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Research Article

Pre-Compact Sets in the Generalized Morrey Spaces in Terms of the Averaging Function

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Abstract: In this paper, sufficient conditions for compactness of sets in generalized Morrey spaces are given in terms of an averaging function. This result is analogous to the well-known Fréchet-Kolmogorov theorem for the pre-compacting of sets in Lebesgue spaces. Our main result consisted of four conditions on the behavior of the function norm and the norm of its averages in generalised Morrey spaces that are sufficient for a set to be pre-compact in these spaces. An example is provided to demonstrate that not all conditions obtained in the main result are necessary for a set to be pre-compacted in generalised Morrey spaces.

***Keywords*:** pre-compact, generalized Morrey spaces, averaging function, commutator, riesz potential

MSC: 42B20, 42B25

1. Introduction

This article addresses the issue of pre-compacted sets in generalised Morrey spaces $M_p^{w(r)}$. The conditions for pre-compacted sets were obtained with regard to the averaging function. The need for having pre-compact conditions provided in the generalised Morrey spaces is also discussed. The criteria for pre-compacting sets in Lebesgue spaces regarding equicontinuity and functional averaging are well-known.

The article further considers the conditions for the pre-compactness of sets in generalised Morrey spaces. An analogue of the Fréchet-Kolmogorov theorem for Morrey spaces concerning equicontinuity was obtained in [1, 2]. There are also applications of this result to prove the compactness of the commutator for the Riesz potential and Calderon-Zygmund singular integral in Morrey spaces, respectively considered in [1, 3]. An analogue of the Fréchet-Kolmogorov theorem for generalised Morrey spaces regarding the equicontinuity and compactness of the commutator of the Riesz potential in generalised Morrey spaces was considered in [4]. The pre-compact of sets in variable-exponent Morrey spaces. were studied by [5]. Pre-compacts of sets and the characterisations of commutators in the ball-Banach function space were obtained by [6]. The pre-compacts of the sets in Orlicz spaces are shown in [7]. Spanne et al. [8], Adams [9], Mizuhara [10], Nakai [11], and Guliyev extended Adams' results and provided sufficient conditions for the boundedness

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of I_α in generalised Morrey spaces. The boundedness of the commutator for the Riesz potential in both Morrey and generalised Morrey spaces was studied in [12, 13]. Compact commutators for the Riesz potential on Morrey spaces and Morrey spaces with non-doubling measures were considered in [1, 14]. The compactness of the commutator for the Riesz potential $[b, I_\alpha]$ in local Morrey-type spaces $M_p^{w(r)}$ is discussed in terms of uniform equity in [15].

The remainder of this paper is organised as follows: Section 2 presents the notation and preliminaries. Section 3 presents the results for the pre-compaction of a set in terms of the uniform equi-continuity averaging function n generalised Morrey spaces. Several theorems are also revisited, and auxiliary lemmas are established. Section 4 concludes the study.

We introduce our conventional notation. Throughout, this study uses C to denote a positive constant that is independent of the main parameters involved, although its value may vary from line to line. Constants with subscripts such as c_p depend on the subscript p. From the difference $A \setminus B$ between the two sets A and B (in that order) denotes the set of elements in A, which do not belong to B, $C(\mathbb{R})$ denotes the set of all continuously bounded functions on $\mathbb{R}$ with a uniform norm.

This study's main goal was to determine the conditions for the pre-compacts of sets in generalised Morrey spaces $M_p^{w(r)}$ based on the averaging function. Function averaging and the properties of averaging were actively used in this study. To prove the main theorem on the pre-compactness of sets in generalised Morrey spaces, we first show that the set of function averages is completely bounded in a certain ring of sets. Subsequently, the total boundedness of the considered sets in the generalised Morrey spaces was established.

2. Preliminaries

In this section, we recall some definitions of various functional spaces, as well as their properties.

For a Lebesgue measurable set $E \subset \mathbb{R}^n$ and $0 < p \leq \infty$, $L_p(E)$ is the standard Lebesgue space for all functions f. on E for which

$$\|f\|_{L_p(E)} = \begin{cases} \left(\int\limits_E |f(y)|^p dy \right)^{\frac{1}{p}} < \infty, & \text{if} \quad 1 \leq p < \infty, \\ ess \sup |f(y)| < \infty, & \text{if} \quad p = \infty. \end{cases}$$

Definition 1 (See [16, p.7]) ($L_p^{\text{loc}}(\mathbb{R}^n)$). We assume that $1 \leq p \leq \infty$. The space $L_p^{\text{loc}}(\mathbb{R}^n)$ collects all $f \in L_0(\mathbb{R}^n)$ such that $f \in L_p(K)$ for each compact set K, where $L_0(\mathbb{R}^n)$ denotes the set of all Lebesgue measurable functions.

