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Research Article

The Control Policy $M/M/c/N$ Interrelated Queue with Manageable Incoming Rates, Reverse Balking and Impatient Customers

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Abstract: This study explores an interconnected $M/M/c/N$ queueing model that incorporates reverse balking, reneging, a control technique, and adjustable arrival rates. In reverse balking, the likelihood of a visitor deciding to join or leave a queue is influenced by the current system size, with a higher probability of joining as the system size grows. Conversely, long wait times can lead to customer impatience, causing them to leave the queue before receiving service (reneging). Service begins with the arrival of the initial customer (T_L) and continues until the system is empty. We are developing a multi-server queueing system with adjustable arrival rates that includes reverse balking and considers impatient customers. For this model, we determine the steady-state condition and performance metrics such as average waiting time and average system size.

Keywords: interrelated queue, incoming rates, departure rates, reverse balking, control policy

MSC: 90B22, 60K30, 60K25

1. Introduction

Queueing concepts have been successfully applied to the formulation and study of numerous systems, including congestion, service, and communications networks. There have been several modifications made to the fundamental queueing models, and new ideas have emerged. The person being served is not usually considered as being in a queue. Queueing theory explains how the queue works and its properties. Queueing problems have become quite important in the decision-making process in the current population growth and globalization of worldwide trade.

Nowadays business environment is extremely difficult because of unpredictability. Unpredictability can be found in numerous aspects of a functioning organization, such as a volatile economy, unpredictable natural disasters and customer behavior. Therefore, business enterprises have extremely restricted possibilities for error. Each business seeks effective risk management and accurate future forecasting. One of the most ambiguous aspects of the corporate environment is customers behavior. Consumer dissatisfaction in a given company rises as a result of greater levels of expectations. As a result, customer dissatisfaction is a major challenge for business.

The current era of progress and liberalization has made it challenging to run a business. Clients are now more selective in their choices. The rate at which customers switch brands has risen. Increased expectations lead to heightened

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frustration with particular companies. In the business realm, customer dissatisfaction has become a significant concern. Various service systems facing impatient clients can utilize queueing theory's multiple models. Numerous studies explore the concept of balking. These models indicate that the size of the system and the length of wait times influence the rate of balking. As the system size grows, instances of balking increase. However, in investment scenarios, a larger number of clients at a particular firm can draw in potential investors. Therefore, in sectors like customer investment, the likelihood of balking decreases with a larger system size, and conversely, this phenomenon is referred to as reverse balking.

Reneging and balking are usual problems in queueing systems. Customers may choose not to join the queue if there is a lengthy line in the system or no waiting space, or they may leave the queue after joining it because they are impatient and cannot wait to be served. Any business that has impatient customers suffers. It results in the loss of potential clients. The control policy queue was investigated by many researchers. The term "control policy", refers to the concept of a server remaining inactive until T_L consumers have joined the queue. The service starts upon the arrival of the Nth customer and continues until the system is empty.

The objectives and scope of the research work are, the article investigates a multi-server queueing model with limited capacity and adjustable arrival rates. It aims to understand the behaviors of customers where customers are more likely to join a queue when it is longer, contrary to typical balking behavior (reverse-balking), those who leave the queue after joining (reneging), and the use of additional servers to manage the system efficiently. The study examines the steady-state solution and characteristics of the queueing model. It includes numerical examples to demonstrate the impact of adjustable arrival rates on system performance and customer behavior. The findings are intended to help optimize queue management in various practical scenarios like investment firms and share markets.

The real world validation (example) of the queueing model: At Prime Investments, a busy firm with three financial advisors (servers), clients encountered a distinctive queueing system. The arrival rate (follows Poisson distribution) of clients was adjusted dynamically, increasing when the queue was short and decreasing as it grew longer. Many clients joined longer queues, viewing them as an indicator of the firm's success (reverse balking). However, some clients would leave if the wait became too lengthy or some personal issues (reneging). To handle this, the firm adopted Threshold-policy, where advisors would start consultations (service) only when a minimum number of clients in the queue and service follows exponential-distribution, were in the queue, ensuring efficient use of their time and maintaining high service standards.

This paper develops a manageable $M/M/c$ queue model with limited capacity, incorporating impatient consumers, reverse balking, and a control policy. An equilibrium solution is produced. The remaining part is organized as in the following sections: the literature review is given in section 2, a list of the presumptions used to create the model is provided in section 3, steady-state solution is created according to the model in section 4. Section 5 outlines the characteristics of the model; performance measures are given in section 6, particular cases are listed in section 7; numerical illustrations are calculated in section 8, conclusions are discussed in section 9.

