-\mathfrak{p})-1}}{(2\mathfrak{e}-1)}\hat{Q}_{\mathscr{L}}^2\left(1+\tilde{n}_1^2\mathrm{b}^2\right)$$

$$+\frac{6M^2}{\Gamma^2(\mathfrak{e})}\mathrm{Tr}(\mathfrak{Q})\frac{\mathrm{b}^{2\mathfrak{e}-1+2(1-\mathfrak{p})}}{(2\mathfrak{e}-1)}\hat{Q}_{\mathscr{N}}^2\left(1+\tilde{n}_2^2\mathrm{b}^2\right)<1. \tag{6}$$

Proof. Let us define $\Gamma:\mathscr{O}_{\mathrm{H}}'\to\mathscr{O}_{\mathrm{H}}'$ as

$$\Gamma u(\mathfrak{r})=\begin{cases}\kappa(\mathfrak{r}),\ \mathfrak{r}\in(-\infty,\ 0),\\ \mathrm{I}^{\rho(\mathfrak{e}-1)}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)[\kappa(0)-\mathfrak{k}(0,\ \kappa)]+\mathfrak{k}\left(\mathfrak{r},\ u_{\mathfrak{r}}\right)+\int_0^{\mathfrak{r}}\mathscr{G}^{\mathfrak{e}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathfrak{k}\left(\theta,\ u_\theta\right)d\theta\\ +\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)y(\theta)d\theta+\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{L}\left(\theta,\ u_\theta,\ \int_0^\theta\mathscr{M}\left(\theta,\ \rho,\ u_\rho\right)d\rho\right)d\theta\\ +\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{N}\left(\theta,\ u_\theta,\ \int_0^\theta\tilde{\mathscr{M}}\left(\theta,\ \rho,\ u_\rho\right)d\rho\right)d\gamma(\theta),\ \mathfrak{r}>0.\end{cases}$$

Consider $\bar{\kappa}$ as

$$\bar{\kappa}(\mathfrak{r})=\begin{cases}\kappa(\mathfrak{r});\mathfrak{r}\in(-\infty,\ 0]\\ \kappa(0);\mathfrak{r}\in(0,\ \mathrm{b}]\end{cases}$$

We know that $\kappa\in\mathscr{O}_{\mathrm{H}}$ then $\bar{\kappa}\in\mathscr{O}_{\mathrm{H}}'$. Set $u(\mathfrak{r})=\mathfrak{z}(\mathfrak{r})+\bar{\kappa}(\mathfrak{r}),\ -\infty<\mathfrak{r}\le\mathrm{b}$. Clearly $\mathfrak{z}_0=0$ otherwise u wont be satisfied.

$$\mathfrak{z}(\mathfrak{r})=-I^{\nu(\mathfrak{e}-1)}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathfrak{k}(0,\ \kappa)+\mathfrak{k}\left(\mathfrak{r},\ \mathfrak{z}\mathfrak{r}+\bar{\kappa}_{\mathfrak{r}}\right)+\int_0^{\mathfrak{r}}\mathscr{G}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathfrak{k}\left(\theta,\ \mathfrak{z}_\theta\right)ds$$

$$+\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)y(\theta)d\theta+\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{L}\left(\theta,\ \mathfrak{z}_\theta+\bar{\kappa}_\theta,\int_0^\theta\mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_\rho+\bar{\kappa}_\rho\right)d\rho\right)d\theta$$

$$+\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{N}\left(\theta,\ \mathfrak{z}_\theta+\bar{\kappa}_\theta\int_0^\theta\tilde{\mathscr{M}}\left(\theta,\ \rho,\ \mathfrak{z}_\rho+\bar{\kappa}_\rho\right)d\rho\right)d\gamma(\theta).$$

Let $\mathscr{O}_{\mathrm{H}}''=\left\{\mathfrak{z}\in\mathscr{O}_{\mathrm{H}}';\ \mathfrak{z}_0=0\in\mathscr{O}_{\mathrm{H}}\right\}$.
For any $\mathfrak{z}\in\mathscr{O}_{\mathrm{H}}''$,

$$\|\mathfrak{z}\|_{\mathfrak{b}}=\|\mathfrak{z}_0\|_{\mathscr{O}_{\mathrm{H}}}+\sup_{0\le\theta\le\mathrm{b}}\left(\mathscr{E}\|\mathfrak{z}(\theta)\|^2\right)^{1/2}=\sup_{0\le\theta\le\mathrm{b}}\left(\mathscr{E}\|\mathfrak{z}(\theta)\|^2\right)^{1/2}.$$

Finally, $\left(\mathscr{O}_{\mathrm{H}}'',\ \|\cdot\|_{\mathfrak{b}}\right)$ is a complete normed linear space. Consider $\mathrm{B}_c=\left\{\mathfrak{z}\in\mathscr{O}_{\mathrm{H}}'':\|\mathfrak{z}\|_{\mathfrak{b}}^2\le c\right\};\ c>0$, such that $\mathrm{B}_c\subseteq\mathscr{O}_{\mathrm{H}}''$ is uniformly bounded, $\forall c$ which leads to with the help of Lemma 2.4,

$$\|\ \mathfrak{z}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\ \|_{\mathscr{O}_H}^2\le c'\ \text{where}\ c'>0.$$

Let us define $\tilde{\zeta}:\mathscr{O}''_{\mathrm{H}}\to\mathscr{O}''_{\mathrm{H}}$ as

$$\tilde{\zeta}\mathfrak{z}(\mathfrak{r})=\begin{cases}0;\ \mathfrak{r}\in(-\infty,\ 0]\\ -\mathrm{I}^{\rho(\mathfrak{e}-1)}(\mathfrak{r})\mathfrak{g}^{\mathfrak{e}-1}T\left(\dfrac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)[\mathfrak{k}(0,\ \kappa)]+\mathfrak{k}\left(\mathfrak{r},\ \mathfrak{z}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\right)+\int_0^{\mathfrak{r}}\mathscr{G}\mathfrak{g}^{\mathfrak{e}-1}T\left(\dfrac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathfrak{k}\left(\theta,\ \mathfrak{z}_{\theta}+\bar{\kappa}_{\theta}\right)d\theta\\ +\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\dfrac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)y(\theta)d\theta+\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\dfrac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\\ \times\mathscr{L}\left(\theta,\ \mathfrak{z}_{\theta}+\bar{\kappa}_{\theta},\ \int_0^{\theta}\mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_{\rho}+\bar{\kappa}_{\rho}\right)d\rho\right)d\theta+\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\dfrac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\\ \times\mathscr{N}\left(\theta,\ \mathfrak{z}_{\theta}+\bar{\kappa}_{\theta},\ \int_0^{\theta}\tilde{\mathscr{M}}\left(\theta,\ \rho,\ \mathfrak{z}_{\rho}+\bar{\kappa}_{\rho}\right)d\rho\right)d\gamma(\theta);\ \mathfrak{r}\in(0,\ b].\end{cases}$$

Step 1 $\tilde{\zeta}\left(\mathrm{B}_c\right)\subset\mathrm{B}_c$.