Charles Morrey introduced Morrey spaces in 1938 (see [17]) for application in elliptic partial differential equations.

Definition 2 For $0 < \lambda < n$, $0 < p \leq \infty$ the Morrey spaces $M_p^\lambda \equiv M_p^\lambda(\mathbb{R}^n)$ are defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$, for which

$$\|f\|_{M_p^\lambda} = \sup_{x \in \mathbb{R}^n,\, r>0} \left(r^{-\lambda} \|f\|_{L_p(B(x,\, r))} \right) < \infty.$$

where $B(x, r)$ is the open ball in $\mathbb{R}^n$ with a centre at point $x \in \mathbb{R}^n$ and a radius $r > 0$.

When $\lambda = 0$ and $\lambda = n$ the Morrey spaces $M_p^0(\mathbb{R}^n)$ and $M_p^n(\mathbb{R}^n)$ respectively coincide (with equality of norms) with the spaces $L_p(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$ (see [18, page 13-14]).

Recent times have seen an increasing interest in applying Morrey spaces to diverse areas of analysis such as the partial differential equations, potential theory, and harmonic analysis [16, 19, 20]. Later, Morrey spaces were found to have numerous important applications, including the Navier-Stokes equations (see [21, 22]), Schrödinger equations in [23, 24], and potential theory [25, 26].

Generalised Morrey spaces $M_p^{w(r)}$ were first considered by Mizuhara [10], Nakai [11], and Guliyev.

Definition 3 Let $1 \leq p \leq \infty$ and w be a Lebesgue measurable non-negative function on $(0, \infty)$ that is not equivalent to zero. Generalised Morrey space $M_p^{w(r)} \equiv M_p^{w(r)}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with $\|f\|_{M_p^{w(r)}} < \infty$, where

$$\|f\|_{M_p^{w(r)}} = \sup_{x \in \mathbb{R}^n,\, r>0} \left(w(r) \, \|f\|_{L_p(B(x,\, r))} \right).$$

Space $M_p^{w(r)}$ coincides with Morrey space M_p^{λ} if $w(r) = r^{-\lambda}$, where $0 < \lambda < n$. $\Omega_{p\infty}$ denotes the set of all positive functions $w(r)$, measurable on $(0, \infty)$ such that for some $t > 0$,

$$\|w(r) r^{\frac{n}{p}}\|_{L_\infty(0,\, t)} < \infty,$$

and

$$\|w(r)\|_{L_\infty(t,\, \infty)} < \infty.$$

If condition $||w(r)||_{L_\theta(t,\, \infty)} < \infty$ is replaced by $||w(r)||_{L_\theta(0,\, \infty)} < \infty$, we say that $w \in \widetilde{\Omega}_{p\theta}$.

The space $M_p^{w(r)}$ is nontrivial and consists of functions equivalent to 0 on $\mathbb{R}^n$ if and only if $w \in \Omega_{p\infty}$ [18, 27].

Definition 4 Let $\mathcal{F}$ be a subset of the function spaces X and $\mathcal{G} \subset \mathcal{F}$. Then, $\mathcal{G}$ is called the ε-net of $\mathcal{F}$ if, for any $f \in \mathcal{F}$ and any $\varepsilon > 0$, $g \in \mathcal{G}$ such that $\|f - g\|_X < \varepsilon$. Moreover, if $\mathcal{G}$ is an ε-net of $\mathcal{F}$ and the cardinality of $\mathcal{G}$ is finite, $\mathcal{G}$ is designated a finite ε-net of $\mathcal{F}$. Furthermore, $\mathcal{F}$ is said to be completely bounded if, for any $\varepsilon \in (0, \infty)$, there exists a finite ε. In addition, if a completely bounded $\mathcal{F}$ has compact closure in X, then it is said to be relatively compact or pre-compact.

From the Hausdorff theorem (see [28, p.13]), it follows that a subset $\mathcal{F}$ of the function space X is relatively compact if and only if $\mathcal{F}$ is completely bounded because of the completeness of X.

The well-known Frechet-Kolmogorov theorem calculates the pre-compactness of sets in $L_p(\mathbb{R}^n)$ in terms of uniform equi-continuity.

Theorem 1 [28] Suppose $1 \leq p < \infty$. The set $S \subset L_p(\mathbb{R}^n)$ is pre-compact in $L_p(\mathbb{R}^n)$ if and only if set S satisfies the following three conditions:

$$\sup_{f \in S} \|f\|_{L_p(\mathbb{R}^n)} < \infty, \tag{1}$$

$$\lim_{\delta \to 0^+} \sup_{|h| \leq \delta} \sup_{f \in S} \|f(\cdot + h) - f(\cdot)\|_{L_p(\mathbb{R}^n)} = 0 \tag{2}$$

and

$$\lim_{R \to \infty} \sup_{f \in S} \|f\|_{L_p({}^c B(0,\, R))} = 0, \tag{3}$$

where ${}^{c}B(0, R)$ is the complement of the ball $B(0, R)$.