2. Literature review

Queuing system with balking and client reneging was studied by [1, 2]. Barrer [3] discusses a queueing problem marked by customers' impatience and personnel apathy. Barrer [4] looks at a wait problem involving impatient customers who receive organised service. Ancker et al. [5] conduct a stationary investigation of a Markovian queue with a finite capacity with balking and reneging. The data used to produce the study results is generated using a single server queueing system with balking. A single server non-Markovian queueing model including interruptions, reneging, and balking was studied by [6]. In [7] studied the stationary workload of the $G/M/1$ queue with impatient customers provides valuable insights into customer impatience. However, it does not consider the impact of controllable arrival rates or the reverse balking. Manoharan and Jose [8] raise the notion of balking in the context of $M/M/1$ queueing. $M/M/1$ queue having state-dependent reneging and finite buffers Check out Balking [9]. They obtain data from a mathematical model that remains steady. In this article [10], the authors look at input, disasters, recovery, and dissatisfied users in addition to a single server queueing problem. Networks for communication employ this concept. Kumar [11] examines a multi-server queuing strategy where users are reneging and balking. A matrix approach is applied to generate the model's transient form. An economic analysis of the model is developed. In [12], a cost model was developed for a single-server, limited-capacity Markovian feedback queue with retention. They maximize the cost of service and system performance. Kumar and Sharma [13] take into account the Markovian queue with numerous channels, balking, and keeping impatient clients as feedback from customers. In [14], an $M/M/1/N$ queue with reverse balking is considered. They compute the stable probability of system size and develop formulas for basic performance measurements. Many different presumptions are frequently made, like the one that customer service and input processes are separate. However there are a lot of unique situations where the arrival and service protocols are connected; therefore, this needs to be considered. A model that links arrivals with services is called an "interrelated queue". The interconnected standard queue with controllable arrival rates has received much attention in the literature. A controlled arrivals rate $M/M/1$ interdependent queuing model is described in [15]. The associated limited area $M/M/1$ queue with controlled arrival rates, $M/M/c$ with controllable arriving rates, and $M/M/c$ with controllable arrival rates, as well as impatient clients and balking consumers, were explored in [16–18]. Nisha and Thiagarajan [19, 20] covered impatient clients, retrial queueing methods with controllable arrival rates on one server, and retrial queueing structures with similar features on various servers. In [21] looked towards a Poisson queue for an N-policy with an average setup time. Through the use of more than one vacation plus a matrix geometric approach, Ayyappan and Subramanian [22] examined the $M/M/1$ retrial orbit queuing system with N-Policy, focusing on non-preemptive prioritized service. The Poisson entry waiting system beneath control-operating policy and with a starting time was studied by [23]. Balaji and Saradha [24] discussed an $M/M/1/K$ queuing model with controllable arrival rates, reverse-balking provide valuable insights into the impact of controllable arrival rates on system performance. A Markovian multi-server controlled limited capacity queuing model that includes reneging and control policy is extended with the notion of reverse balking. We identify the model's steady state and run an analytical evaluation on it.

Critical appraisal of the existing approaches: Queuing system with balking and reneging laid the groundwork for understanding customer behavior in queues discussed by [1, 2]. However, the models are limited to simple scenarios and do not account for more complex systems with multiple servers or controllable arrival rates. Barrer [3, 4] studied the impatient customers and indifferent clerks introduced the concept of customer impatience in queuing systems. While this models are insightful, they do not consider the impact of controllable arrival rates. Ancker and Gafarian [5] studied a queuing problems with balking and reneging expanded the understanding of these behaviors in more complex systems. However, their models still lack the consideration of controllable arrival rates and the reverse-balking. Rao [6] introduced the concept of interruptions in queuing systems. While this adds a layer of complexity, their models do not address the controllability of arrival rates. Bae and Kim [7] studied the stationary workload of the $G/M/1$ queue with impatient customers provides valuable insights into customer impatience. However, it does not consider the impact of controllable arrival rates or the reverse balking. Manoharan and Jose [8] worked on Markovian queuing systems with random balking adds randomness to customer behavior. However, it does not address the controllability of arrival rates or the reverse-balking. The study on balking and reneging in finite buffer queues provides insights into customer behavior in limited capacity systems was studied by [9]. However, it does not consider the impact of controllable arrival rates or the reverse balking. Kumar et al. [10, 12] worked on queuing with reneging, balking, and retention of reneged customers introduces the concept of retaining customers who have reneged. While this adds a new dimension to queuing models, it does not address the controllability of arrival rates or reverse balking concept. Srinivasan and Thiagarajan [16, 17] worked on interdependent and independent queuing models with controllable arrival rates provide valuable insights into the impact of controllable arrival rates on system performance. However, their models do not consider the use of reverse balking and reneging. Nisha and Thiagarajan [19, 20] developed a retrial queuing models with impatient customers and controllable arrival rates adds a layer of complexity to queuing models. However, it does not address the use of reverse balking. Balaji and Saradha [24] discussed an $M/M/1/K$ queuing model with controllable arrival rates, reverse-balking provide valuable insights into the impact of controllable arrival rates on system performance. However, their models do not consider the reneging and multiple server concepts.

