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Research Article

Commutative Rings and Corresponding V-Graphs

Nasr Zeyada[1,2*], S. A. Bashamakh[1], Ohoud Almalawi[1]

[1] Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah, 23218, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Cairo University, Giza, 12613, Egypt
E-mail: nzeyada@uj.edu.sa

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Abstract: Algebraic graph theory, which applies algebraic techniques to graph problems, is a pivotal area of study. Many ring algebraic properties have representations in graph theory. In this paper, we introduce an innovative type of graph related to rings that we call the V-graph. Let T be a ring with identity. The V-graph of T, denoted by $V(T)$, consists of a set of vertices equal to all non-zero elements of T. Two different vertices u and v are adjacent if and only if uv is a regular element in T. We present several examples to demonstrate how this form of graph differs from the known ring-based graphs, such as zero-divisor graphs, unit graphs, and quasi-regular graphs. We calculate the diameter and independent number, as well as the domination number for the V-graph of the ring of integers $\mathbb{Z}_n$.

***Keywords*:** V-graph, quasi-regular graph, unit graph, zero-divisor graph

MSC: 05C25, 13A99, 13M99

1. Introduction

Algebraic graph theory has greatly benefited from the investigation of the connections between graph theory and rings. As early as 1988 [1]. Beck proposed examining the theoretical facets of a graph over a ring graph. He presented the idea of a commutative ring T with a zero-divisor graph $\Gamma(T)$. $Z(T)^* = Z(T) - \{0\}$, the set of non-zero zero divisors of T, was his definition of an undirected graph where two vertices w and z are adjacent if $wz = 0$ [1]. Beck's zero-divisor graph was first significantly reorganized by Anderson and Livingston [2]. The quasi-regular graph was initially analyzed by Zeyada et al. [3]. To construct the quasi-regular graph of T, denoted by $Q(T)$, we make all elements of the $T - \{0\}$ vertices and define two vertices a and b adjacent if and only if $1 - ab$ is a unit of T. Diameter, as an invariant of graphs, is essential. The size of the generated graph is the subject of numerous academic studies; for examples, see [3–5]. Zeyada et al. [3] showed that the girth of $diam(Q(T)) \leq 3$ whenever T is an arbitrary ring, and the girth of the quasi-regular graphs is 3 whenever $Q(T)$ is not a star graph. Von Neumann defined a regular von Neumann element for the first time in 1936. The unit element serves as inspiration for the definition of a regular element; that is, if a is a unit element, then there exists a^1 such that $aa^1 = 1$. In other words, $aa^1a = a$. The definition of a regular element is derived from that unit element. If an element e satisfies $e^2 = e$, it is termed an idempotent element. In a ring T, an element a is said to be a regular element if there exist $x \in T$ and $axa = a$. Clearly, the concept of unit elements is generalized to the definition of regular elements. If each element in a ring T is a regular element, then the ring is said to be Von Neumann regular. If every right ideal is an

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intersection of maximal right ideals of T, then the ring T is called a right V-ring. Villamayor is credited with the initial study of V-rings and with demonstrating that these rings are distinguished by the characteristic that each right module has a zero Jacobson radical [6]. Since its creation, academics have focused a lot of attention on V-rings because of these and other intriguing features. As an illustration, Kaplansky demonstrated that a commutative ring is only von Neumann regular if and when it is a V-ring [6].

Using the notation $V(T)$, where T is a commutative ring with nonzero identity, we define a new concept that we call a V-graph. We present a new graph associated with rings that is different from all the graphs associated with rings before. We prove when this graph is one of the known graphs, such as the completed graph, the path graph, the cycle graph, and others. We are also looking at finding the diameter of this graph and finding the girth. In addition, we are looking for the independent number in some cases and the domination number in the case of $\mathbb{Z}_n$. Further concepts and results relevant to our study have been explored in [6–13].

2. V-ring graph

Definition 1 In the case that every pair of vertices x and y is adjacent if and only if xy is a regular element in T, we say that $V(T)$ is a V-ring graph (V-graph for abbreviation) with vertex set $V(V(T)) = T\backslash\{0\}$. The edge set of $V(T)$ is $E(V(T)) = \{xy : xy \text{ is a regular in } T \text{ for } x, y \in T\}$.