For the function $f \in L_1^{loc}(\mathbb{R}^n)$, the averaging function $(A_\delta f)(x)$ depends on $\delta > 0$ with

$$(A_\delta f)(x) = \frac{1}{|B(x, \delta)|} \int\limits_{B(x, \delta)} f(y)dy,$$

where $|B|$ is the Lebesgue measure of set B.

The following theorem is a well-known result that establishes the conditions for the pre-compactness of sets in $L_p(\mathbb{R}^n)$ using the averaging function $(A_\delta f)(x)$:

Theorem 2 [28] Let $1 \leq p < \infty$. The set $S \subset L_p(\mathbb{R}^n)$ is pre-compact in $L_p(\mathbb{R}^n)$ if and only if set S satisfies the following conditions:

$$\sup_{f \in S} \|f\|_{L_p(\mathbb{R}^n)} < \infty, \tag{4}$$

$$\lim_{\delta \to 0^+} \sup_{f \in S} \|A_\delta f - f\|_{L_p(\mathbb{R}^n)} = 0, \tag{5}$$

and

$$\lim_{R \to \infty} \sup_{f \in S} \|f\|_{L_p({}^{c}B(0, R))} = 0. \tag{6}$$

From the definitions of $A_\delta f$, and the symmetry of the circle, we have

$$(A_\delta f)(x) = \int\limits_{B(0, \delta)} \psi_\delta(y) f(x-y)dy = (\psi_\delta * f)(x), \ x \in \mathbb{R}^n, \tag{7}$$

where

$$\psi_\delta(x) = \frac{\chi_{B(0, \delta)}(x)}{|B(0, \delta)|}.$$

Indeed,

$$(\psi_\delta * f)(x) = \int\limits_{B(0, \delta)} \psi_\delta(y) f(x-y)dy = \frac{1}{|B(x, \delta)|} \int\limits_{B(x, \delta)} \chi_{B(0, \delta)}(x-y) f(y)dy$$

$$= \frac{1}{|B(x, \delta)|} \int\limits_{B(x, \delta)} f(y)dy = (A_\delta f)(x).$$

We denote the maximum Hardy-Littlewood function Mf as

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)|dy.$$

Note that, founded on the basic sense of an average, for any $x \in \mathbb{R}^n$ and $\delta > 0$

$$|(A_\delta f)(x)| \leq \frac{1}{|B(x, \delta)|} \int_{B(x, \delta)} |f(y)|dy \leq \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)|dy = (Mf)(x). \tag{8}$$

The following theorem provides the conditions for the boundedness of operator M in space $M_p^{w(r)}$:

Theorem 3 [29] Let $1 < p < \infty$, $w \in \Omega_{p\infty}$. For the maximal operator M to be bounded in space $M_p^{w(r)}$, it is sufficient that for some c_1

$$\left\| w(r) \left(\frac{r}{r+t} \right)^{\frac{n}{p}} \right\|_{L_\infty(0, \infty)} \leq c_1 \|w(r)\|_{L_\infty(t, \infty)}, \tag{9}$$

for any $t > 0$.

Note that condition (9) is equivalent to the next condition. For $c_1 > 0$

$$t^{-\frac{n}{p}} \left\| w(r) r^{\frac{n}{p}} \right\|_{L_\infty(0, t)} \leq c_1 \|w(r)\|_{L_\infty(t, \infty)}. \tag{10}$$

Corollary 1 Let $1 < p < \infty$. If $w \in \Omega_{p\infty}$ and condition (10) are satisfied, then $c_2 > 0$ exists such that for any $\delta > 0$ and $f \in M_p^{w(r)}$

$$\|A_\delta f\|_{M_p^{w(r)}} \leq c_2 \|f\|_{M_p^{w(r)}}. \tag{11}$$

Proof. It suffices to use the inequality (8) and Theorem 3. □

3. Pre-compact of sets in generalized Morrey spaces in terms averaging function

In this section, we demonstrate the sufficient conditions for the precompactness of sets in generalised Morrey spaces. $M_p^{w(r)}$ in terms of the averaging function and establish auxiliary lemmas.