3. Model assumptions

Consider $M/M/c/N$ where customers are admitted based on a Poisson input flow with the rates Δ_a and Δ_b and where holding durations are based on a distribution that is exponential with rate υ. Having a joint probability mass function of the form, assumed to be interrelated, and following a bivariate Poisson process is the system's arrival process $[Y_1(m_0)]$ and service process $[Y_2(m_0)]$.

$$P(Y_1 = y_1,\ Y_2 = y_2;\ m_0) = e^{-(\Delta_d + \upsilon_g - \zeta)m_0} \sum_{g=0}^{\min(y_1,\ y_2)} \frac{(\zeta m_0)^g \left[(\Delta_d - \zeta)\, m_0\right]^{y_1 - g} \left[(\upsilon_g - \zeta)\, m_0\right]^{(y_2 - g)}}{g!\,(y_1 - g)!\,(y_2 - g)!} \tag{1}$$

Where,

$$y_1,\ y_2 = 0,\ 1,\ 2,\ \ldots,\ \ 0 < \Delta_d,\ \upsilon_g;\ 0 \le \zeta < \min(\Delta_d,\ \upsilon_g),\ \ d = 0$$

$$g = 0,\ 1,\ 2,\ \ldots\ T_L - 1,\ T_L,\ T_L + 1\ \ldots\ c - 1,\ c,\ c + 1,\ \ldots\ m_0 - 1,\ m_0,\ m_0 + 1,\ \ldots\ M_0 - 1,\ M_0,\ M_0 + 1,\ \ldots\ N$$

with parameters $\Delta_a,\ \Delta_b,\ \upsilon_g$ and ζ as the expected rate of departures, input, and expected dependence rates. It is supposed that $c < r$.

$$\upsilon_g = \begin{cases} g\upsilon; & 0 \le g < c \\ c\upsilon; & c \le g \le T_L \end{cases}$$

The system space is supposed to be finite, first-come, first-served is the queue order policy. Customers are not balking with probability $r' = 1 - q'$. Customers balk probability $1 - \dfrac{g}{N-1}$ and not balking $\dfrac{g}{N-1}$ when the service facility is not free. It is named to be reverse balking. Each consumer waits in queue for a while before receiving service. When that time comes and he still hasn't received service, he reneges and exits the queue. The times at which reneging occurs follow an exponential distribution with the parameter α.

The both incoming and outgoing processes are assumed to be associated and follow a Poisson process. The arrival rates are defined as Δ_a for a higher rate and Δ_b for a moderate rate. When the queue length reaches a specified threshold (T_L) (where $(T_L > N)$), the arrival rate decreases from Δ_a to Δ_b and remains at this lower rate until the queue size exceeds another specified number $m_0(m_0 \ge 0\ \&\ m_0 < T_L)$ and $(m_0 < T_L)$. Once the queue size drops below (T_L), the arrival rate reverts to Δ_a, and this cycle continues. Figure 1 shows this model.

The model's underlying assumptions are:

i) There is a chance that no arrivals with reverse balking will happen during any given period of length w. If the system detects an increase in arrival frequency, no service will be provided.

$$1 - \left[(\Delta_a - \zeta)\, r' + (\upsilon_g - \zeta)\right] f + v(f)$$

ii) There is a chance that one arrival with reverse balking will happen during any given period of length w. If the system detects an increase in arrival frequency, no service will be provided.

$$(\Delta_a - \zeta)\, r' f + v(f).$$

iii) The probability that no arrival with reverse balking will occur for any given length of period w and no service will be provided if the system observes an upward arrival frequency is

$$1-\left[\left(\frac{g}{N-1}\right)(\Delta_a-\zeta)r'+(\upsilon_g-\zeta)\right]f+v(f)$$

and downward arrival frequency is

$$1-\left[\left(\frac{g}{N-1}\right)(\Delta_b-\zeta)r'+(\upsilon_g-\zeta)\right]f+v(f)$$

iv) There is a potential that zero arrivals with reverse balking will happen for a particular period of time f at state g and there will only be one service available when the system is either in upward or downward arrival rate is $(\upsilon_g-\zeta)f+v(f)$.

v) The chances that there is one arrival with reverse balking and one service completion during a small interval of time f when the system is either in upward or downward arrival frequency is $\zeta f+v(f)$.