The following illustrations contrast the concept of a V-graph with that of previously established graphs over a ring.

Example 1 The ring $\mathbb{Z}_6 = \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$ is a Von Neumann regular ring, it follows that, every element in $\mathbb{Z}_6$ is regular. Hence, the V-graph $V(\mathbb{Z}_6)$ with a vertex set $\{1,\ 2,\ 3,\ 4,\ 5\}$ is a complete shown in Figure 1.

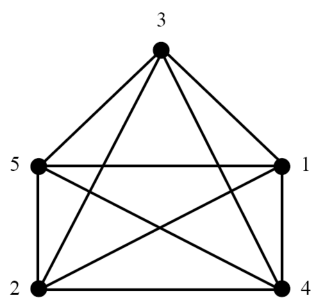

Figure 1. V-graph of $\mathbb{Z}_6$

Example 2 The zero divisors of the ring $\mathbb{Z}_6 = \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$ are $2,\ 3$ and 4 where $2*3 = 0 = 4*3$. Hence, the zero-divisor graph $\Gamma(\mathbb{Z}_6)$ is a path graph shown in Figure 2.

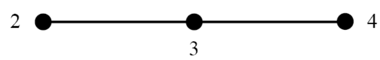

Figure 2. Zero divisor graph of $\mathbb{Z}_6$

Example 3 The vertex set of the quasi-regular graph of the ring $\mathbb{Z}_6$ is $\mathbb{Z}_6^* = \{1,\ 2,\ 3,\ 4,\ 5\}$. Since $\{1,\ 5\}$ is the set of unit elemets in $\mathbb{Z}_6$, the set of edges of $Q(\mathbb{Z}_6)$ is $E(Q(\mathbb{Z}_6)) = \{ab : 1 - wz \textit{ is a unit in } \mathbb{Z}_6\} = \{12,\ 23,\ 24,\ 34,\ 45\}$. Hence, the quasi-regular graph of the ring $Q(\mathbb{Z}_6)$ shown instantly in Figure 3.

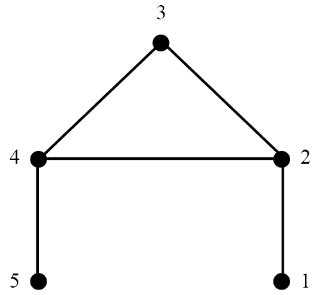

Figure 3. Quasi-regular graph of $\mathbb{Z}_6$

Example 4 The vertex set of the V-graph of the ring $\mathbb{Z}_8$ is $\mathbb{Z}_8^* = \{1, 2, 3, 4, 5, 6, 7\}$. Since $\{0, 1, 3, 5, 7\}$ is the set of unit elements in $\mathbb{Z}_8$, the set of edges of $V(\mathbb{Z}_8)$ is $E(V(\mathbb{Z}_8)) = \{ab : ab \textit{ is regular in } \mathbb{Z}_8\} = \{13, 15, 17, 35, 37, 57, 24, 46\}$. Hence, the V-graph of the ring $V(\mathbb{Z}_8)$ is immediately shown in Figure 4.

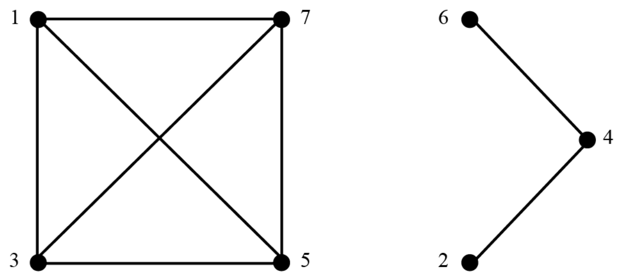

Figure 4. V-graph of $\mathbb{Z}_8$

Example 5 The vertex set of the V-graph of the ring $\mathbb{Z}_{12}$ is $\mathbb{Z}_{12}^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Since $\{0, 1, 3, 4, 5, 7, 8, 9, 11,\}$ is the set of regular elemets in $\mathbb{Z}_{12}$, the set of edges of $V(\mathbb{Z}_{12})$ is $E(V(\mathbb{Z}_{12})) = \{ab : ab \textit{ is regular in } \mathbb{Z}_{12}\} = \{13, 14, 15, 17, 18, 19, 1(11), 34, 35, 37, 38, 39, 3(11), 57, 58, 59, 5(11), 78, 79, 7(11), 89, 8(11), 9(11), 24, 26, 28, 2(10), 64, 68, 6(10), (10)4, (10)8\}$. Hence, the V-graph of the ring $V(\mathbb{Z}_{12})$ is immediately shown in Figure 5.