Lemma 1 We assume that $1 \leq p \leq \infty$. Then, for any $R > 0$, $\delta > 0$ and $f \in L_p(B(0, R+\delta))$

$$||A_\delta f||_{L_p(B(0, R))} \leq ||f||_{L_p(B(0, R+\delta))}. \tag{12}$$

Lemma 2 We assume that $1 \leq p \leq \infty$. Subsequently, for any $0 < \delta < R_1 < R_2 < \infty$ and $f \in L_p(B(0, R_2+\delta) \setminus B(0, R_1-\delta))$

$$||A_\delta f||_{L_p(B(0,R_2)\setminus B(0,R_1))} \le ||f||_{L_p(B(0,R_2+\delta)\setminus B(0,R_1-\delta))}. \tag{13}$$

Lemmas 1 and 2 are particular cases of the following variants of Young's inequality [30] for convolution:

$$\|(k*f)(x)\|_{L_p(H)} = \left\| \int_G k(y)f(x-y)dy \right\|_{L_p(H)} \le \|k\|_{L_1(G)} \|f\|_{L_p(G-H)},$$

where $G-H=\{x-y\colon x\in G,\ y\in H\}$, (see formula (7)) with $k=\psi_\delta$, $G=B(0,\ \delta)$ and $H=B(0,\ R)$ in Lemma 1, $H=B(0,R_2)\setminus B(0,R_1)=1$ in Lemma 2. Since $\|\psi_\delta\|_{L_1(B(0,\delta))}=1$ and $B(0,\delta)-B(0,R)=\{x-y,$ where $x\in B(0,\delta), y\in B(0,R)\}$ then in Lemma 1 the result is as follows $|x-y|\le|y|+|x|\le R+\delta$. In the case of Lemma 2 $B(0,\delta)-B(0,R_2)\setminus B(0,R_1)=\{x-y,$ where $x\in B(0,\delta)$, $y\in B(0,R_2)\setminus B(0,R_1)\}$ it turns out the following $|x-y|\le|y|+|x|\le R_2+\delta$ and $|x-y|\ge|y|-|x|\ge R_1-\delta$.

Lemma 3 Let $1\le p\le\infty$, $w\in\Omega_{p\infty}$. Then, for any $0<R_1<R_2<\infty$, $\delta>0$ and for any function $f,\ \varphi\in M_p^{w(r)}$

$$\begin{aligned}||f-\varphi||_{M_p^{w(r)}} \le{}& ||f\chi_{B(0,R_1)}||_{M_p^{w(r)}} + ||\varphi\chi_{B(0,R_1)}||_{M_p^{w(r)}} \\ &+ ||(A_\delta f-f)\chi_{B(0,R_2)\setminus B(0,R_1)}||_{M_p^{w(r)}} + ||(A_\delta f-A_\delta\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)}||_{M_p^{w(r)}} \\ &+ ||(A_\delta\varphi-\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)}||_{M_p^{w(r)}} + ||f\chi_{^cB(0,R_2)}||_{M_p^{w(r)}} + ||\varphi\chi_{^cB(0,R_2)}||_{M_p^{w(r)}}.\end{aligned} \tag{14}$$

Proof. By adding and subtracting the corresponding summands and applying the triangle inequality, we obtain

$$\begin{aligned}||f-\varphi||_{M_p^{w(r)}} ={}& ||(f-\varphi)\chi_{B(0,R_1)} + (f-\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)} + (f-\varphi)\chi_{^cB(R_2,\infty)}||_{M_p^{w(r)}} \\ ={}& ||f\chi_{B(0,R_1)} - \varphi\chi_{B(0,R_1)} + f\chi_{^cB(R_2,\infty)} - \varphi\chi_{^cB(R_2,\infty)} - (A_\delta f-f)\chi_{B(0,R_2)\setminus B(0,R_1)} \\ &+ (A_\delta\varphi-\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)} - (A_\delta f-A_\delta\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)}||_{M_p^{w(r)}}. \\ \le{}& \left\|f\chi_{B(0,R_1)}\right\|_{M_p^{w(r)}} + \left\|\varphi\chi_{B(0,R_1)}\right\|_{M_p^{w(r)}} + \left\|f\chi_{^cB(R_2,\infty)}\right\|_{M_p^{w(r)}} \\ &+ \left\|\varphi\chi_{^cB(R_2,\infty)}\right\|_{M_p^{w(r)}} + \left\|(A_\delta f-f)\chi_{B(0,R_2)\setminus B(0,R_1)}\right\|_{M_p^{w(r)}} \\ &+ \left\|(A_\delta\varphi-\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)}\right\|_{M_p^{w(r)}} + \left\|(A_\delta f-A_\delta\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)}\right\|_{M_p^{w(r)}}.\end{aligned}$$