By the method of contradiction, we may assume that for any $c>0$ there exists $\mathfrak{r}^{1-\mathfrak{p}}\mathfrak{z}^c(\cdot)\in\mathrm{B}_c$ and $\tilde{\zeta}\left(\mathfrak{r}^{1-p}\mathfrak{z}^c\right)\notin\mathrm{B}_c$, (i.e), $\mathscr{E}\left\|\mathfrak{r}^{1-\mathfrak{p}}\left(\tilde{\zeta}\left(\mathfrak{z}^c\right)\right)(\mathfrak{r})\right\|^2>c$ for some $\mathfrak{r}\in(0,\ \mathrm{b}]$.

$$\begin{aligned}c&\le\mathfrak{r}^{2(1-\mathfrak{p})}\mathscr{E}\left\|\left(\bar{\zeta}\mathfrak{z}\right)(\mathfrak{r})\right\|^2,\\ &\le6\mathfrak{r}^{2(1-\mathfrak{p})}\mathscr{E}\left\|\mathrm{I}^{\rho(\mathfrak{e}-1)}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)[\mathfrak{k}(0,\kappa)]\right\|^2+6\mathfrak{r}^{2(1-\mathfrak{p})}\mathscr{E}\left\|\mathfrak{k}\left(\mathfrak{r},\ \mathfrak{z}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\right)\right\|^2\\ &+6\mathrm{r}^{2(1-\mathfrak{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}}\mathscr{G}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathfrak{k}\left(\theta,\ \mathfrak{z}_{\theta}+\bar{\kappa}_{\theta}\right)d\theta\right\|^2+6\mathfrak{r}^{2(1-\mathfrak{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)y(\theta)d\theta\right\|^2\\ &+6\mathfrak{r}^{2(1-\mathfrak{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{L}\left(\theta,\ \mathfrak{z}_{\theta}+\bar{\kappa}_{\theta},\ \int_0^{\theta}\mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_{\rho}+\bar{\kappa}_{\rho}\right)d\rho\right)d\theta\right\|^2\\ &+6\mathfrak{r}^{2(1-\mathfrak{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{N}\left(\theta,\ \mathfrak{z}_{\theta}+\bar{\kappa}_{\theta},\ \int_0^{\theta}\tilde{\mathscr{M}}\left(\theta,\ \rho,\ \mathfrak{z}_{\rho}+\bar{\kappa}_{\rho}\right)d\rho\right)d\gamma(\theta)\right\|^2,\\ &\le\mathrm{S}_1+\mathrm{S}_2+\mathrm{S}_3+\mathrm{S}_4+\mathrm{S}_5+\mathrm{S}_6,\end{aligned}$$

where

$$\begin{aligned}\mathrm{S}_1&=6\mathfrak{r}^{2(1-\mathfrak{p})}\mathscr{E}\left\|\mathrm{I}^{\rho(\mathfrak{e}-1)}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)[\mathfrak{k}(0,\ \kappa)]\right\|^2,\\ &\le\frac{6M^2}{\Gamma^2(\mathfrak{e})}\left[\left\|\mathscr{G}^{-\beta}\right\|^2\left\|\mathscr{G}^{\beta}\mathrm{I}^{\rho(\mathfrak{e}-1)}(\mathfrak{k}(0,\ \kappa))\right\|^2\right],\\ &\le\frac{6M^2}{\Gamma^2(\mathfrak{e})}\frac{M_0^2}{\Gamma^2(\rho(1-\mathfrak{e}))}\hat{Q}_{\mathfrak{k}}^2\left(1+\|\kappa\|_{\mathscr{O}_H}^2\right);\ \text{where}\ \left(\left\|\mathscr{G}^{-\beta}\right\|=M_0\right).\end{aligned}$$

$$\mathrm{S}_2 = 6\mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\left\|\mathfrak{k}\left(\mathfrak{r},\ \mathfrak{z}_{\mathfrak{r}} + \bar{\kappa}_{\mathfrak{r}}\right)\right\|^2 \le 6\mathfrak{r}^{2(1-\mathrm{p})} M_0^2 \hat{Q}_{\mathfrak{k}}^2\left(1 + \left\|\mathfrak{z}_{\mathfrak{r}} + \bar{\kappa}_{\mathfrak{r}}\right\|^2\right) \le 6\mathrm{b}^{2(1-\mathrm{p})} M_0^2 \hat{Q}_{\mathfrak{k}}^2\left(1 + c_1\right).$$

$$\mathrm{S}_3 = 6\mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}} \mathscr{G}\mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathfrak{k}\left(\theta,\ \mathfrak{z}_\theta\right) d\theta\right\|^2,$$

$$\le 6\mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\left(\int_0^{\mathfrak{r}} (\mathfrak{r} - \theta)^{\mathfrak{e}-1}\left\|\mathscr{G}^{1-\beta} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\right\|\left\|\mathscr{G}^{\beta}\mathfrak{k}\left(\theta,\ \mathfrak{z}_\theta\right)\right\| d\theta\right)^2,$$

$$\le 6\left(\frac{\mathfrak{e}^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta + 2(1-\mathrm{p})}}{2\mathfrak{e}\beta - 1}\right)\hat{Q}_{\mathfrak{k}}^2\left(1 + c_1\right).$$