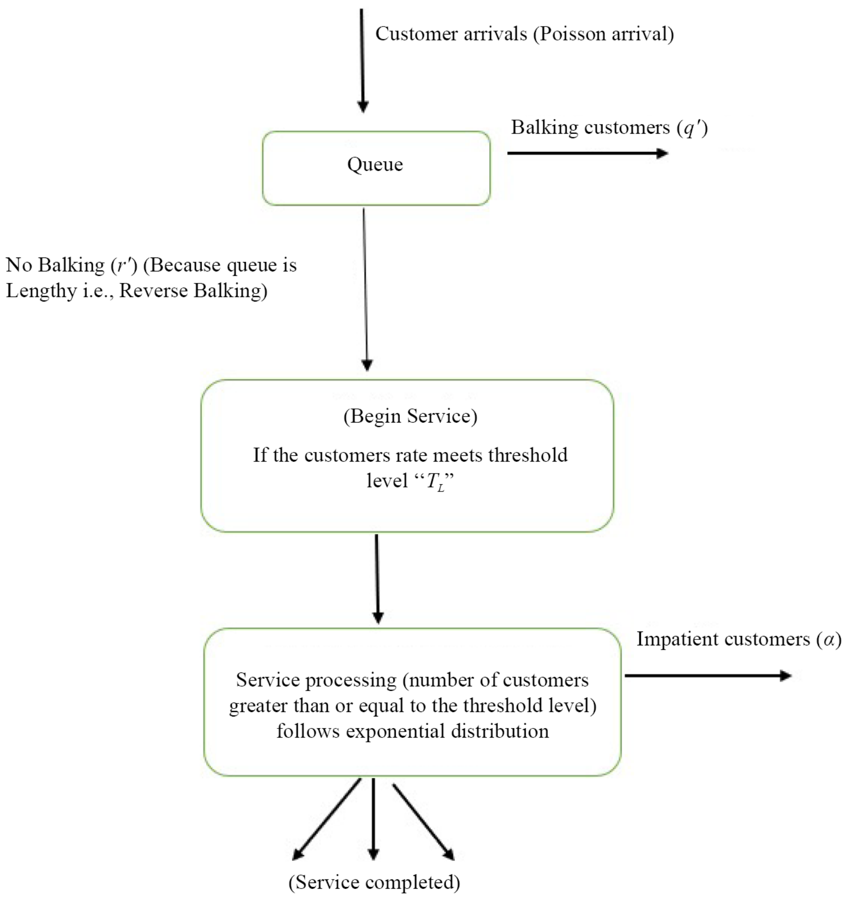

Figure 1. Flow chart of the queueing model

4. Problem formulation and solution

If the system has an upward arrival frequency and the server is not in use, then let $P_{0,\ g,\ 0}$ signify the probability of "g" users in the system. $P_{1,\ g,\ 0}$ represents that there are g clients in the system when the server is busy and the system has an upward arrival frequency. $P_{1,\ g,\ 1}$ denotes the probability of g clients in the service facility when the server is overloaded and the system is experiencing a downward arrival frequency. We notice that $P_{0,\ g,\ 0}$ occurs when $g = 0,\ 1,\ 2,\ \ldots T_L - 1$ and $P_{1,\ g,\ 0}$ occurs when $g = 0,\ 1,\ 2,\ \ldots T_L - 1,\ T_L \ldots\ m_0 - 1,\ m_0$ both $P_{1,\ g,\ 0}$ and $P_{1,\ g,\ 1}$ occurs when $g = m_0 + 1,\ m_0 + 2,\ \ldots,\ M_0 - 1$ and $P_{1,\ g,\ 1}$ occurs when $g = M_0,\ M_0 + 1,\ \ldots,\ N$. These dependence patterns are present in the following steady-state equations.

$$0 = -(\Delta_a - \zeta) P_{0,\ g,\ 0} + (\Delta_a - \zeta) P_{0,\ g-1,\ 0} \quad 1 \le g \le T_L - 1 \tag{2}$$

$$0 = -(\Delta_a - \zeta) r' P_{0,\ 0,\ 0} + (\upsilon - \zeta) P_{1,\ g,\ 0} \tag{3}$$

$$0 = -\left[\left(\frac{1}{N-1}\right)(\Delta_a - \zeta) + (\upsilon - \zeta)\right] P_{1,\ g,\ 0} + 2(\upsilon - \zeta) P_{1,\ 2,\ 0} + (\Delta_a - \zeta) r' P_{0,\ 0,\ 0} \tag{4}$$

$$0 = -\left[\left(\frac{g}{N-1}\right)(\Delta_a - \zeta) + g(\upsilon - \zeta)\right] P_{1,\ g,\ 0} + (g+1)(\upsilon - \zeta) P_{1,\ g+1,\ 0}$$

$$+\left(\frac{g-1}{N-1}\right)(\Delta_a - \zeta) P_{0,\ g-1,\ 0} \quad g = 2,\ 3,\ \ldots\ldots,\ T_L - 1 \tag{5}$$

$$0 = -\left[\left(\frac{T_L}{N-1}\right)(\Delta_a - \zeta) + T_L(\upsilon - \zeta)\right] P_{1,\ T_L,\ 0} + (T_L + 1)(\upsilon - \zeta) P_{1,\ T_L+1,\ 0}$$

$$+\left(\frac{T_L - 1}{N-1}\right)(\Delta_a - \zeta) P_{0,\ T_L-1,\ 0} + \left(\frac{T_L - 1}{N-1}\right)(\Delta_a - \zeta) P_{1,\ T_L-1,\ 0} \tag{6}$$

$$0 = -\left[\left(\frac{g}{N-1}\right)(\Delta_a - \zeta) + g(\upsilon - \zeta)\right] P_{1,\ g,\ 0} + (g+1)(\upsilon - \zeta) P_{1,\ g+1,\ 0}$$

$$\left(\frac{g-1}{N-1}\right)(\Delta_a - \zeta) P_{0,\ g-1,\ 0} \quad T_L < g < c \tag{7}$$

$$0 = -\left[\left(\frac{g}{N-1}\right)(\Delta_a - \zeta) + c(\upsilon - \zeta) + (g - c)\alpha\right] P_{1,\ g,\ 0}$$