□

Theorem 4 Let $1 < p < \infty$ and let $w \in \widetilde{\Omega}_{p\infty}$ satisfy the condition (9). Suppose that the set $S \subset M_p^{w(r)}(\mathbb{R}^n)$ satisfies the following four conditions, where $0 < R_1 < R_2$ and $\delta > 0$.

$$\sup_{f \in S} \|f\|_{M_p^{w(r)}(\mathbb{R}^n)} < \infty, \tag{15}$$

$$\lim_{R_1 \to 0^+} \sup_{f \in S} \left\| f \chi_{B(0,\, R_1)} \right\|_{M_p^{w(r)}(\mathbb{R}^n)} = 0, \tag{16}$$

$$\lim_{\delta \to 0^+} \sup_{f \in S} \|A_\delta f - f\|_{L_p(B(0,\, R_2) \setminus B(0,\, R_1))} = 0 \tag{17}$$

and

$$\lim_{R_2 \to \infty} \sup_{f \in S} \left\| f \chi_{{}^c B(0,\, R_2)} \right\|_{M_p^{w(r)}(\mathbb{R}^n)} = 0. \tag{18}$$

Then, set S as a precompact in $M_p^{w(r)}$.

Proof of Theorem 4

Let $S \subset M_p^{w(r)}$ and suppose that conditions (15)-(18) are satisfied.

Step 1 In the first step, we prove that for any $0 < \delta < R_1 < R_2 < \infty$ the set $S_\delta = \{A_\delta f : f \in S\}$ is precompact in $L_p(B(0,\, R_2) \setminus B(0,\, R_1))$.

Note that

$$||f||_{M_p^{w(r)}} \geq \left\| w(r) ||f||_{L_p(B(0,\, r))} \right\|_{L_\infty(R_2,\, \infty)} \geq \|w(r)\|_{L_\infty(R_2,\, \infty)} \, \|f\|_{L_p(B(0,\, R_2) \setminus B(0,\, R_1))},$$

hence

$$\|f\|_{L_p(B(0,\, R_2) \setminus B(0,\, R_1))} \leq \|w(r)\|^{-1}_{L_\infty(R_2,\, \infty)} \, ||f||_{M_p^{w(r)}}. \tag{19}$$

Therefore, according to inequalities (19), (11), and condition (15) for $g \in S_\delta$, we have

$$\begin{aligned}
\sup_{g \in S_\delta} \|g\|_{L_p(B(0,\, R_2) \setminus B(0,\, R_1))} &= \sup_{f \in S} \|A_\delta f\|_{L_p(B(0,\, R_2) \setminus B(0,\, R_1))} \\
&\leq \|w(r)\|^{-1}_{L_\infty(R_2,\, \infty)} \sup_{f \in S} \|A_\delta f\|_{M_p^{w(r)}} \\
&\leq c_2 \, \|w(r)\|^{-1}_{L_\infty(R_2,\, \infty)} \sup_{f \in S} \|f\|_{M_p^{w(r)}} < \infty.
\end{aligned} \tag{20}$$

Additionally, according to inequality (13) and condition (17),

$$\lim_{\tau\to 0^+}\sup_{g\in S_\delta}\|A_\tau g-g\|_{L_p(B(0,R_2)\setminus B(0,R_1))}=\lim_{\tau\to 0^+}\sup_{f\in S}\|A_\tau A_\delta f-A_\delta f\|_{L_p(B(0,R_2)\setminus B(0,R_1))}$$

$$=\lim_{\tau\to 0^+}\sup_{f\in S}\|A_\delta(A_\tau f-f)\|_{L_p(B(0,R_2)\setminus B(0,R_1))}$$

$$\leq\lim_{\tau\to 0^+}\sup_{f\in S}\|A_\tau f-f\|_{L_p(B(0,R_2+\delta)\setminus B(0,R_1-\delta))}=0.$$

Therefore, since the set $B(0, R_2)\setminus B(0, R_1)$ satisfies conditions (4), (5) and (6) of Theorem 2, it follows that the set S_δ is precompact in $L_p(B(0, R_2)\setminus B(0, R_1))$, or equivalently, totally bounded in $L_p(B(0, R_2)\setminus B(0, R_1))$.

Hence, for any $\varepsilon>0$, there exists $f_1, \ldots, f_m\in S$ (depending on ε, r and R) such that $\{A_\delta f_1, A_\delta f_2, \ldots, A_\delta f_m\}$ is a finite ε-net in S_δ with respect to norm of $L_p(B(0, R_2)\setminus B(0, R_1))$. Therefore, for any $f\in S$, $1\leq j\leq m$ such that

$$\left\|A_\delta f-A_\delta f_j\right\|_{L_p(B(0,R_2)\setminus B(0,R_1))}<\varepsilon.$$

Hence,

$$\min_{j=1,\ldots,m}\left\|A_\delta f-A_\delta f_j\right\|_{L_p(B(0,R_2)\setminus B(0,R_1))}<\varepsilon.$$

Step 2 In the second step, we prove that set S is completely bounded by $M_p^{w(r)}$.