$$\mathrm{S}_4 = 6\mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}} \mathscr{O}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right) y(\theta) d\theta\right\|^2,$$

$$\le \frac{6M^2}{\Gamma^2(\mathfrak{e})}\mathfrak{r}^{2(1-\rho)(1-\mathfrak{e})}\left\|y\right\|^2 \int_0^{\mathfrak{r}} \mathfrak{g}^{2(\mathfrak{e}-1)} d\theta,$$

$$\le \frac{6M^2}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}\left\|y\right\|^2.$$

$$\mathrm{S}_5 = 6\mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1}\mathscr{L}\left(\theta,\ \mathfrak{z}_\theta + \bar{\kappa}_\theta,\ \int_0^s \mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) ds\right\|^2,$$

$$\le 6\mathfrak{r}^{2(1-\mathrm{p})}\frac{M^2}{\Gamma^2(\mathfrak{e})}\mathscr{E}\left\|\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1}\mathscr{L}\left(\theta,\ \mathfrak{z}_\theta + \bar{\kappa}_\theta,\ \int_0^s \mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) d\theta\right\|^2,$$

$$\le 6\mathfrak{r}^{2(1-\mathrm{p})}\frac{M^2}{\Gamma^2(\mathfrak{e})}\frac{\mathfrak{r}^{2\mathfrak{e}-1}}{(2\mathfrak{e}-1)}\mathscr{E}\left\|\int_0^{\mathfrak{r}} \mathscr{L}\left(\theta,\ \mathfrak{z}_\theta + \bar{\kappa}_\theta,\ \int_0^\theta \mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) d\theta\right\|^2,$$

$$\le 6\frac{M^2}{\Gamma^2(\mathfrak{e})}\frac{\mathfrak{r}^{2\mathfrak{e}+2(1-\mathrm{p})-1}}{(2\mathfrak{e}-1)}\hat{Q}_{\mathscr{L}}^2\left(1 + \left\|\mathfrak{z}_\theta + \bar{\kappa}_\theta\right\|_{\mathscr{O}_H}^2 + \tilde{n}_1^2\mathrm{b}^2\left(1 + \left\|\mathfrak{z}_\rho + \bar{\kappa}_\rho\right\|_{\mathscr{O}_H}^2\right)\right),$$

$$\le 6\frac{M^2}{\Gamma^2(\mathfrak{e})}\frac{\mathrm{b}^{2\mathfrak{e}+2(1-\mathrm{p})-1}}{(2\mathfrak{e}-1)}\hat{Q}_{\mathscr{L}}^2\left(1 + c_1 + \tilde{n}_1^2\mathrm{b}^2\left(1 + c_1\right)\right).$$

$$\mathrm{S}_6 = 6\mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}} \mathfrak{g}^{\mathfrak{e}-1} T\left(\frac{\mathfrak{r}^{\mathfrak{e}} - \theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\mathscr{N}\left(\theta,\ \mathfrak{z}_\theta + \bar{\kappa}_\theta,\ \int_0^\theta \tilde{\mathscr{M}}\left(\theta,\ \rho,\ \mathfrak{z}_\rho + \bar{\kappa}_\rho\right) d\rho\right) d\gamma(s)\right\|^2,$$

$$\le 6\frac{M^2}{\Gamma^2(\mathfrak{e})}\mathrm{Tr}(\mathfrak{Q})\frac{\mathrm{b}^{2\mathfrak{e}-1+2(1-\mathrm{p})}}{(2\mathfrak{e}-1)}\hat{Q}_{\mathscr{N}}^2\left(1 + c_1 + \tilde{n}_2^2\mathrm{b}^2\left(1 + c_1\right)\right).$$

Therefore,

$$c \leq 6\frac{M^2}{\Gamma^2(\mathfrak{e})}\frac{M_0^2}{\Gamma^2(\rho(1-\mathfrak{e}))}\hat{Q}_{\mathfrak{k}}^2\left(1+\|\kappa\|_{\mathscr{C}_H}^2\right)+6\mathrm{b}^{2(1-\mathrm{p})}M_0^2\hat{Q}_{\mathfrak{k}}^2\left(1+c_1\right)$$

$$+6\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)\hat{Q}_{\mathfrak{k}}^2\left(1+c_1\right)+\frac{6M^2}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}\|y\|^2$$

$$+6\frac{M^2}{\Gamma^2(\mathfrak{e})}\frac{\mathrm{b}^{2\mathfrak{e}+2(1-\mathrm{p})-1}}{(2\mathfrak{e}-1)}\hat{Q}_{\mathscr{L}}^2\left(1+c_1+\tilde{n}_1^2\mathrm{b}^2\left(1+c_1\right)\right)+6\frac{M^2}{\Gamma^2(\mathfrak{e})}\mathrm{Tr}(\mathfrak{Q})\frac{\mathrm{b}^{2\mathfrak{e}-1+2(1-\mathrm{p})}}{(2\mathfrak{e}-1)}$$

$$\times\hat{Q}_{\mathscr{N}}^2\left(1+c_1+\tilde{n}_2^2\mathrm{b}^2\left(1+c_1\right)\right).$$

Dividing by c throughout and let $c\to\infty$ we obtain,

$$1\leq 6\mathrm{b}^{2(1-\mathrm{p})}M_0^2\hat{Q}_{\mathfrak{k}}^2+6\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)\hat{Q}_{\mathfrak{k}}^2$$

$$+6\left(\frac{M^2}{\Gamma^2(\mathfrak{e})}\frac{\mathrm{b}^{2\mathfrak{e}+2(1-\mathrm{p})-1}}{(2\mathfrak{e}-1)}\right)\hat{Q}_{\mathscr{L}}^2\left(1+\tilde{n}_1^2\mathrm{b}^2\right)+6\left(\frac{M^2}{\Gamma^2(\mathfrak{e})}Tr(\mathfrak{Q})\frac{\mathrm{b}^{2\mathfrak{e}-1+2(1-\mathrm{p})}}{(2\mathfrak{e}-1)}\right)\hat{Q}_{\mathscr{N}}^2\left(1+\tilde{n}_2^2\mathrm{b}^2\right),$$

which contradicts the assumption. So, there must be some c > 0, such that $\tilde{\zeta}\left(\mathrm{B}_c\right)\subset\mathrm{B}_c$.