$$+[c(\upsilon - \zeta) + (g + 1 - c)\alpha] P_{1,\ g+1,\ 0} + \left(\frac{g-1}{N-1}\right)(\Delta_a - \zeta) P_{0,\ g-1,\ 0} \quad c - 1 < g < m_0 \tag{8}$$

$$0 = -\left[\left(\frac{m_0}{N-1}\right)(\Delta_a - \zeta) + c(\upsilon - \zeta) + (m_0 - c)\alpha\right] P_{1,\ m_0,\ 0}$$

$$+ [c(\upsilon - \zeta) + (m_0 + 1 - c)\alpha] P_{1,\ m_0+1,\ 0} + [c(\upsilon - \zeta) + (m_0 + 1 - c)]\alpha P_{1,\ m_0+1,\ 1}$$

$$+ \left(\frac{m_0 - 1}{N-1}\right)(\Delta_a - \zeta) P_{0,\ m_0-1,\ 0} \tag{9}$$

$$0 = -\left[\left(\frac{g}{N-1}\right)(\Delta_a - \zeta) + c(\upsilon - \zeta) + (g - c)\alpha\right] P_{1,\ g,\ 0}$$

$$+ [c(\upsilon - \zeta) + (g + 1 - c)\alpha] P_{1,\ g+1,\ 0} + \left(\frac{g-1}{N-1}\right)(\Delta_a - \zeta) P_{0,\ g-1,\ 0} \quad m_0 < g < M_0 - 1 \tag{10}$$

$$0 = -\left[\left(\frac{M_0 - 1}{N-1}\right)(\Delta_a - \zeta) + c(\upsilon - \zeta) + (M_0 - 1 - c)\alpha\right] P_{1,\ M_0-1,\ 0}$$

$$+ \left(\frac{M_0 - 2}{N-1}\right)(\Delta_a - \zeta) P_{0,\ M_0-2,\ 0} \tag{11}$$

$$0 = -\left[\left(\frac{m_0 + 1}{N-1}\right)(\Delta_b - \zeta) + c(\upsilon - \zeta) + (m_0 + 1 - c)\alpha\right] P_{1,\ m_0+1,\ 1}$$

$$+ [c(\upsilon - \zeta) + (m_0 + 2 - c)\alpha] P_{1,\ m_0+2,\ 1} \tag{12}$$

$$0 = -\left[\left(\frac{g}{N-1}\right)(\Delta_b - \zeta) + c(\upsilon - \zeta) + (g - c)\alpha\right] P_{1,\ g,\ 1}$$

$$+ [c(\upsilon - \zeta) + (g + 1 - c)\alpha] P_{1,\ g+1,\ 1} + \left(\frac{g-1}{N-1}\right)(\Delta_b - \zeta) P_{0,\ g-1,\ 1} \quad m_0 + g < g < M_0 \tag{13}$$

$$0 = -\left[\left(\frac{M_0}{N-1}\right)(\Delta_b - \zeta) + c(\upsilon - \zeta) + (M_0 - c)\alpha\right] P_{1,\ M_0,\ 1} + [c(\upsilon - \zeta) + (M_0 + 1 - c)\alpha] P_{1,\ M_0+1,\ 1}$$

$$+ \left(\frac{M_0 - 1}{N-1}\right)(\Delta_a - \zeta) P_{0,\ M_0-1,\ 0} + \left(\frac{M_0 - 1}{N-1}\right)(\Delta_b - \zeta) P_{0,\ M_0-1,\ 1} \tag{14}$$

$$0 = -\left[\left(\frac{g}{N-1}\right)(\Delta_b - \zeta) + c(\upsilon - \zeta) + (g-c)\alpha\right] P_{1,\ g,\ 1} + [c(\upsilon - \zeta) + (g+1-c)\alpha] P_{1,\ g+1,\ 1}$$

$$+\left(\frac{g-1}{N-1}\right)(\Delta_b - \zeta) P_{g,\ 0-1,\ 1} \tag{15}$$

$$0 = \left(\frac{g-1}{N-1}\right)(\Delta_b - \zeta) P_{0,\ N-1,\ 1} - [c(\upsilon - \zeta) + (g+1-c)\alpha] P_{1,\ N,\ 1} \tag{16}$$

Let $H_0 = (\Delta_a - \varepsilon)$; $H_1 = (\Delta_b - \varepsilon)$; $G = (\upsilon - \varepsilon)$.
From (2) to (11) we get

$$P_{0,\ g,\ 0} = P_{0,\ 0,\ 0} \quad 0 \le g < T_L \tag{17}$$

$$P_{1,\ g,\ 0} = \left[\frac{(g-1)!}{(N-1)^{g-1}} \prod_{n=1}^{g} \frac{H_0}{nG}\right] r' P_{0,\ 0,\ 0} \quad 0 < g < T_L \tag{18}$$

$$P_{1,\ g,\ 0} = \left[\frac{(g-1)!}{(N-1)^{g-1}} \prod_{n=1}^{g} \frac{H_0}{nG}\right] r' P_{0,\ 0,\ 0} - \frac{(g-1)H_0}{(N-1)(g+1)G} P_{1,\ g-1,\ 0} \quad g-1 < g < c-1 \tag{19}$$