Let us show that set S is a precompact set in $M_p^{w(r)}$. For any f, $\varphi\in S$, we have

$$\left\|(A_\delta f-A_\delta\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)}\right\|_{M_p^{w(r)}}=\sup_{x\in\mathbb{R}^n,R_1>0}\Big(w(r)||(A_\delta f-A_\delta\varphi)\chi_{B(0,R_2)\setminus B(0,R_1)}||_{L_p(B(x,r))}\Big)$$

$$\leq\sup_{r>0}(w(r))\,\|(A_\delta f-A_\delta\varphi)\|_{L_p(B(0,R_2)\setminus B(0,R_1))}\tag{21}$$

and the inequality (14). It follows that for any f, $\varphi\in S$

$$||f-\varphi||_{M_p^{w(r)}}\leq 2\sup_{g\in S}||g\chi_{B(0,R_1)}||_{M_p^{w(r)}}+2\sup_{g\in S}||(A_\delta g-g)\chi_{B(0,R_2)\setminus B(0,R_1)}||_{M_p^{w(r)}}$$

$$+2\sup_{g\in S}||g\chi_{{}^cB(R_2,\infty)}||_{M_p^{w(r)}}+\|w\|_{L_\infty(R_1,\infty)}\|A_\delta f-A_\delta\varphi\|_{L_p(B(0,R_2)\setminus B(0,R_1))}$$

$$=E_1+E_2+E_3+E_4,$$

where E_1, E_2, E_3 and E_4 are defined as follows: We assume that $\varepsilon > 0$. Using condition (16), the radius of the ball $R_1 = R_1(\varepsilon)$ such that

$$E_1 = 2\sup_{g\in S} ||g\chi_{B(0, R_1)}||_{M_p^{w(r)}} < \frac{\varepsilon}{4}.$$

By condition (18), the radius of the ball $R_2 = R_2(\varepsilon)$, such that

$$E_3 = 2\sup_{g\in S} ||g\chi_{{}^cB(R_2, \infty)}||_{M_p^{w(r)}} < \frac{\varepsilon}{4}.$$

Using inequality (21) and condition (17), we can choose $\delta = \delta(\varepsilon)$ such that

$$E_2 = 2\sup_{g\in S} ||(A_\delta g - g)\chi_{B(0, R_2)\backslash B(0, R_1)}||_{M_p^{w(r)}}$$

$$\leq 2||w||_{L_\infty(R, \infty)} \sup_{g\in S} \|A_\delta g - g\|_{L_p(B(0, R_2)\backslash B(0, R_1))} < \frac{\varepsilon}{4}.$$

$$E_4 = \|w\|_{L_{\infty(R_1, \infty)}} \|A_\delta f - A_\delta \varphi\|_{L_p(B(0, R_2)\backslash B(0, R_1))}.$$

Subsequently, for any f, $\varphi \in S$

$$||f-\varphi||_{M_p^{w(r)}} \leq \frac{3\varepsilon}{4} + E_4 = \frac{3\varepsilon}{4} + \|w\|_{L_{\infty(R_1, \infty)}} \|A_\delta f - A_\delta \varphi\|_{L_p(B(0, R_2)\backslash B(0, R_1))}.$$

Hence, for any φ_1, φ_2, ..., $\varphi_m \in S$

$$\min_{j=1, 2, ..., m} ||f-\varphi_j||_{M_p^{w(r)}} \leq \frac{3\varepsilon}{4} + \|w\|_{L_{\infty(R_1, \infty)}} \min_{j=1, 2, ..., m} \left\|A_\delta f - A_\delta \varphi_j\right\|_{L_p(B(0, R_2)\backslash B(0, R_1))}. \tag{22}$$

Finally, by the pre-compact of the set $S_{\delta(\varepsilon)}$ in $L_p(B(0, R_2(\varepsilon)) \setminus B(0, R_1(\varepsilon)))$, for any $f \in S$ there exist $m(\varepsilon) \in \mathbb{N}$ and $f_{1, \varepsilon}$, ..., $f_{m(\varepsilon), \varepsilon} \in S$, such that

$$\min_{j=1, 2, ..., m(\varepsilon)} \left\|A_{\delta(\varepsilon)} f - A_{\delta(\varepsilon)} f_{j, \varepsilon}\right\|_{L_p(B(0, R_2(\varepsilon))\backslash B(0, R_1(\varepsilon)))} \leq \frac{\varepsilon}{4} \left(\|w\|_{L_{\theta(0, \infty)}}\right)^{-1}.$$

Therefore, by setting $\varphi_j = f_{j, \varepsilon}$, $j = 1$, ..., $m(\varepsilon)$ by inequality (22) for any $f \in S$, we find that

$$\min_{j=1, 2, ..., m} ||f - f_{j, \varepsilon}||_{M_p^{w(r)}} \leq \varepsilon.$$

Thus, we have $\varphi_j = f_{j,\,\varepsilon}$, $j = 1, \ldots, m(\varepsilon)$ is a finite ε-net in S with respect to the norm $M_p^{w(r)}$.