Step 2 To claim that the map $\tilde{\zeta}$ is a contraction map, for $\mathfrak{z},\hat{\mathfrak{z}}\in\mathrm{B}_c$, the norm of expectation set be defined as,

$$\mathscr{E}\left\|\mathfrak{r}^{1-\mathrm{p}}\left(\tilde{\zeta}_{\mathfrak{z}}(\mathfrak{r})-\tilde{\zeta}\hat{\mathfrak{z}}(\mathfrak{r})\right)\right\|^2$$

$$=\mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\,\|\left[\mathfrak{k}\left(\mathfrak{r},\,\mathfrak{z}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\right)-\mathfrak{k}\left(\mathfrak{r},\,\hat{\mathfrak{z}}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\right)\right]+\int_0^{\mathfrak{r}}\mathscr{G}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\left[\mathfrak{k}\left(\theta,\,\mathfrak{z}_\theta+\bar{\kappa}_\theta\right)-\mathfrak{k}\left(\theta,\,\hat{\mathfrak{z}}_\theta+\bar{\kappa}_\theta\right)\right]d\theta$$

$$+\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\left[\mathscr{L}\left(\theta,\,\mathfrak{z}_\theta+\bar{\kappa}_\theta,\,\int_0^\theta\mathscr{M}\left(\theta,\,\rho,\mathfrak{z}_\rho+\bar{\kappa}_\rho\right)d\rho\right)-\mathscr{L}\left(\theta,\,\hat{\mathfrak{z}}_\theta+\bar{\kappa}_\theta,\,\int_0^\theta\mathscr{M}\left(\theta,\,\rho,\,\hat{\mathfrak{z}}_\rho+\bar{\kappa}_\rho\right)d\rho\right)\right]d\theta$$

$$+\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\left[\mathscr{N}\left(\theta,\,\mathfrak{z}_\theta+\bar{\kappa}_\theta,\,\int_0^\theta\mathscr{M}\left(\theta,\,\rho,\,\mathfrak{z}_\rho+\bar{\kappa}_\rho\right)d\rho\right)\right.$$

$$\left.-\mathscr{N}\left(\theta,\,\hat{\mathfrak{z}}_\theta+\bar{\kappa}_\theta,\,\int_0^\theta\mathscr{M}\left(\theta,\,\rho,\,\hat{\mathfrak{z}}_\rho+\bar{\kappa}_\rho\right)d\rho\right)\right]d\gamma(\theta)\Bigg\|^2$$

$$=4[\mathrm{S}_1+\mathrm{S}_2+\mathrm{S}_3+\mathrm{S}_4].$$

$$\mathrm{S}_1=\mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\left\|\mathfrak{k}\left(\mathfrak{r},\,\mathfrak{z}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\right)-\mathfrak{k}\left(\mathfrak{r},\,\hat{\mathfrak{z}}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\right)\right\|^2\leq\mathrm{b}^{2(1-\mathrm{p})}M_0^2Q_{\mathfrak{k}}^2\left\|\mathfrak{z}_{\mathfrak{r}}-\hat{\mathfrak{z}}_{\mathfrak{r}}\right\|^2.$$

$$\mathrm{S}_2 = \mathfrak{r}^{2(1-\mathrm{p})}\mathscr{E}\left\|\int_0^{\mathfrak{r}}\mathscr{G}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\left(\mathfrak{k}\left(\theta,\ \mathfrak{z}_\theta+\overline{\kappa}_\theta\right)-\mathfrak{k}\left(\theta,\ \hat{\mathfrak{z}}_\theta+\overline{\kappa}_\theta\right)\right)d\theta\right\|^2$$

$$\leq\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)Q_{\mathfrak{k}}^2\left\|\mathfrak{z}_\theta-\hat{\mathfrak{z}}_\theta\right\|^2.$$

$$\mathrm{S}_3\leq\frac{M^2\mathfrak{r}^{2(1-\mathrm{p})}}{\Gamma^2(\mathfrak{e})}\int_0^{\mathfrak{r}}\mathfrak{g}^{2(\mathfrak{e}-1)}\mathscr{E}\left\|\mathfrak{z}_\theta-\hat{\mathfrak{z}}_\theta\right\|^2d\theta\leq\frac{M^2\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}Q_{\mathscr{L}}^2\left(1+n_1^2\mathrm{b}^2\right)\left\|\mathfrak{z}_\theta-\hat{\mathfrak{z}}_\theta\right\|^2.$$

$$\mathrm{S}_4\leq\mathrm{Tr}(\mathfrak{Q})\frac{M^2\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}Q_{\mathscr{N}}^2\left(1+n_2^2\mathrm{b}^2\right)\left\|\mathfrak{z}_\theta-\hat{\mathfrak{z}}_\theta\right\|^2.$$

Substituting above values, we get

$$\mathscr{E}\left\|\tilde{\Upsilon}\mathfrak{z}(\mathfrak{r})-\tilde{\mathfrak{r}}\hat{\mathfrak{z}}(\mathfrak{r})\right\|^2\leq 4\mathrm{b}^{2(1-\mathrm{p})}M_0^2Q_{\mathfrak{k}}^2\left\|\mathfrak{r}-\hat{\mathfrak{z}}\mathfrak{r}\right\|^2+4\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)Q_{\mathfrak{k}}^2\left\|\mathfrak{z}_\theta-\hat{\mathfrak{z}}_\theta\right\|^2$$

$$+4\left(\frac{M^2\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\right)Q_{\mathscr{L}}^2\left(1+n_1^2\mathrm{b}^2\right)\left\|\mathfrak{z}_\theta-\hat{\mathfrak{z}}_\theta\right\|^2$$

$$+4\mathrm{Tr}(\mathfrak{Q})\left(\frac{M^2\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\right)Q_{\mathscr{N}}^2\left(1+n_2^2\mathrm{b}^2\right)\left\|\mathfrak{z}_\theta-\hat{\mathfrak{z}}_\theta\right\|^2,$$

