On the Sums Running over Reduced Residue Classes Evaluated at Polynomial Arguments

Mehdi Hassani*, Mahmoud Marie

Department of Mathematics, University of Zanjan, 45371-38791, Zanjan, Iran

E-mail: mehdi.hassani@znu.ac.ir

Abstract: For a given polynomial G we study the sums $\varphi_m(n) := \sum' k^m$ and $\varphi_G(n) = \sum' G(k)$ where $m \geq 0$ is a fixed integer and $\sum'$ runs through all integers k with $1 \leq k \leq n$ and $\gcd(k, n) = 1$. Although, for $m \geq 1$ the function φ_m is not multiplicative, analogue to the Euler function, we obtain expressions for $\varphi_m(n)$ and $\varphi_G(n)$. Also, we estimate the averages $\sum_{n\leq x}\varphi_m(n)$ and $\sum_{n\leq x}\varphi_G(n)$, the alternative averages $\sum_{n\leq x}(-1)^{n-1}\varphi_m(n)$ and $\sum_{n\leq x}(-1)^{n-1}\varphi_G(n)$.

Keywords: Euler function, residue systems, arithmetic function, alternating sum

1. Introduction

The Euler function $\varphi(n)$ is defined as the number of positive integers k with $k \leq n$ and $(k, n) = 1$, where (k, n) denotes the greatest common divisor of the integers k and n. For a given integer $m \geq 0$ let

$$\varphi_m(n) := \sum_{\substack{1\leqslant k\leqslant n\\(k,n)=1}} k^m.$$

In 2006, Apostol [1] obtained a formula for $\varphi_m(n)$ (relation (8) on the page 278 of [1], where the inner sum should corrected as product). In this note we provide a classical study of $\varphi_m(n)$ as an arithmetic function in n, more precisely focusing on its asymptotic behaviour and its average order. To keep completeness of our note, first we reprove Apostol's result in the following neat form.

Theorem 1.1 Let $m \geq 0$ be fixed integer. Then for $n \geq 1$ we have

$$\varphi_m(n) = \sum_{j=0}^{m} \beta_j n^{m+1-j} \mathcal{K}_j(n), \tag{1}$$

where for $0 \leq j \leq m$,

$$\beta_j = (-1)^j \binom{m}{j} \frac{B_j}{m+1-j},$$

with B_j denoting the j^{th} Bernoulli number, and for any integer $j \geq 0$ the arithmetic function $\mathcal{K}_j$ is defined by

$$\mathcal{K}_j(n) = \sum_{d\mid n} \mu(d) d^{j-1} = \prod_{p\mid n}\left(1 - p^{j-1}\right). \tag{2}$$

For any integer $n \geq 1$ let $\delta(n) = [\frac{1}{n}]$. We will use this expression for the arithmetic function δ throughout the paper. Since $\mathcal{K}_0(n) = \frac{\varphi(n)}{n}$ and $\mathcal{K}_1(n) = \delta(n)$, we have $\varphi_0(n) = \varphi(n)$ and $\varphi_1(n) = \frac{1}{2}n\varphi(n) + \frac{1}{2}n\delta(n)$. Moreover,

DOI: https://doi.org/10.37256/cm.152020546

$$\varphi_2(n)=\frac{1}{3}n^2\varphi(n)+\frac{1}{2}n^2\delta(n)+\frac{1}{6}n\mathcal{K}_2(n),$$

$$\varphi_3(n)=\frac{1}{4}n^3\varphi(n)+\frac{1}{2}n^3\delta(n)+\frac{1}{4}n^2\mathcal{K}_2(n),$$

$$\varphi_4(n)=\frac{1}{5}n^4\varphi(n)+\frac{1}{2}n^4\delta(n)+\frac{1}{3}n^3\mathcal{K}_2(n)-\frac{1}{30}n\mathcal{K}_4(n),$$

$$\varphi_5(n)=\frac{1}{6}n^5\varphi(n)+\frac{1}{2}n^5\delta(n)+\frac{5}{12}n^4\mathcal{K}_2(n)-\frac{1}{12}n^2\mathcal{K}_4(n).$$

Remark 1.2 In the above expressions the term $\delta(n)$ will drop very soon for $n > 1$. Also, for $m \geq 2$ we have

$$\varphi_m(n)=\frac{1}{m+1}n^m\varphi(n)+\frac{1}{2}n^m\delta(n)+\sum_{j=2}^{m}\beta_j n^{m+1-j}\mathcal{K}_j(n). \tag{3}$$