It follows from this that set S is completely bounded in $M_p^{w(r)}$ or, equivalently, set S is pre-compacted in $M_p^{w(r)}$.

The Proof of Theorem 4 is completed.

The strength of the obtained results is that the theorems considered wider spaces. The Frechet-Kolmogorov theorem for the precompacting of sets in a Lebesgue space in terms of average functions contains three necessary and sufficient conditions. This proves a similar result for generalised Morrey spaces, four conditions for the precompactness of the sets are derived in terms of of averaging functions. An example of a set of functions for which not all the specified conditions are necessary is also considered. Consequently, the questions of determining necessary and sufficient conditions in the generalised Morrey spaces remain open.

Remark Condition (15) in Theorem 4 is necessary because any precompact set in a normed space is bounded.

Conditions (17), (16), and (18) are unnecessary, at least when $n = 1$ and $w(r) = r^{-\lambda}$, where $0 < \lambda < \dfrac{1}{p}$. In this case, the set S, consisting of only one function $|x|^{\lambda - \frac{1}{p}} \in M_p^\lambda$, is precompact. However, the conditions (17) and (16) are not satisfied. This follows from the following example.

Thus, the question of determining the necessary and sufficient conditions for the set $S \subset M_p^{w(r)}$ to be precompact remains open.

Example For $n = 1$ and $w(r) = r^{-\lambda}$, $1 \le p < \infty$, $0 < \lambda < \dfrac{1}{p}$, can state that

$$\lim_{r \to 0^+} \left\| A_r(|\cdot|^{\lambda - \frac{1}{p}}) - |\cdot|^{\lambda - \frac{1}{p}} \right\|_{M_p^\lambda} \neq 0 \tag{23}$$

in M_p^λ as $r \to 0^+$ and

$$\lim_{u \to 0} ||f_0(x+u) - f_0(x)||_{M_p^\lambda} \neq 0 \tag{24}$$

in M_p^λ as $u \to 0$ and finaly

$$\lim_{r \to +\infty} \left\| |x|^{\lambda - \frac{1}{p}} \chi_{{}^c B(0,\, r)}(x) \right\|_{M_p^\lambda} \neq 0 \tag{25}$$

in M_p^λ because $r \to +\infty$.

Indeed, for $x > 0$ and $0 < r < x$,

$$A_r(|\cdot|^{\lambda - \frac{1}{p}})(x) = \frac{1}{2r} \int_{x-r}^{x+r} y^{\lambda - \frac{1}{p}} dy$$

$$= \left(\lambda - \frac{1}{p} + 1\right)^{-1} \frac{1}{2r} \left((x+r)^{\lambda - \frac{1}{p} + 1} - (x-r)^{\lambda - \frac{1}{p} + 1}\right)$$

$$= \left(\lambda - \frac{1}{p} + 1\right)^{-1} \frac{x^{\lambda - \frac{1}{p} + 1}}{2r} \left[\left(1 + \frac{r}{x}\right)^{\lambda - \frac{1}{p} + 1} - \left(1 - \frac{r}{x}\right)^{\lambda - \frac{1}{p} + 1}\right]$$

and

$$A_r(|\cdot|^{\lambda-\frac{1}{p}})(x)-x^{\lambda-\frac{1}{p}}$$

$$=\left(\lambda-\frac{1}{p}+1\right)^{-1}\frac{x^{\lambda-\frac{1}{p}+1}}{2r}\left[\left(1+\frac{r}{x}\right)^{\lambda-\frac{1}{p}+1}-\left(1-\frac{r}{x}\right)^{\lambda-\frac{1}{p}+1}-2\left(\lambda-\frac{1}{p}+1\right)\frac{r}{x}\right]$$