which implies

$$\leq\Bigg[4\mathrm{b}^{2(1-\mathrm{p})}M_0^2Q_{\mathfrak{k}}^2+4\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)Q_{\mathfrak{k}}^2+4\left(\frac{M^2\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\right)Q_{\mathscr{L}}^2\left(1+n_1^2\mathrm{b}^2\right)$$

$$+4\,\mathrm{Tr}(\mathfrak{Q})\left(\frac{M^2\,\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\right)Q_{\mathscr{N}}^2\left(1+n_2^2\mathrm{b}^2\right)\Bigg]\left(l^2\sup_{\theta\in\mathrm{J}}\mathscr{E}\left\|\mathfrak{z}(\theta)-\hat{\mathfrak{z}}(\theta)\right\|^2+\left\|\mathfrak{z}_0\right\|_{\mathscr{C}_{\mathrm{H}}}^2+\left\|\hat{\mathfrak{z}}_0\right\|_{\mathscr{C}_{\mathrm{H}}}^2\right)$$

$$\leq l^2\Bigg[4\mathrm{b}^{2(1-\mathrm{p})}M_0^2Q_{\mathfrak{k}}^2+4\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)Q_{\mathfrak{k}}^2+4\left(\frac{M^2\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\right)$$

$$\times Q_{\mathscr{L}}^2\left(1+n_1^2\mathrm{b}^2\right)+4\,\mathrm{Tr}(\mathfrak{Q})\left(\frac{M^2\,\mathrm{b}^{2(1-\mathrm{p})+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\right)Q_{\mathscr{N}}^2\left(1+n_2^2\mathrm{b}^2\right)\Bigg]\times\left(\sup_{\theta\in\mathrm{J}}\mathscr{E}\left\|\mathfrak{z}(\theta)-\hat{\mathfrak{z}}(\theta)\right\|^2\right).$$

Substitute

$$c^*=l^2\Bigg[4\mathrm{b}^{2(1-\mathrm{p})}M_0^2Q_{\mathfrak{k}}^2+4\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)Q_{\mathfrak{k}}^2$$

$$+4\left(\frac{M^2 b^{2(1-p)+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\right)Q^2_{\mathscr{L}}\left(1+n_1^2 b^2\right)+4\,\mathrm{Tr}(\mathfrak{Q})\left(\frac{M^2 b^{2(1-p)+2\mathfrak{e}-1}}{(2\mathfrak{e}-1)\Gamma^2(\mathfrak{e})}\right)Q^2_{\mathscr{N}}\left(1+n_2^2 b^2\right)\Bigg],$$

which becomes

$$\mathscr{E}\left\|\mathfrak{r}^{1-p}\left(\tilde{\zeta}\mathfrak{z}(\mathfrak{r})-\tilde{\zeta}\hat{\mathfrak{z}}(\mathfrak{r})\right)\right\|^2=c^*\left(\sup_{\theta\in J}\mathscr{E}\left\|\mathfrak{z}(\theta)-\hat{\mathfrak{z}}(\theta)\right\|^2\right).$$

Since $\mathfrak{r}\in(0, b]$, the function $\tilde{\zeta}$ attains its maximum at $\mathfrak{r}=b$ when taking supremum over $\mathfrak{r}$ (i.e) $\left\|\tilde{\zeta}\mathfrak{z}-\tilde{\zeta}\hat{\mathfrak{z}}\right\|_b^2=\left\|\mathfrak{z}-\hat{\mathfrak{z}}\right\|_b^2$. By concluding now that $\tilde{\zeta}$ is a contraction map, there exist a fixed point of the mild solution of the system. □

5. Trajectory controllability

Trajectory curves are the curves which carries solution curve from initial state to final state. They accompany and steer the curves along the prescribed path (which might be more efficient in many categories like, cost reduction, accurate path or accurate direction). In this section, we concerntrate on proving the Conformable Hilfer fractional neutral stochastic integro-differential system is $\mathfrak{T}$-controllable using Gronwall's inequality.

Theorem 5.1 The system (4) is $\mathfrak{T}$-controllable if the suppositions (C1)-(C5) are fulfilled.

Proof. Let $w(\mathfrak{r})$ be the trajectory and $y(\mathfrak{r})$ be the control function on (0, b] as defined by

$$y(\mathfrak{r})=\mathrm{D}_{0^+}^{\rho,\mathfrak{e}}\left[w(\mathfrak{r})-\mathfrak{k}\left(\mathfrak{r},\ w_{\mathfrak{r}}\right)\right]-\mathscr{G}[w(\mathfrak{r})]-\mathscr{L}\left(\mathfrak{r},\ w_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\mathscr{M}\left(\mathfrak{r},\ \theta,\ w_\theta\right)d\theta\right)$$

$$-\mathscr{N}\left(\mathfrak{r},\ w_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\tilde{\mathscr{M}}\left(\mathfrak{r},\ \theta,\ w_\theta\right)d\theta\right)\frac{d\gamma(\mathfrak{r})}{d\mathfrak{r}},$$

where $\mathfrak{e}\in[0, 1]$ and $\rho\in(0, 1)$. The system (4) becomes

$$\mathrm{D}_{0^+}^{\rho,\mathfrak{e}}\left[u(\mathfrak{r})-\mathfrak{k}\left(\mathfrak{r},\ u_{\mathfrak{r}}\right)\right]$$

$$=\mathscr{G}[u(\mathfrak{r})]+\Bigg[\mathscr{D}^{\rho,\mathfrak{e}}\left[w(\mathfrak{r})-\mathfrak{k}\left(\mathfrak{r},\ w_{\mathfrak{r}}\right)\right]-\mathscr{G}[w(\mathfrak{r})]-\mathscr{L}\left(\mathfrak{r},\ w_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\mathscr{M}\left(\mathfrak{r},\ \theta,\ w_\theta\right)d\theta\right)$$

$$-\mathscr{N}\left(\mathfrak{r},\ w_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\tilde{\mathscr{M}}\left(\mathfrak{r},\ \theta,\ w_\theta\right)d\theta\right)\frac{d\gamma(\mathfrak{r})}{d\mathfrak{r}}\Bigg]+\mathscr{L}\left(\mathfrak{r},\ u_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\mathscr{M}\left(\mathfrak{r},\ \theta,\ u_\theta\right)d\theta\right)$$

$$+\mathscr{N}\left(\mathfrak{r},\ u_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\tilde{\mathscr{M}}\left(\mathfrak{r},\ \theta,\ u_\theta\right)d\theta\right)\frac{d\gamma(\mathfrak{r})}{d\mathfrak{r}}.$$