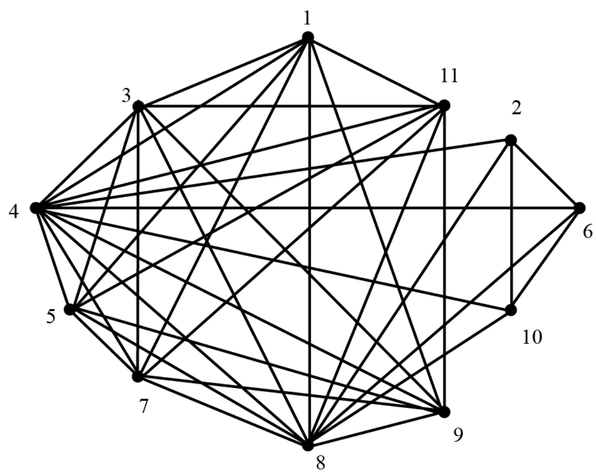

Figure 5. V-graph of $\mathbb{Z}_{12}$

Example 6 The vertex set of the V-graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4$ is $(\mathbb{Z}_2 \times \mathbb{Z}_4)^* = \{(0,\ 1),\ (0,\ 2),\ (0,\ 3),\ (1,\ 0), (1,\ 1),\ (1,\ 2),\ (1,\ 3)\}$. Since $\{(0,\ 0),\ (0,\ 1),\ (0,\ 3),\ (1,\ 0),\ (1,\ 1),\ (1,\ 3)\}$ is the set of regular elements in $\mathbb{Z}_2 \times \mathbb{Z}_4$, the set of edges of $\mathbb{Z}_2 \times \mathbb{Z}_4$ is $E(V(\mathbb{Z}_2 \times \mathbb{Z}_4)) = \{ab : ab \textit{ is regular in } \mathbb{Z}_2 \times \mathbb{Z}_4\} = \{(0,\ 1)(0,\ 3),\ (0,\ 1)(1,\ 0),\ (0,\ 1)(1.1),\ (0,\ 1)(1,\ 3),\ (0,\ 3)(1,\ 0),\ (0,\ 3)(1,\ 1),\ (0,\ 3)(1,\ 3),\ (1,\ 0)(1,\ 1),\ (1,\ 0)(1,\ 3),\ (1,\ 1)(1,\ 3),\ (1,\ 0)(0,\ 2),\ (1,\ 0)(1,\ 2),\ (0,\ 2)(1,\ 2)\}$. Hence, the V-graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4$ is shown in Figure 6.

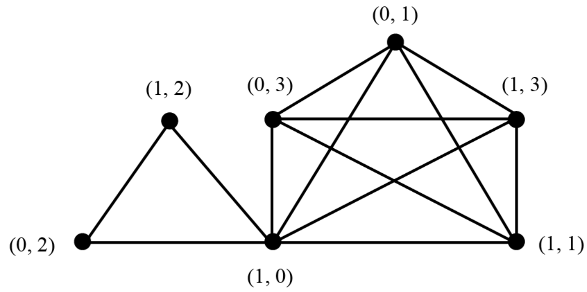

Figure 6. V-graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4$

Example 7 Let $\mathbb{F}_4$ be the extension field over $\mathbb{Z}_2[x]$ irreducible polynomial $P(x) = x^2 + x + 1$ over $\mathbb{Z}_2$. $\mathbb{F}_4 = \{0,\ 1,\ u,\ 1+u\}$. The vertex set of the V-graph of the ring $\mathbb{F}_4$ is $\mathbb{F^*}_4 = \{1,\ u,\ 1+u\}$. Since $\{0,\ 1,\ u,\ 1+u\}$ is the set of regular elemets in $\mathbb{F}_4$, the set of edges of $V(\mathbb{F}_4)$ is $E(V(\mathbb{F}_4)) = \{ab : ab \textit{ is regular in } \mathbb{F}_4\} = \{1u,\ 1(1+u),\ u(1+u)\}$. Hence, the V-graph of the ring $V(\mathbb{F}_4)$ is the cyclic shown in Figure 7.