Moreover, $B_{2j-1} = 0$ for $j \geq 2$. Thus, $\beta_{2j-1} = 0$, and

$$\varphi_m(n)=\frac{1}{m+1}n^m\varphi(n)+\frac{1}{2}n^m\delta(n)+\sum_{1\leqslant j\leqslant \frac{m}{2}}\beta_{2j} n^{m+1-2j}\mathcal{K}_{2j}(n).$$

Proposition 1.3 Let $m \geq 0$ be fixed integer. As $n \to \infty$ we have

$$\varphi_m(n)=\frac{1}{m+1}n^m\varphi(n)+O\left(n^m\right), \tag{4}$$

where the constant of O-term doesn't exceed $\sum_{j=0}^{m}|\beta_j|$.

While the function φ_0, which coincides with the Euler function φ is known to be multiplicative, we deduce the following result examining the multiplicative property of the function φ_m for a given integer $m \geq 1$.

Corollary 1.4 Let $m \geq 1$ be fixed integer. The function $\varphi_m(n)$ is not multiplicative.

The relation (4) is the key to study average order of the arithmetic function $\varphi_m(n)$. Because of the deep connection to the Euler function, we expect a result similar to

$$\sum_{n\leqslant x}\varphi(n)=\frac{3}{\pi^2}x^2+O\left(xw(x)\right), \tag{5}$$

which holds as $x \to \infty$, with

$$w(x)=(\log x)^{\frac{2}{3}}(\log\log x)^{\frac{4}{3}},$$

providing the best error term known to date, due to Walfisz [2]. We will use this expression for $w(x)$ throughout the paper. Indeed, the following generalizations is an immediate corollary of (4) and (5).

Theorem 1.5 Let $m \geq 0$ be fixed integer. Then, as $x \to \infty$,

$$\sum_{n\leqslant x}\varphi_m(n)=\frac{6}{(m+1)(m+2)\pi^2}x^{m+2}+O\left(x^{m+1}w(x)\right). \tag{6}$$

In 2017, Tóth [3] developed a method to obtain asymptotic expansion for alternating sums of certain arithmetical functions, including $\sum_{n\le x}(-1)^{n-1}\varphi(n)$, for which he proved that

$$\sum_{n\leqslant x}(-1)^{n-1}\varphi(n)=\frac{1}{\pi^2}x^2+O\big(xw(x)\big). \tag{7}$$

In this paper, we use his result to prove the following.

Theorem 1.6 Let $m \ge 0$ be fixed integer. Then, as $x \to \infty$,

$$\sum_{n\leqslant x}(-1)^{n-1}\varphi_m(n)=\frac{2}{(m+1)(m+2)\pi^2}x^{m+2}+O\Big(x^{m+1}w(x)\Big). \tag{8}$$

The following corollary asserts that the sum of the values of $\varphi_m(n)$ over odd arguments is approximately twice the sum of the values of $\varphi_m(n)$ over even arguments.

Corollary 1.7 Let $m \ge 0$ be fixed integer. Then, as $x \to \infty$,

$$\sum_{\substack{n\leqslant x\\ n\ odd}}\varphi_m(n)=\frac{4}{(m+1)(m+2)\pi^2}x^{m+2}+O\Big(x^{m+1}w(x)\Big),$$

and

$$\sum_{\substack{n\leqslant x\\ n\ even}}\varphi_m(n)=\frac{2}{(m+1)(m+2)\pi^2}x^{m+2}+O\Big(x^{m+1}w(x)\Big).$$

Finally, as a generalization, for a given polynomial $G(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ with $a_m \ne 0$ we define

$$\varphi_G(n):=\sum_{\substack{1\leqslant k\leqslant n\\ (k,n)=1}}G(k).$$

Corollary 1.8 We have $\varphi_G(n) = a_m\varphi_m(n) + a_{m-1}\varphi_{m-1}(n) + \cdots + a_1\varphi_1(n) + a_0\varphi_0(n)$, and as $n \to \infty$,

$$\varphi_G(n)=\frac{a_m}{m+1}n^m\varphi(n)+O\big(n^m\big).$$

Also, as $x \to \infty$,

$$\sum_{n\leqslant x}\varphi_G(n)=\frac{6a_m}{(m+1)(m+2)\pi^2}x^{m+2}+O\Big(x^{m+1}w(x)\Big),$$

and

$$\sum_{n\leqslant x}(-1)^{n-1}\varphi_G(n)=\frac{2a_m}{(m+1)(m+2)\pi^2}x^{m+2}+O\Big(x^{m+1}w(x)\Big).$$

2. Proofs

To prove Theorem 2.1, and then other average results, we follow Apostol [1] to obtain the following more general and key result.