Let $\tilde{\Upsilon}(\mathfrak{r})=u(\mathfrak{r})-w(\mathfrak{r})$, we obtain

$$\mathrm{D}_{0^+}^{\rho,\mathfrak{e}}\left[\tilde{\Upsilon}(\mathfrak{r})-\left[\mathfrak{k}\left(\mathfrak{r},\ u_{\mathfrak{r}}\right)-\mathfrak{k}\left(\mathfrak{r},\ w_{\mathfrak{r}}\right)\right]\right]$$

$$=\mathscr{G}[\tilde{\Upsilon}(\mathfrak{r})]+\mathscr{L}\left(\mathfrak{r},\ u_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\mathscr{M}\left(\mathfrak{r},\ \theta,\ u_\theta\right)d\theta\right)-\mathscr{L}\left(\mathfrak{r},\ w_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\mathscr{M}\left(\mathfrak{r},\ \theta,\ w_\theta\right)d\theta\right)$$

$$+\left[\mathscr{N}\left(\mathfrak{r},\ u_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\tilde{\mathscr{M}}\left(\mathfrak{r},\ \theta,\ u_\theta\right)d\theta\right)-\mathscr{N}\left(\mathfrak{r},\ w_{\mathfrak{r}},\ \int_0^{\mathfrak{r}}\tilde{\mathscr{M}}\left(\mathfrak{r},\ \theta,\ w_\theta\right)d\theta\right)\right]\frac{d\gamma(\mathfrak{r})}{d\mathfrak{r}};\ \mathfrak{r}\in(0,\ \mathrm{b}],$$

here $\tilde{\Upsilon}(\mathfrak{r}) = 0$ for $\mathfrak{r} \in (-\infty, 0]$. For $\mathfrak{r} \in (0, \mathrm{b}]$, the expectation as follows,

$$\mathscr{E}\left\|\Upsilon(\mathfrak{r})\right\|^2 \le \mathfrak{r}^{2(1-\mathrm{p})}\Bigg[4\mathscr{E}\left\|\mathfrak{k}\left(\mathfrak{r},\ \mathfrak{z}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\right)-\mathfrak{k}\left(\mathfrak{r},\ \tilde{\mathfrak{z}}_{\mathfrak{r}}+\bar{\kappa}_{\mathfrak{r}}\right)\right\|^2+4\mathscr{E}\left\|\int_0^{\mathfrak{r}}\mathscr{G}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\left[\mathfrak{k}\left(\theta,\ \mathfrak{z}_\theta+\bar{\kappa}_\theta\right)\right.\right.$$

$$\left.\left.-\mathfrak{k}\left(\theta,\ \tilde{\mathfrak{z}}_\theta+\bar{\kappa}_\theta\right)\right]d\theta\right\|^2+4\mathscr{E}\left\|\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\left[\mathscr{L}\left(\theta,\ \mathfrak{z}_\theta+\bar{\kappa}_\theta,\ \int_0^\theta\mathscr{M}\left(\theta,\ \rho,\ \mathfrak{z}_\rho+\bar{\kappa}_\rho\right)d\rho\right)\right.\right.$$

$$\left.\left.-\mathscr{L}\left(\theta,\ \tilde{\mathfrak{z}}_\theta+\bar{\kappa}_\theta,\ \int_0^\theta\mathscr{M}\left(\theta,\ \rho,\ \tilde{\mathfrak{z}}_\rho+\bar{\kappa}_\rho\right)d\rho\right)\right]d\theta\right\|^2+4\mathscr{E}\left\|\int_0^{\mathfrak{r}}\mathfrak{g}^{\mathfrak{e}-1}T\left(\frac{\mathfrak{r}^{\mathfrak{e}}-\theta^{\mathfrak{e}}}{\mathfrak{e}}\right)\left[\mathscr{N}\left(\theta,\ \mathfrak{z}_\theta+\bar{\kappa}_\theta,\right.\right.\right.$$

$$\left.\left.\left.\int_0^s\tilde{\mathscr{M}}\left(s,\ \rho,\ \mathfrak{z}_\rho+\bar{\kappa}_\rho\right)d\rho\right)-\mathscr{N}\left(\theta,\ \tilde{\mathfrak{z}}_\theta+\bar{\kappa}_\theta\int_0^\theta\tilde{\mathscr{M}}\left(\theta,\ \rho,\ \tilde{\mathfrak{z}}_\rho+\bar{\kappa}_\rho\right)d\rho\right)\right]d\gamma(\theta)\right\|^2\Bigg]$$

$$\le 4\frac{M^2}{\Gamma^2(\mathfrak{e})}Q_{\mathfrak{k}}^2\mathscr{E}\left\|\Upsilon(\mathfrak{r})\right\|^2+4\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)Q_{\mathfrak{k}}^2\int_0^{\mathfrak{r}}\mathfrak{g}^{2(\mathfrak{e}\beta-1)}\left(\mathscr{E}\left\|\mathfrak{z}_\theta-\tilde{\mathfrak{z}}_\theta\right\|^2\right)d\theta$$

$$+4\frac{M^2}{\Gamma^2(\mathfrak{e})}Q_{\mathscr{L}}^2\int_0^{\mathfrak{r}}\mathfrak{g}^{2(\mathfrak{e}-1)}\left(\left\|\left(\mathfrak{z}_\theta+\bar{\kappa}_\theta\right)-\left(\tilde{\mathfrak{z}}_\theta+\bar{\kappa}_\theta\right)\right\|_{\mathscr{C}_{\mathrm{H}}}^2+\mathscr{E}\left\|\int_0^\theta\mathscr{M}\left(s,\ \rho,\ \mathfrak{z}_\rho+\bar{\kappa}_\rho\right)d\rho\right.\right.$$