$$P_{1,\ g,\ 0} = \left[\frac{(g-1)!}{(N-1)^{g-1}} \prod_{d=c}^{g} \frac{H_0}{cG + (d-c)\alpha} \prod_{e=1}^{c-1} \frac{H_0}{eG}\right] r' P_{0,\ 0,\ 0}$$

$$- \frac{(g-1)H_0}{(N-1)(g+1)G} P_{1,\ g-1,\ 0} \quad c < s < m_0 + 1 \tag{20}$$

$$P_{1,\ g,\ 0} = \left[\frac{(g-1)!}{(N-1)^{g-1}} \prod_{d=c}^{g} \frac{H_0}{cG + (d-c)\alpha} \prod_{e=1}^{c-1} \frac{H_0}{eG}\right] r' P_{0,\ 0,\ 0}$$

$$- \frac{(g-1)H_0}{(N-1)(g+1)G} P_{1,\ g-1,\ 0}$$

$$- \left[\sum_{s=p'+1}^{Q-1} \left(\frac{H_0}{(N-1)cG + (p'+1-c)\alpha}\right)^{g-s} (g-1) P_{g-s}\right] P_{1,\ p'+1,\ 1} \quad m_0 < g < M_0 \tag{21}$$

From (12) to (16) we get

$$P_{1,\ g,\ 1} = \left[\sum_{s=m_0+1}^{M_0-1} \left(\frac{H_1}{(N-1)cG+(s-c)\alpha}\right)^{g-s} (g-1)P_{g-s}\right] P_{1,\ m_0+1,\ 1}, \quad m_0 < g \leq N \tag{22}$$

Where,

$$P_{1,\ m_0+1,\ 1} = \frac{P' \dfrac{(M_0-1)! \qquad H_0^Q}{(N-1)^{M_0-1}\left[c^{M_0-c}G^{M_0}+(M_0-1-c)\gamma\right]}}{\displaystyle\sum_{h=m_0}^{M_0-1} \frac{(M_0-1)!}{h!}\left(\frac{H_0}{(N-1)cG+(h-c)\alpha}\right)^{Q-h-1}} P_{0,\ 0,\ 0}$$

Using the normalizing condition $P_{0,\ g,\ 0} + P_{1,\ g,\ 0} + P_{1,\ g,\ 1} = 1$, the probability $P_{0,\ 0,\ 0}$ of the free system can be considered.

$$P_{0,\ 0,\ 0} = \left[\frac{1}{P_{1,\ g,\ 0}+P_{1,\ g,\ 0}+P_{1,\ g,\ 1}}\right] \tag{23}$$

$P_{0,\ g,\ 0}$ is used to find the probability that g customers in the system and it has an upward arrival frequency and the server is not in use. It occurs when $(g = 0,\ 1,\ 2,\ \ldots\ T_L - 1)$.

$P_{1,\ g,\ 0}$ is used to find that there are g clients in the system when the server is busy and the system has an upward arrival frequency. It occurs only when $(g = 0,\ 1,\ 2,\ \ldots\ T_L - 1,\ T_L,\ \ldots\ m_0 - 1,\ m_0)$ and also occurs $(g = m_0 + 1,\ m_0 + 2,\ \ldots\ T_L - 1)$.

$P_{1,\ g,\ 1}$ is used to find the probability of g clients in the service facility when the server is overloaded and the system is experiencing a downward arrival frequency. Occurs when $(g = T_L,\ T_L + 1,\ \ldots\ K)$ and also occurs $(g = m_0 + 1,\ m_0 + 2,\ \ldots\ T_L - 1)$.

$P_{0,\ 0,\ 0}$ denotes the normalization condition, states that the sum of all probabilities of all possible states in the system equals to 1.

The server is not in use with probability assumed by $P_{0,\ g,\ 0} = \sum_{g=0}^{T_L-1} P_{0,\ g,\ 0}$. The probability of the system reaching N and the server being busy with increasing upward arrival frequency is

$$P_{1,\ g,\ 0} = \sum_{g=1}^{T_L} P_{1,\ g,\ 0} \tag{24}$$

The system is in upward arrival frequency with probability

$$P_{1,\ g,\ 0} = \sum_{g=T_L}^{M_0-1} P_g(0)$$

When g reaches $0,\ 1,\ 2,\ \ldots.\ T_L - 1,\ T_L,\ T_L + 1\ \ldots,\ c - 1,\ c,\ c + 1,\ \ldots,\ m_0,\ m_0 + 1,\ \ldots,\ M_0 - 1$, we get

$$P(0) = \sum_{g=0}^{T_L-1} P_{0,\ g,\ 0} + \sum_{g=1}^{T_L} P_{1,\ g,\ 0} + \sum_{g=T_L+1}^{c} P_{1,\ g,\ 0} + \sum_{g=c+1}^{m_0} P_{1,\ g,\ 0} + \sum_{g=m_0+1}^{M_0-1} P_{1,\ g,\ 0} \tag{25}$$