Indeed, according to Taylor's formula, there exist ξ and η such that $1-x<\eta<1<\xi<1+x$ and

$$\begin{aligned}(1+y)^{\mu}-(1-y)^{\mu}-2\mu x &= 1+\mu x+\frac{\mu(\mu-1)}{2}x^2+\frac{\mu(\mu-1)(\mu-2)}{6}x^3\\ &+\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{24}\xi^{\mu-4}x^4\\ &-\left(1-\mu x+\frac{\mu(\mu-1)}{2}x^2-\frac{\mu(\mu-1)(\mu-2)}{6}x^3\right.\\ &\left.+\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{24}\eta^{\mu-4}x^4\right)-2\mu x\\ &=\frac{\mu(\mu-1)(\mu-2)}{3}x^3+\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{24}\left(\xi^{\mu-4}-\eta^{\mu-4}\right)x^4\\ &\geq\frac{\mu(\mu-1)(\mu-2)}{3}x^3.\end{aligned}$$

Using the inequality

$$(1+y)^{\mu}-(1-y)^{\mu}-2\mu y\geq\frac{\mu(1-\mu)(2-\mu)}{3}y^3$$

which is valid for any $0<\mu<1$ and $0<y<1$. Hence, assuming $\mu=\lambda-\frac{1}{p}+1$ and $y=\frac{r}{x}$, for some $c_{\lambda}>0$, depending only on λ and p, for any $0<r<x$:

$$A_r(|\cdot|^{\lambda-\frac{1}{p}})(x)-x^{\lambda-\frac{1}{p}}\geq c_{\lambda}x^{\lambda-\frac{1}{p}}\left(\frac{r}{x}\right)^2.$$

Therefore,

$$\left\| A_r(|\cdot|^{\lambda-\frac{1}{p}})(x) - |x|^{\lambda-\frac{1}{p}} \right\|_{M_p^\lambda} = \sup_{z\in\mathbb{R},\, r>0} r^{-\lambda} \left\| A_r(|\cdot|^{\lambda-\frac{1}{p}})(x) - |x|^{\lambda-\frac{1}{p}} \right\|_{L_p(z-r,\, z+r)}$$

$$\geq r^{-\lambda} \left\| A_r(|\cdot|^{\lambda-\frac{1}{p}})(x) - |x|^{\lambda-\frac{1}{p}} \right\|_{L_p(2r,\, 4r)}$$

$$\geq c_\lambda r^{-\lambda} \left\| x^{\lambda-\frac{1}{p}} \left(\frac{r}{x}\right)^2 \right\|_{L_p(2r,\, 4r)}$$

$$\geq c_\lambda\, r^{-\lambda} (4r)^{\lambda-\frac{1}{p}} \left(\frac{1}{4}\right)^2 (2r)^{\frac{1}{p}} = c_\lambda\, 4^{\lambda-\frac{1}{p}-2} 2^{\frac{1}{p}} > 0,$$

which follows from equation (23).

From (23) and Lemma 9, we obtain (24):

Finally,

$$\left\| |x|^{\lambda-\frac{1}{p}} \chi_{\underset{B(0,\, r)}{c}}(x) \right\|_{M_p^\lambda} = \sup_{z\in\mathbb{R}^n,\, \rho>0} \left\| |x|^{\lambda-\frac{1}{p}} \chi_{\underset{B(0,\, r)}{c}}(x) \right\|_{L_p(x-\rho,\, x+\rho)}$$

$$\geq \sup_{\rho>r} \rho^{-\lambda} \left\| |x|^{\lambda-\frac{1}{p}} \right\|_{L_p((0,\, \rho)\cap(r,\, \infty))} = \sup_{\rho>r} \rho^{-\lambda} \left(n v_n \int_r^\rho x^{\lambda p-1} dx \right)^{\frac{1}{p}}$$

$$\geq \lim_{\rho\to+\infty} \rho^{-\lambda} \left(\frac{n v_n}{\lambda p} \left(\rho^{\lambda p-1} - r^{\lambda p-1} \right) dx \right)^{\frac{1}{p}} = \left(\frac{n v_n}{\lambda p} \right)^{\frac{1}{p}},$$

from which (25) is as follows.

4. Discussion and conclusion

In this study, sufficient conditions for the precompactness of sets in generalised Morrey spaces were provided in terms of the averaging function. This result serves as an analogue of the well-known Fréchet-Kolmogorov theorem, which provides the conditions for the compactness of sets in Lebesgue spaces. Generalisation to Morrey spaces extends the applicability of compact criteria to a broader class of functional spaces, reflecting the interplay between local regularities and global behaviour in these spaces. The obtained results can be used to prove the compactness of the integral operators in generalised Morrey spaces. A similar theorem in terms of uniform equicontinuity, specifically, the pre-compact theorem for sets in Morrey spaces, was considered in [1] and applied to prove the compactness of the commutator for the singular Calderón-Zygmund integral operator in [3].

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Conflict of interest

The authors declare no competing financial interest.

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