$$\left.\left.-\int_0^\theta\mathscr{M}\left(\theta,\ \rho,\ \tilde{\mathfrak{z}}_\rho+\bar{\kappa}_\rho\right)d\rho\right\|^2\right)d\theta+4\frac{M^2}{\Gamma^2(\mathfrak{e})}Q_{\mathscr{N}}^2\ \mathrm{Tr}(\mathfrak{Q})\int_0^{\mathfrak{r}}\mathfrak{g}^{2(\mathfrak{e}-1)}\left(\left\|\left(\mathfrak{z}_\theta+\bar{\kappa}_S\right)-\left(\tilde{\mathfrak{z}}_\theta+\bar{\kappa}_\theta\right)\right\|_{\mathscr{C}_{\mathrm{H}}}^2\right.$$

$$\left.+\mathscr{E}\left\|\int_0^\theta\tilde{\mathscr{M}}\left(\theta,\ \rho,\ \mathfrak{z}_\rho+\bar{\kappa}_\rho\right)d\rho-\int_0^\theta\tilde{\mathscr{M}}\left(\theta,\ \rho,\ \tilde{\mathfrak{z}}_\rho+\bar{\kappa}_\rho\right)d\rho\right\|^2\right)d\theta$$

$$\le 4\frac{M^2}{\Gamma^2(\mathfrak{e})}Q_{\mathfrak{k}}^2\mathscr{E}\left\|\mathrm{r}(\mathfrak{r})\right\|^2+4\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mathfrak{e}\beta)}\frac{\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{2\mathfrak{e}\beta-1}\right)Q_{\mathfrak{k}}^2\int_0^{\mathfrak{r}}\mathfrak{g}^{2(\mathfrak{e}\beta-1)}\mathscr{E}\left\|\Upsilon(\theta)\right\|^2d\theta$$

$$+4\frac{M^2}{\Gamma^2(\mathfrak{e})}\int_0^{\mathfrak{r}}\mathfrak{g}^{2(\mathfrak{e}-1)}\left(Q_{\mathscr{L}}^2\left(1+n_1^2\mathrm{b}^2\right)+Q_{\mathscr{N}}^2\ \mathrm{Tr}(\mathfrak{Q})\left(1+n_2^2\mathrm{b}^2\right)\right)\mathscr{E}\left\|\mathrm{r}(\theta)\right\|^2d\theta$$

Since β is arbitrary and $\beta \in (0, 1)$, by taking maximum value, the above inequality is identical to Gronwall's inequality.

$$\mathscr{E}\left\|\Upsilon(\mathfrak{r})\right\|^2 \le \left[\frac{4Q_{\mathscr{L}}^2\left(1+n_1^2\mathrm{b}^2\right)+4Q_{\mathscr{N}}^2\ \mathrm{Tr}(\mathfrak{Q})\left(1+n_2^2\mathrm{b}^2\right)+4Q_{\mathfrak{k}}^2\left(\frac{\mathfrak{e}^2C_{1-\beta}^2\Gamma^2(1+\beta)\mathrm{b}^{2\mathfrak{e}\beta+2(1-\mathrm{p})}}{(2\mathfrak{e}\beta-1)\Gamma^2(1+\mathfrak{e}\beta)}\right)}{\left(1-4\frac{M^2}{\Gamma^2(\mathfrak{e})}Q_{\mathfrak{k}}^2\right)}\right]$$

$$\times\int_0^{\mathfrak{r}}\mathfrak{g}^{2(\mathfrak{e}-1)}\mathscr{E}\left\|\Upsilon(\theta)\right\|^2 d\theta,$$

which yields $\mathscr{E}\left\|\Upsilon(\mathfrak{r})\right\| = 0$ by generalised Gronwall's inequality, that is $u_{\mathfrak{r}} = w_{\mathfrak{r}}$. Thus system (4) is Trajectory Controllable on [0, b], (ie). the trajectory curves controls the solution curve along the prescribed path. □

6. Illustration

Hilfer fractional neutral stochastic integro-differential system is given as follows,

$$\begin{cases} D_{0^+}^{\frac{1}{2},\frac{1}{2}}\left[w(\mathfrak{r},\ \delta)-\int_{-\infty}^{0}\mathrm{k}(\mathfrak{r},\ \delta)w(\mathfrak{r},\ \delta)d\delta\right]=\dfrac{\partial^2}{\partial\delta^2}w(\mathfrak{r},\ \delta)+y(\mathfrak{r},\ \delta) \\ \quad +\mathscr{L}\left(\mathfrak{r},\ \int_{-\infty}^{\mathfrak{r}}\mathscr{L}_1(\theta-\mathfrak{r})w(s,\ \delta)d\theta,\ \int_0^{\mathfrak{r}}\int_{-\infty}^{0}\mathscr{L}_2(\theta,\ \delta,\ \varepsilon-\theta)w(\varepsilon,\ \delta)d\varepsilon d\theta\right) \\ \quad +\mathscr{N}\left(\mathfrak{r},\ \int_{-\infty}^{\mathfrak{r}}\mathscr{L}_1(\theta-\mathfrak{r})w(s,\ \delta)d\theta,\ \int_0^{\mathfrak{r}}\int_{-\infty}^{0}\mathscr{L}_3(\theta,\ \delta,\ \varepsilon-\theta)w(\varepsilon,\ \delta)d\varepsilon d\theta\right)\dfrac{d\gamma(\mathfrak{r})}{d\mathfrak{r}}, & \mathfrak{r}\in[0,\ b] \\ w(\mathfrak{r},\ 0)=w(\mathfrak{r},\ \pi)=0, & \mathfrak{r}\geq 0, \\ \mathcal{I}^{\frac{1}{4}}w(0,\ \delta)=w_0=\kappa(\mathfrak{r},\ \delta), & \delta\in[0,\ \pi],\ -\infty<\mathfrak{r}<0, \end{cases} \tag{7}$$

here $b \leq \pi$, κ (t, δ) ∈ $H = L^2([0, \pi])$ which is the Hilbert space.

Let Y = U = H and γ(t) is 1-D Brownian motion on the filtered probability space (Λ, $\mathscr{K}$, P). Let the function κ(t, δ) ∈ (-∞, 0) × [0, π] be continuous and $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ be continuous on [0, b] and measurable in $H = L_2([0, \pi])$ while the memory functions or history functions $\mathscr{L}_2$, $\mathscr{L}_3$: [0, b] × [0, π] × (-∞, 0) → [0, π] satisfies Lipschitz condition.