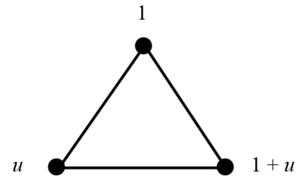

Figure 7. V-graph of extension field $\mathbb{F}_4$

Several examples of V-graphs over an infinite ring are shown.

Example 8 Referring to the set of integers known as $\mathbb{Z}$. Because of this, we conclude that there is only one edge $(1)(-1)$ in the edge set of the V-graph of $\mathbb{Z}$, and all elements in $\mathbb{Z}\backslash\{1,\ -1\}$ of the graph are not connected to any other elements.

Example 9 The V-graph of rational numbers $\mathbb{Q}$. For all $a,\ b \in \mathbb{Q}^*$, a is adjacent to b since $\mathbb{Q}$ is a field and so a Von Neumann regular ring.

Proposition 1 Let S and R be commutative rings. If $S \cong R$ as rings, then $V(S) \cong V(R)$ as graphs.

Proof. Assume that $f : S \to R$ is an isomorphism of rings. This isomorphism induces an isomorphism $h : V(S) \to V(R)$ where $h(a) = f(a)$ for all $a \in S^*$. In fact, if $a,\ b \in S$ are adjacent vertices in $V(S)$, then ab is a regular element in S and $(ab)^2x = ab$ for some $x \in S$. Therefore, $h((ab)^2x) = h(ab)$ and $h(ab) = h(a)h(b)$ is a regular element in R and $V(S) \cong V(R)$ as graphs. □

In the following example, we construct the quasi-regular graphs of two isomorphic rings $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.

Example 10 By proposition 1, since $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$ as rings $V(\mathbb{Z}_2 \times \mathbb{Z}_3) \cong V(\mathbb{Z}_6)$ as graphs. See Figure 8.

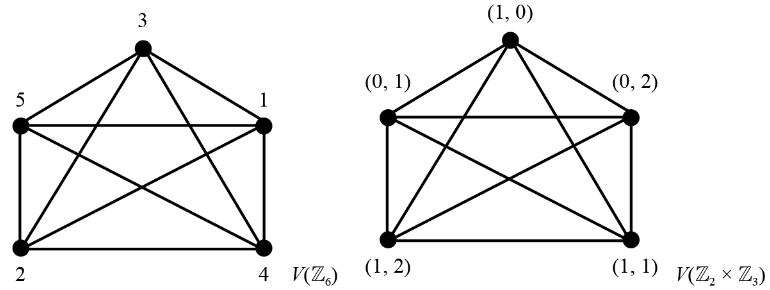

Figure 8. V-graphs of two isomorphic rings

Example 11 In Figure 9, the quasi-regular graph over $\mathbb{Z}_3$ is complete.

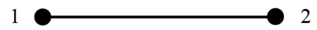

Figure 9. Complete quasi-regular graph $Q(\mathbb{Z}_3)$

The following proposition is based on ideas from Zeyada, Muthana, and Elrashidi [3].

Proposition 2 For a commutative ring T with identity, the quasi-regular graph $Q(T)$ is a complete graph if and only if either $T \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

We establish the following for V-graphs over a ring T.

Proposition 3 Any commutative ring T with identity is a Von Neumann regular ring if and only if its corresponding V-graph $V(T)$ is a complete graph.

Proof. Let T be a commutative ring with identity. If $V(T)$ is a complete graph, then every pair u and v in $T \backslash \{0\}$ are adjacent, and every element in T is regular. It follows that T is a Von Neumann regular ring. Now, if T is a Von Neumann regular ring, then every element in T is regular and every pair u and v in $T \backslash \{0\}$ are adjacent, and the graph $V(T)$ is complete. □

Example 12 In Figure 10, the quasi-regular graph over $\mathbb{Z}_4$ is the path graph.

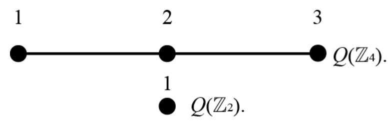

Figure 10. Quasi-regular graphs of $\mathbb{Z}_4$ and $\mathbb{Z}_2$

Proposition 4 For a commutative ring T with identity, the V-graph $V(T)$ is a star graph if and only if $T \cong \mathbb{Z}_2$ or $T \cong \mathbb{Z}_3$.