Lemma 2.1 Assume that f is an arbitrary arithmetic function. Then for $n \ge 1$,

$$\sum_{\substack{1\leqslant k\leqslant n\\(k,n)=1}} f(k)=\sum_{d|n}\mu(d)\sum_{1\leqslant q\leqslant\frac{n}{d}} f(dq). \tag{9}$$

Proof. The result is valid for $n = 1$. We assume that $n > 1$, for which we have $\delta(n) = 0$. Hence

$$\sum_{\substack{1\leqslant k\leqslant n\\(k,n)=1}} f(k)=\sum_{k=1}^{n-1} f(k)\delta((k,n))=\sum_{k=1}^{n-1} f(k)\sum_{d|(k,n)}\mu(d)=\sum_{k=1}^{n-1}\sum_{d|k,d|n}\mu(d)f(k).$$

By taking $k = dq$, we get

$$\sum_{k=1}^{n-1}\sum_{d|k,d|n}\mu(d)f(k)=\sum_{1\leqslant dq<n}\sum_{d|n}\mu(d)f(dq)=\sum_{1\leqslant q<\frac{n}{d}}\sum_{d|n}\mu(d)f(dq)=\sum_{d|n}\mu(d)\sum_{1\leqslant q<\frac{n}{d}} f(dq).$$

Now, we note that if $q=\frac{n}{d}$, then $f(dq) = f(n)$, and since $n > 1$, we imply that

$$\sum_{d|n}\mu(d)f(n)=f(n)\left\lfloor\frac{1}{n}\right\rfloor=0.$$

Thus, we obtain (9), and the proof is completed.

Proof of Theorem 1.1. The classical identity

$$\sum_{q=1}^{N} q^m=\sum_{j=0}^{m}\beta_j N^{m+1-j}, \tag{10}$$

which is known in literature holds for integers $m \geq 0$ and $N \geq 1$. By using the relations (10) and (9) we get

$$\varphi_m(n)=\sum_{d|n}\mu(d)\sum_{1\leqslant q\leqslant\frac{n}{d}}(dq)^m=\sum_{d|n}\mu(d)d^m\sum_{1\leqslant q\leqslant\frac{n}{d}} q^m=\sum_{d|n}\mu(d)\sum_{j=0}^{m}\beta_j d^{j-1}n^{m+1-j}.$$

Changing the order of sums gives (1).

Proof of Proposition 1.3. Let $m \geq 2$. We write

$$\mathcal{K}_j(n)=(-1)^{\omega(n)}\kappa(n)^{j-1}\prod_{p|n}\left(1-\frac{1}{p^{j-1}}\right),$$

where as usual $\omega(n)$ denotes the number of distinct prime divisors of n, and $\kappa(n)=\prod_{p|n} p$ denotes the square-free kernel of n. Thus, for $j \geq 2$ we have $|\mathcal{K}_j(n)| \leq n^{j-1}$, and

$$\left|\sum_{j=2}^{m}\beta_j n^{m+1-j}\mathcal{K}_j(n)\right|\leqslant n^m\sum_{j=2}^{m}|\beta_j|.$$

Considering (3) we get

$$\left|\varphi_m(n)-\frac{1}{m+1}n^m\varphi(n)\right|\leqslant\frac{1}{2}n^m\delta(n)+n^m\sum_{j=2}^{m}|\beta_j|.$$

Note that for each integer $m \geq 1$ we have $\beta_0 = \frac{1}{m+1}$ and $\beta_1 = \frac{1}{2}$. Hence, for any integer $m \geq 0$ we obtain

$$\left|\varphi_m(n) - \frac{1}{m+1} n^m \varphi(n)\right| \leqslant n^m \sum_{j=0}^{m} |\beta_j| .$$

This completes the proof.

Proof of Corollary 1.4. For each odd prime p the relation (4) gives

$$\varphi_m(p) = \frac{1}{m+1} p^m (p-1) + O\left(p^m\right),$$

and

$$\varphi_m(2p) = \frac{1}{m+1} (2p)^m (p-1) + O\left((2p)^m\right).$$

Thus,

$$\lim_{\substack{p \to \infty \\ p \text{ prime}}} \frac{\varphi_m(2p)}{\varphi_m(p)} = 2^m.$$

Since $m \geq 1$ is fixed integer and $\varphi_m(2) = 1$, thus $\varphi_m(2p) > \varphi_m(2)\varphi_m(p)$ for prime p sufficiently large.