Assumptions that helps to write the system (7) as (1).

Let $\mathscr{G}$: D($\mathscr{G}$) ⊂ Y → Y be defined as $\mathscr{G}$w = -w'' with D($\mathscr{G}$) = {w ∈ Y; w, w' are absolutely continuous, w'' ∈ Y, $w(0) = w(\pi) = 0$} and it produces self-adjoint, compact, analytic semigroup $S(\mathfrak{r})$.

Therefore properties (C1)-(C4) satisfied.

$\mathscr{G}w$ = -w'' which is a Strum-Liovullie boundary value problem, therefore $\mathscr{G}$ acts as a generator of eigenfunctions. So the Eigenvalues of $\mathscr{G}$ are n^2 where n ∈ N with the eigen functions $w_n(\delta) = B_n \sin(n\delta)$, when normalising it, $w_n(\delta) = \sqrt{\dfrac{2}{\pi}}\sin(n\delta)$ which brings some properties with it as follows:

- For any $v\in D(\mathscr{G})$, $\mathscr{G}_v = \sum_{n=1}^{\infty} n^2\left\langle v,\ w_n\right\rangle w_n$.
- The operator $\mathscr{G}^{\frac{1}{2}}v$ is defined by $\mathscr{G}^{\frac{1}{2}}v = \sum_{n=1}^{\infty} n\left\langle v,\ w_n\right\rangle w_n$ on

$$D\left(\mathscr{G}^{\frac{1}{2}}\right) = \left\{v(\cdot) \text{ such that } \sum_{n=1}^{\infty} n\left\langle v,\ w_n\right\rangle w_n \text{ on H}\right\}.$$

$\mathrm{H}(s) = e^{4s}$, $s < 0$, $\Rightarrow\ l = \int_{-\infty}^{0}\mathrm{H}(s)ds = \dfrac{1}{4}$. The abstract phase space $\mathscr{O}_{\mathrm{H}}$ provided with

$$\|\kappa\|_{\mathscr{O}_{\mathrm{H}}} = \int_{-\infty}^{0}\mathrm{H}(s)\sup_{s\leq\mathfrak{r}\leq 0}\left(\mathrm{H}\left\|\kappa\left((t)\right\|^2\right)^{1/2} ds. \tag{8}$$

Thus, $\left(\mathscr{O}_{\mathrm{H}},\ \|\cdot\|_{\mathscr{O}_{\mathrm{H}}}\right)$ is a complete normed linear space.

For neutral term k(𝔯, δ) let us assume the followings

- The function k is Lebesgue square integrable and bounded.
- The function $\frac{\partial}{\partial\delta}\mathrm{k}(t,\ \delta)$ is measurable such that k(𝔯, 0) = k(𝔯, π) = 0 and bounded by M_1 where M_1 is lebesgue square integrable.

Define $\mathfrak{k}$: $[0, b] \times L^2([0, \pi]) \to L^2([0, \pi])$ by $\mathfrak{k}(t, w) = \mathscr{V}_1(w)$ where $\mathscr{V}_1(w)(\delta) = \int_0^\pi \mathrm{k}(\mathfrak{r}, \delta) w(\mathfrak{r}) d\mathfrak{r}$.

Obviously $\mathscr{V}_1$ is a bounded linear operator and $\| \mathscr{G}^{\frac{1}{2}} \mathscr{V}_1 \|^2 \le M_1^2$ since $\mathscr{V}_1 \in D\left(\mathscr{G}^{\frac{1}{2}}\right)$.

Therefore by Riez Representation theorem,

$$\left\langle \mathscr{V}_1(w),\ w_n \right\rangle = \int_0^\pi w_n(\delta) \left[\int_0^\pi \mathrm{k}(\mathfrak{r}, \delta) w(\mathfrak{r}) d\mathfrak{r} \right] d\delta$$

$$= \sqrt{\frac{2}{\pi}} \sin(n\delta) \left[\int_0^\pi \mathrm{k}(\mathfrak{r}, \delta) w(\mathfrak{r}) d\mathfrak{r} \right] d\delta$$

$$= \left\langle \mathscr{V}_2(w),\ \frac{1}{n} \sqrt{\frac{2}{\pi}} \cos(n\delta) \right\rangle$$

$$= \frac{1}{n} \sqrt{\frac{2}{\pi}} \left\langle \mathscr{V}_2(w,\ \cos(n\delta) \right\rangle$$

Obtained by simply taking partial derivative by without loss of generality since the norm is preserved according to the Riez Representation theorem, here $\mathscr{V}_2(w) = \int_0^\pi \frac{\partial}{\partial \delta} \mathrm{k}(\mathfrak{r}, \delta) w(\mathfrak{r}) d\mathfrak{r}$ and $\| \mathscr{V}_2 \|^2 \le M_1^2$ therefore (C5) is fulfilled. Furthermore, the control y: $\mathscr{O}_{\mathrm{H}}([0,\ \pi]) \to \mathscr{R}$ where $y \in L^2\left(\mathscr{O}_{\mathrm{H}}([0,\ \pi])\right)$ and $\mathfrak{r} \to y(\mathrm{t})$ is measurable, continuous and do not vanish on $(0, b]$.

Consider

$$\mathscr{G} = \left\{ y \in \mathrm{U} : \|y\| \le \mathrm{b} \ \text{ where } \ \mathrm{b} \in L^2\left(\delta,\ \mathscr{R}^+\right) \right\}.$$

For any trajectory w which is continuously differentiable and meets with initial data of w, there exists a control y such that $\mathscr{G}_w = -w''$ and $w(0) = w(\pi) = 0$. Therefore the system (7) is $\mathfrak{T}$-controllable on (0, b] by theorem (5.1).

7. Conclusion

This manuscript has addressed the mild solution and trajectory controllability of conformable Hilfer fractional neutral stochastic integro-differential equation with infinite delay by assuming Lipschitz and growth conditions and using Banach contraction mapping theorem and Gronwall inequality. An example was given to support the theoretical approach.

Discussing the same system with impulse and Rosenblatt process will be our future work.

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Conflict of interest

The authors declare no competing interests.

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