Proof. Let T be a commutative ring with identity and $V(T)$ be a star graph. We consider the following cases:

Case 1: T has more than 2 elements and 1 is adjacent to every element in $T \backslash \{0, 1\}$. It follows that a is regular for every $a \in T \backslash \{0, 1\}$ and T is the Von Neumann regular ring. From proposition 3 $V(T)$ is a complete graph, which means that T has only 3 elements and $T \cong \mathbb{Z}_3$.

Case 2: T has more than 2 elements and c is adjacent to every element in $T \backslash \{0, c\}$. One can see c is adjacent to 1 and c is the only regular element in $T \backslash \{0, 1\}$. We get the following sub-cases:

Sub case i: c is a unit. If there is $a \in T \backslash \{0, 1, c\}$, then ca is regular and $ca = 1$ or $ca = 0$ or $ca = c$. It follows that a is a unit or $a = 0$ or $a = 1$, which is a contradiction.

Sub case ii: c is not a unit. Note that -1 is a unit and regular, so $1=-1$ and $charT=2$. Now, it's clear that every idempotent element is regular. Since c is regular, there exists $x \in T$ such that $c^2x=c$ and cx is an idempotent. Moreover, $1+cx$ is an idempotent ($(1+cx)^2 = 1+2cx+c^2x^2 = 1+(c^2x)x = 1+cx$) and $1+cx$ is regular. Therefore, $1+cx=0$ or $1+cx=1$ or $1+cx=c$ and it follows that $cx=1$ or $cx=0$ or ($c+c^2x=c^2$ and $2c=c^2=0$). But this means that c is invertible or $c=0$, which is a contradiction. Therefore, $S \cong \mathbb{Z}_2$ or $S \cong \mathbb{Z}_3$. The converse is clear by the definition of the V-graph. □

Proposition 5 Let T be a ring and $|T| \geq 4$. $V(T)$ is a cyclic graph if and only if $T \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $T \cong \mathbb{F}_4$.

Proof. Let T be a commutative ring with $|T| \geq 4$ and $V(T)$ be a cyclic graph. Then 1 is adjacent to different vertices x and y and x and y are regular elements. It follows that x and y are adjacent and T has only 4 elements. Moreover, we have four possibilities $xy=1$ or $xy=0$ or $xy=x$ or $xy=y$. The first possibility $xy=1$ implies that x and y are invertible elements and $T=\{0,\ 1,\ x,\ y\}$ is a field of 4 elements. The case $xy=0$ implies that $T \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. For the last case $xy=x$, since 1 is the only invertible element in T so $1=-1$ and $char(T)=2$. Now note that $x+y=1$ and $(1+x)x=x$, so $x^2=0$, which is a contradiction to the regularity of x. Hence $T \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $T \cong \mathbb{F}_4$. The converse is clear. □

Proposition 6 Let T be a ring with more than two elements. Then $V(T)$ is a path graph if and only if $T \cong \mathbb{Z}_3$.

Proof. Assume that $V(T)$ is a path $x_1x_2x_3...$ so we have two cases:

First case: 1 has two different adjacent elements x_1 and x_2. It follows that x_1 and x_2 are regular elements and are adjacent, which is a contradiction.

Second case: 1 is adjacent to only one element and the graph must be started by 1. Then we have the path $1x_1x_2...$ and $0,\ 1,\ x_1$ are the only regular elements in T. It is clear that -1 is regular element, so $-1=1$ or $-1=x_1$. First, assume that $x_1=-1$ and $x_1 \neq x_2$, so $-x_2$ is regular and $-x_2=1$ or $-x_2=0$ or $-x_2=-1$, which gives a contradiction. Secondly, assume that $-1=1$ and $x_1 \neq x_2$, so $x_1x_2=1$ or $x_1x_2=0$ or $x_1x_2=x_1$. But $x_1x_2=1$ means that x_2 is regular and adjacent to 1 which is a contradiction. When $x_1x_2=0$ we find that $x_1(x_2+1)=x_1x_2+x_1=x_1$ is regular and x_1 adjacent to x_2+1 which implies that $x_2+1=1$ or $x_2+1=x_1$ and $x_2=0$ or $x_2=x_1+1$. Thus, $x_1(x_1+1)=0$ and x_1 and $x_2=x_1+1$ are idempotent. Therefore, x_2 is regular and gives a contradiction. Now, if $x_1x_2=x_1$, then x_2+1 is adjacent to x_1 and $x_2=0$ or $x_2+1=x_1$. It follows that, x_1 and $x_1+1=x_2$ are idempotent, so x_2 is regular, which is a contradiction. Therefore, T has only three elements and $T \cong \mathbb{Z}_3$. The converse is clear. □