The system is in downward arrival frequency with probability

$$P(1) = \sum_{g=p'+1}^{N} P_g(1) \tag{26}$$

When g attains $m_0+1,\ m_0+2,\ \ldots,\ M_0-2,\ M_0-1,\ \ldots,\ N$ we get

$$P(1) = \sum_{g=0}^{m_0} P_{1,\ g,\ 1} + \sum_{g=p'+1}^{M_0-1} P_{1,\ g,\ 1} + \sum_{g=M_0}^{N} P_{1,\ g,\ 1} \tag{27}$$

Usefulness of the proposed model: The queueing model explored in the control policy $M/M/c/N$ interrelated queue with manageable incoming rates, reverse balking and impatient customers offers significant benefits across various sectors, including investment firms, telecommunications, computer networks, manufacturing, and service industries. By analyzing and controlling arrival rates reneging and reverse balking behavior, organizations can better allocate resources, increase customer satisfaction, and improve overall system efficiency.

4.1 *Performance measures*

The rationale behind the four key queueing performance measures L_s, W_s, L_q, and W_q lies in their ability to provide a comprehensive understanding of a queuing system's efficiency and effectiveness. L_s represents the average number of customers in the system, helping to gauge overall load and capacity planning. W_s indicates the average time a customer spends in the system, crucial for assessing customer satisfaction and service efficiency. L_q measures the average number of customers waiting in the queue, highlighting congestion levels and aiding in queue management. Finally, W_q reflects the average waiting time in the queue, essential for identifying bottlenecks and improving service processes. Together, these metrics offer valuable insights for optimizing resource allocation and enhancing system performance.

(i) The predicted system size is provided by

$$L_s = L_{s'} + L_{s''} \tag{28}$$

Where

$$L_{s'} = \sum_{g=0}^{T_L-1} gP_{0,\ g,\ 0} + \sum_{g=1}^{T_L} gP_{1,\ g,\ 0} + \sum_{g=T_L+1}^{c} gP_{1,\ g,\ 0} + \sum_{g=c+1}^{m_0} gP_{1,\ g,\ 0} + \sum_{g=m_0+1}^{M_0-1} gP_{1,\ g,\ 0}$$

$$L_{s''} = \sum_{g=m_0+1}^{M_0-1} gP_{1,\ g,\ 1} + \sum_{g=M_0}^{N} gP_{1,\ g,\ 1}$$

(ii) The expected system holding time

$$W_s = \frac{L_s}{\bar{\Delta}}$$

Where,

$$\bar{\Delta} = \Delta_a P(0) + \Delta_b P(1) \tag{29}$$

(iii) The expected size of the queue is

$$L_q = \Delta W_q \tag{30}$$

(iv) The expected holding time in the queue

$$W_q = W_s - \frac{1}{\upsilon} \tag{31}$$

5. Particular cases

When c (server) = 1, this system becomes to a single server limited capacity interconnected queueing model with an threshold policy, reverse balking, and impatient customers. When Δ_a tends to Δ_b, $c = 1$, $N = \infty$ and ζ = 0, this model reduces to a single server with control-arrival policy, impatient customer queueing mechanism. When T_L tends to c (server), $\zeta = 0$, and N (capacity) = T_L (threshold level), this type turns to a multiple processor queuing model with impatient customers and reverse balking. If T_L tends to c, and α (reneging) = 0 this model reduces to multiple server queues with controllable arrival rates and reverse balking.

6. Model sensitivity analysis

In this below Table 1 represents the values for c, $q^{'}$, r', υ, m_0, M_0, g, N, Δ_a, and then Table 2 shows the variation occurs when the empty system ($P_{0,\ 0,\ 0}$), probability of the system is in upward arrival frequency-$P(0)$, probability that the system is in downward arrival frequency-$P(1)$. It increases in the rate of reneging results increase in $P_0(0)$, $P(0)$ and decrease in $P(1)$, the size of the expected system, waiting time of the system, expected queue length and waiting time of the queue with respect to υ, Δ_a, Δ_b

Table 1. Values for c, $q^{'}$, r', υ, m_0, M_0, g, N, Δ_a

c	$q^{'}$	r'	υ	m_0	M_0	g	N	Δ_a
3	0.8	0.2	3	5	12	2	15	8

Table 2. Variation occurs when the empty system ($P_{0,\,0,\,0}$), probability of the system is in upward arrival frequency-$P(0)$, probability that the system is in downward arrival frequency-$P(1)$, queue length-L_q, queue waiting time-W_q, length of the system-L_s and waiting time of the system-W_s with respect to υ, Δ_a, Δ_b