Proof of Theorem 1.5. By using the expansion (5) and Abel summation [4], for each $m \geq 0$ we obtain

$$\sum_{n \leqslant x} n^m \varphi(n) = \frac{6}{(m+2)\pi^2} x^{m+2} + O\left(x^{m+1} w(x)\right). \tag{11}$$

Considering (4) concludes the proof.

Proof of Theorem 1.6. By using the expansion (7) and Abel summation, for each $m \geq 0$ we obtain

$$\sum_{n \leqslant x} (-1)^{n-1} n^m \varphi(n) = \frac{2}{(m+2)\pi^2} x^{m+2} + O\left(x^{m+1} w(x)\right).$$

Considering (4) concludes the proof.

Remark 2.2 We take a deep look at the error term in (6) and the missing role of the function μ in its estimating. Since $\varphi_0(n) = \varphi(n)$ and $\varphi_1(n) = \frac{1}{2} n\varphi(n) + \frac{1}{2} n\delta(n)$, the approximation (6) holds for $m = 0$ and $m = 1$ due to (11). Let $m \geq 2$ and $x \geq 1$. The relation (3) gives

$$\sum_{n \leqslant x} \varphi_m(n) = \frac{1}{m+1} \sum_{n \leqslant x} n^m \varphi(n) + \frac{1}{2} \sum_{n \leqslant x} n^m \delta(n) + \sum_{n \leqslant x} \sum_{j=2}^{m} \beta_j n^{m+1-j} \mathcal{K}_j(n).$$

The first sum is approximated due to (11). Clearly, $\sum_{n \leq x} n^m \delta(n) = 1$. For the third double sum, for which $j \geq 2$, we write

$$\sum_{n \leqslant x} n^{m+1-j} \mathcal{K}_j(n) = \sum_{n \leqslant x} n^{m+1-j} \sum_{d \mid n} d^{j-1} \mu(d) = \sum_{d \leqslant x} \sum_{k \leqslant \frac{x}{d}} (kd)^{m+1-j} d^{j-1} \mu(d) = \sum_{d \leqslant x} d^m \mu(d) \sum_{k \leqslant \frac{x}{d}} k^{m+1-j}.$$

Note that $2 \leq j \leq m$, or $1 \leq m + 1 - j \leq m - 1$. The truncated version of (10) gives

$$\sum_{k\leqslant \frac{x}{d}} k^{m+1-j} = \frac{1}{m+2-j}\left(\frac{x}{d}\right)^{m+2-j} + O\left(\left(\frac{x}{d}\right)^{m+1-j}\right).$$

So,

$$\sum_{d\leqslant x} d^m \mu(d) \sum_{k\leqslant \frac{x}{d}} k^{m+1-j} = \frac{x^{m+2-j}}{m+2-j}\sum_{d\leqslant x} d^{j-2}\mu(d) + O\left(x^{m+1}\right).$$

We recall the approximation $\sum_{d\leqslant x}\mu(d) \ll x\mathrm{e}^{-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}}$, where $c > 0$ is constant, and the error term is best known to date, due to Walfisz [2]. This expansion and Abel summation give

$$\sum_{d\leqslant x} d^{j-2}\mu(d) \ll x^{j-1}\,\mathrm{e}^{-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}}.$$

Therefore,

$$\sum_{n\leqslant x} n^{m+1-j}\mathcal{K}_j(n) = \sum_{d\leqslant x} d^m \mu(d) \sum_{k\leqslant \frac{x}{d}} k^{m+1-j} \ll x^{m+1}\,\mathrm{e}^{-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}} + x^{m+1} \ll x^{m+1},$$

and

$$\sum_{n\leqslant x}\sum_{j=2}^{m}\beta_j n^{m+1-j}\mathcal{K}_j(n) = \sum_{j=2}^{m}\beta_j \sum_{n\leqslant x} n^{m+1-j}\mathcal{K}_j(n) \ll x^{m+1}.$$

References

[1] T. M. Apostol. Bernoulli's power-sum formulas revisited. *Math. Gaz*. 2006; 90: 276-279.

[2] A. Walfisz. *Weylsche Exponentialsummen in der neueren Zahlentheorie*. Berlin: VEB Deutscher Verlag der Wissenschaften; 1963.

[3] L. Tóth. Alternating sums concerning multiplicative arithmetic functions. *J. Integer Seq*. 2017; 20: 41.

[4] G. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*. 3rd ed. American Mathematical Society; 2015.