Example 13 In Figure 11 the domination number of the V-graph of $\mathbb{Z}_9$ is 2 .

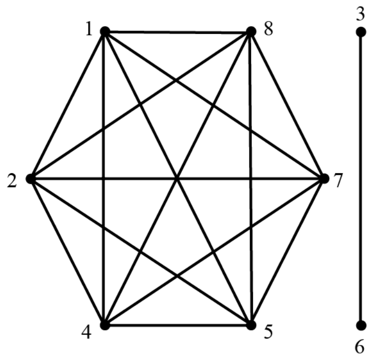

Figure 11. V-graph of $\mathbb{Z}_9$

The following Proposition provides the diameter of the V-graph of the rings of integers modulo n.

Proposition 7 For any positive integer $n \neq p^r$, where p is prime,

$$diam(V(\mathbb{Z}_n)) = \begin{cases} 1, \text{ if } n \text{ is a square free.} \\ 2, \text{ if } n = pq^r \text{ where } p \text{ and } q \text{ are different primes and } r \neq 1. \\ 3, \text{ otherwise.} \end{cases}$$

Proof. Suppose that n is a square-free positive integer, so $\mathbb{Z}_n$ is a semisimple ring and Von Neumann regular ring. It follows that $V(\mathbb{Z}_n)$ is a complete graph and $diam(V(\mathbb{Z}_n)) = 1$. Now, if $n = pq^l$ where p and different primes and $l \neq 1$, the set of nonzero elements in $\mathbb{Z}_n$ is divided into regular elements $A = \{r : (r, n) = 1 \; or \; r = tp \; or \; r = tq^r\}$ and non-regular elements $B = \{tq : tq \neq 0\}$. The elements in A are connected to each other and every element in B is connected to q^l but not to 1, so the diameter is 2. Lastly, if $n = (p_1)^{r_1}(p_2)^{r_2}...(p_l)^{r_l}$ where $p_1, p_2, ..., p_l$ are different primes and $r_i \neq 1$ for at least two of i. We may assume that $r_1 \neq 1$ and $r_2 \neq 1$, note that the elements p_1 and p_2 are not adjacent and there is no common adjacent element. In fact, we have the following path.

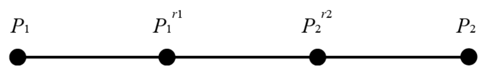

Figure 12

Therefore, $diam(V(\mathbb{Z}_n)) = 3$. □

Definition 2 [9] An independent set is a collection of vertices in a graph that are not adjacent to one another.

The size of an independent set is the number of vertices it contains.

Definition 3 [9] A *maximum independent set* is the largest feasible independent set for a given graph G. This size is known as *G's independence number* and is commonly indicated by $\alpha(G)$.

The following result will discuss the maximum independent set of V-graphs of $\mathbb{Z}_n$, denoted by $\alpha(V(\mathbb{Z}_n))$.

Proposition 8 Let n be a positive integer and p be a prime number. Then

$$\alpha(V(\mathbb{Z}_n)) = \begin{cases} 1, \text{ if } n \text{ is a square free.} \\ p^{r-1} - p^{\frac{r}{2}} + 2, \text{ if } n = p^r \text{ and } r \text{ is even.} \\ p^{r-1} - p^{\frac{r-1}{2}} + 1, \text{ if } n = p^r \text{ and } r \text{ is odd.} \end{cases}$$