S. No	Δ_a	Δ_b	υ	α	ζ	$P_{0,\,0,\,0}$	$P(0)$	$P(1)$	L_q	W_q	L_s	W_s
1	8	10	3.0	0.1	0.3	0.3622	0.9574	0.0426	0.5868	0.1972	0.6979	0.2564
2	8	10	3.2	0.1	0.3	0.4102	0.9926	0.0074	0.5390	0.1627	0.6410	0.1706
3	8	10	3.3	0.1	0.3	0.5293	0.9954	0.0046	0.4665	0.1506	0.5776	0.1546
4	8	10	3.4	0.1	0.3	0.5417	0.9981	0.0019	0.4398	0.1202	0.5398	0.1275
5	8	10	3.5	0.1	0.3	0.5690	0.9992	0.0008	0.4125	0.1432	0.5804	0.1551
6	8	11	3	0.1	0.3	0.3899	0.9765	0.0235	0.4878	0.1134	0.6487	0.1620
7	8	12	3	0.1	0.3	0.3769	0.9654	0.0346	0.5134	0.1345	0.7202	0.2145
8	8	13	3	0.1	0.3	0.3643	0.9542	0.0458	0.6276	0.2658	0.8165	0.3467
9	8	15	3	0.1	0.3	0.3467	0.9323	0.0677	0.8567	0.4942	1.0061	0.6654
10	8	16	3	0.1	0.3	0.3156	0.9145	0.0855	0.8789	0.6843	1.0093	0.8734

In this below Table 3 represents the values for c, Δ_a, Δ_b, m_0, M_0, g, N, Δ_a, and then Table 4 shows the probability of the system is in upward arrival frequency-$P(0)$, probability that the system is in downward arrival frequency-$P(1)$ and variation occurs when the system is empty ($P_{0,\,0,\,0}$). The size of the expected system, waiting time of the system, expected queue length and waiting time of the queue with respect to υ, Δ_a, Δ_b and it increases in the rate of reneging results increase in $P_0(0)$, $P(0)$ and decrease in $P(1)$.

Table 3. Values for c, Δ_a, Δ_b, m_0, M_0, g, N Δ_a

c	Δ_a	Δ_b	m_0	M_0	g	N	Δ_a
3	8	10	5	12	2	15	8

Table 4. Variation occurs when the system is empty ($P_{0,\,0,\,0}$), probability that the system is in upward arrival frequency-$P(0)$ defines the probability of system is in downward arrival frequency-$P(1)$, queue Length-L_q, queue waiting time-W_q and L_s system length, system waiting time-W_s respect to ζ, q', r', α

S. No	α	ζ	q'	r'	$P_{0,\,0,\,0}$	$P(0)$	$P(1)$	L_q	W_q	L_s	W_s
1	0.05	0.3	0.8	0.2	0.5408	0.9970	0.0030	0.3767	0.1412	0.4067	0.1392
2	0.06	0.3	0.8	0.2	0.5281	0.9955	0.0045	0.3654	0.1179	0.3943	0.1187
3	0.07	0.3	0.8	0.2	0.5173	0.9923	0.0077	0.3545	0.1106	0.3922	0.1140
4	0.08	0.3	0.8	0.2	0.4047	0.9873	0.0127	0.3492	0.1095	0.3910	0.1113
5	0.09	0.3	0.8	0.2	0.3908	0.9831	0.0169	0.3421	0.1046	0.3901	0.1509
6	0.1	0.3	0.1	0.9	0.3865	0.9763	0.0257	0.3509	0.1112	0.3927	0.1601
7	0.1	0.3	0.2	0.8	0.3963	0.9879	0.0221	0.3421	0.1098	0.3821	0.1545
8	0.1	0.3	0.3	0.7	0.4031	0.9929	0.0171	0.3384	0.1071	0.3734	0.1421
9	0.1	0.3	0.4	0.6	0.4671	1.0934	0.0366	0.3202	0.0952	0.3698	0.1403
10	0.1	0.3	0.5	0.5	0.4878	1.1267	0.0233	0.3152	0.0994	0.3541	0.1364
11	0.1	0.5	0.5	0.5	0.2289	0.5256	0.4074	0.9248	0.5142	0.9884	0.9432

7. Conclusion

From the preceding table, it is visible that L_s, W_s, L_q, W_q and $P(1)$ rises as the input rate rises while the further parameters remain constant and $P_0(0)$, $P(0)$ drops. As long as the other parameters are held constant, $P_0(0)$, and $P(0)$ increases, $P(1)$, L_q, W_q, L_s and W_s drops as the mean dependence rate rises, L_q, W_q, L_s and W_s grows on a regular basis when the balking rate falls, $P_0(0)$, $P(0)$, decreases and other factors remain constant. It is evident that an increase in the rate of reneging results increase in $P_0(0)$, $P(0)$ and decrease in $P(1)$, the size of the expected system, waiting time of the system, predicted queue length and waiting period of the queue. This is due to the fact that the rate of reneging is rising, which indicates that a growing number of clients are leaving the system before having their service completed. In Table 4 the performance measures L_s and L_q is high (that is, the queue is lengthy), the balking rate decreases (due to reverse balking), while W_q and W_s also increase when the queue size and system size. This study deals the principles of reverse balking and control policy into an $M/M/c/N$ interrelated queueing system with adjustable arrival rates. The model's steady-state analysis provides key performance metrics, and a sensitivity analysis was performed. This approach is especially useful for investment firms dealing with impatient clients. Future research could investigate the application of this concept in a $M/G/1$ queue environment.

Conflict of interest

The authors declare no competing financial interest.

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