Proof. If n is a square free integer, then $V(\mathbb{Z}_n)$ is a complete graph and $\alpha(V(\mathbb{Z}_n)) = 1$. In the situation $n = p^r$ and r is even, the set of nonzero elements in $\mathbb{Z}_n$ is divided into regular elements $A = \{r : (r, p) = 1\}$ and non-regular elements $B = \{tp : tp \neq 0\}$. The set $I = \{1\} \cup \{rp : (r, p) = 1\} \cup ... \cup \{rp^{\frac{r}{2}-1} : (r, p) = 1\} \cup \{p^{\frac{r}{2}}\}$ is a maximal independent set. $|I| = 2 + \sum_{t=1}^{\frac{r}{2}-1} \phi(p^{r-t}) = 2 + (P^{r-1} - P^{r-2}) + (P^{r-1} - P^{r-2}) + ... + (p^{\frac{r}{2}+1} - p^{\frac{r}{2}}) = 2 + p^{r-1} - p^{\frac{r}{2}}$. The odd case is similar.

Definition 4 [9] For a graph $G = (V, E)$ the dominating set is a subset D of V where every vertex not in D is adjacent to at least one of its members. The domination number $\gamma(G)$ is the number of vertices in the smallest dominating set.

The following results show the values of the domination number of V-graphs for $\mathbb{Z}_n$ rings, indicated as $\gamma(V(\mathbb{Z}_n))$.

Proposition 9 Let $n \geq 2$ be a positive integer and p be a prime number. Then

$$\gamma(V(\mathbb{Z}_n)) = \begin{cases} 1, \text{ if } n \text{ is a square free} \\ 2, \text{ if } n = p^r \text{ where } p \text{ is prime, and } r > 1. \\ 2^l, \text{ if } n = p_1^{r_1} p_2^{r_2} \; ... \; p_l^{r_l} \text{ and } 2 \leq r_i \text{ for all } i, \; 1 \leq i \leq l. \end{cases}$$

□

Proof. Let $n \geq 2$ be a positive integer. If n is square-free, then $V(\mathbb{Z}_n)$ is a complete graph, and $\gamma(V(\mathbb{Z}_n)) = 1$. Whenever $n = p^r$ where p is a prime number and $r \geq 2$, so $\mathbb{Z}^*_{p^r} = E \cup D$ where E is the set of inverible elements and $D = \{tp : 0 < t < p^{r-1}\}$. One can see that 1 is adjacent to every element in E and p^{r-1} is adjacent to every element in D. Thus, $\{1,\ p^{r-1}\}$ is a minimal domination set. Therefore, $\gamma(V(\mathbb{Z}_n)) = 2$. In the last case, consider $n = p_1^{r_1} p_2^{r_2} \ldots p_l^{r_l}$ where $p_i,\ 1 \leq i \leq l$ are different primes and $2 \leq r_i$ for all $i,\ 1 \leq i \leq l$. Consider the multiplicative set of regular elements $P = \langle \{p_i^{r_i} : 1 \leq i \leq l\} \rangle$, $P^* \cup \{p_1^{r_1-1} p_2^{r_2-1} \ldots p_l^{r_l-1}\}$ is a minimal-domination set. In fact, all regular elements are adjacent to every element in P^*. And if a is an irregular element, $a = cp_1^{t_1} p_2^{t_2} \ldots p_l^{t_l}$ where $0 \leq t_i < r_i$ for some t_i, not all t_i are zeros, and $(c,\ p_i) = 1$ for all i, then a is adjacent to an element in $P^* \cup \{p_1^{r_1-1} p_2^{r_2-1} \ldots p_l^{r_l-1}\}$. Therefore, $\gamma(V(\mathbb{Z}_n)) = |P^* \cup \{p_1^{r_1-1} p_2^{r_2-1} \ldots p_l^{r_l-1}\}| = 2^l$. □

3. Conclusions

In this research, we created a new type of graph associated with rings, which we call the V-graph. We provided many examples to illustrate that this type of graph is different from the graphs you know using rings, such as zero-divisor graphs, unit graphs, and quasi-regular graphs. We found the properties of the rings in which the graph we created satisfies the conditions of being a complete graph, cyclic, star, or path. We were also able to calculate the diameter and the independent number in some cases, as well as the domination number. Furthermore, in certain circumstances, we are searching for the independent number and the dominance number of the V-graph of $\mathbb{Z}_n$. Of course, many of the issues and aspects surrounding this graph remain unsolved and will require more research and effort to resolve. And in the next studies, we could have an answer to that question.

Conflict of interest

The authors declare no competing financial interest.

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