Contemporary Mathematics
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Research Article

Families of Gracefuls Spiders with $\ell(2k+1)-k$, $\ell(2k+1)-k+1$ and $\ell(2k+1)+k+1$ Legs

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Received: 15 August 2024; **Revised:** 16 October 2024; **Accepted:** 25 October 2024

Abstract: We say that a tree is a spider if has at most one vertex of degree greater than two. We obtain existence of families of gracefuls spiders with $\ell(2k+1)-k$, $\ell(2k+1)-k+1$ and $\ell(2k+1)+k+1$ legs. We provide specific labels for each spider graph, these labels are constructed from graceful path graphs that have a particular label, so there is a correspondence between some paths and graceful spiders that we are studying, this correspondence is described in an algorithm outlined in the preliminaries.

Keywords: graceful labeling, graph labeling, tree, spider

MSC: 05C78

1. Introduction

A graceful labeling f of a tree $T := (E(T),\ V(T))$, is a bijective function from the set of vertices $V(T)$ of T to the set $\{0,\ 1,\ 2,\ \cdots,\ |E(T)|\}$ such that the set $\{|f(u)-f(v)|:\ \{u,\ v\}\in E(T)\}$ is equal to $\{1,\ 2,\ \cdots,\ |E(T)|\}$, where $E(T)$ is the set of edges of T and $|E(T)|$ is its cardinality.

A tree T is graceful if there is some graceful labeling for T. In 1964, Ringel and Rosa [1, 2] proposed the famous and still unsolved *graceful tree conjecture*, which states that all trees are graceful.

A tree T is a *spider* if it has at most on branch point, that is, at most one vertex v such that its degree $d(v)$ satisfies $d(v)>2$. Let v^* be the unique branch point of a spider T.

Gallian in [3] observed that conjecture for the case of spider graphs is still open; regarding this, there are the following advances. Huang et al. [4] proved that all spider graphics with three or four legs are graceful. Poljak et al. and Bahls et al. [5, 6] also proved that every spider in which lengths of any two its legs differ by at most one is graceful. Jampachon et al. and A. Panpa et al. [7, 8] proved that spiders with three legs (four legs in [8]) of any length and arbitrary legs of length one are graceful. In general this paper is a generalization of the theorems of Huamaní et al. [9].

In this paper, three new results are demonstrated, which are in the Theorems 2, 3 and 4.

We will also give an alternative proof of Theorem 1 proved in [5, 6].

See [10–14] for recent work on graceful graphs.

DOI: https://doi.org/10.37256/cm.6120255497

1.1 *Preliminaries*

Definition 1 [2] Path is a spider with only one leg. A caterpillar is a tree with the property that the removal of its endpoints leaves a path.

Rosa proved in [2] that all caterpillars are graceful. Since paths are also caterpillars, it will follow that paths are also graceful. From the proof for caterpillars, we can write the following lemma for paths.

Lemma 1 Every path L of length m is graceful.

Let's describe the label that make L graceful. For this, let us denote v^* one of the vertices of degree 1 from path L, and let us denote by v_j the vertex in L of distance j from v^*.

Let f be the labeling defined as follows:

$$\text{i) } f(v^*) = 0; \text{ ii) if } j \text{ is odd, } f(v_j) = m - \frac{j-1}{2}; \text{ iii) if } j \text{ is even, } f(v_j) = \frac{j}{2}.$$

where $j = 1, 2, \cdots, m$.

From the label f for paths of Lemma 1, it is possible to build graceful spider. We will describe this way of building with example in which an algorithm will be given, which can be extended to find several families of graceful spiders.

1.2 *Algorithm for the construction of a graceful labeling for a spider with legs of the same length*

Let T be a spider of ℓ legs all of length m, let us build a graceful label for this spider.

Step I: Let's build a path P from length ℓm, we have that this path P is graceful with the label f described in the Lemma 1.

Step II: Considering the label f from path L, all its edges are labeled with the set $\{1, 2, \cdots, \ell m\}$. Let's remove all edges that have label im where $i = 1, 2, \cdots, \ell$. Doing this we get $\ell + 1$ disjoint labeled paths, of which ℓ paths have length m and one path (from zero length) that vertex v^* with label 0.

Step III: From step II in each of the ℓ disjoint paths, one of the extreme vertices has label im and we will denote this path by L_i, where $i = 1, 2, \cdots, \ell$. Then, we connect with an edge the vertices of label im of L_i with the vertex v^* for each $i = 1, 2, \cdots, \ell$. When finished, we obtain an graceful spider graph that we will denote by $T(L)$ and this spider has ℓ legs each of length m. Figure 1 shows an example for $\ell = 5$ and $m = 4$.

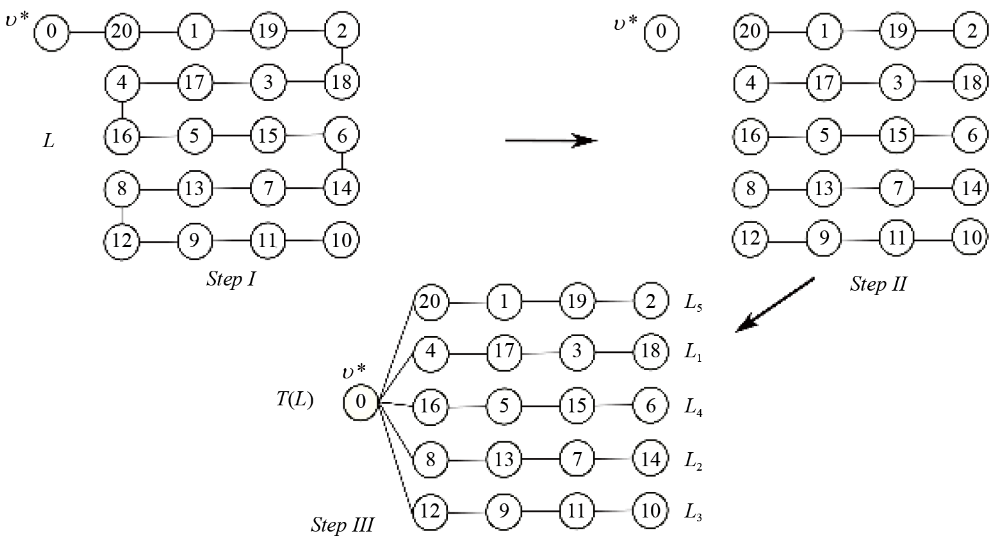

Figure 1. Steps of the construction of a graceful label, for a spider with $\ell = 5$ legs, all with length $m = 4$

Remark 1 Taking into account the previous algorithm, it is valid to ask:

Given a graceful path L with the Lemma label 1, What kind of graceful spiders can be obtained by removing edges from L and rejoining them with v^*, similar to the previous algorithm?

We give a partial answer to this question, for the moment, we find three families of graceful spiders, which are given Theorems 1, 2, 3 and 4.

2. Main results

Theorem 1 Let T be a spider with ℓ legs, each of which has length m, for some $m > 1$. Then T is graceful.

Proof. Since ℓ is the number of legs of length m. Note that T has $n+1 = \ell m+1$ vertices, to be labeled by the set $\{0,\ 1,\ 2,\ \cdots,\ n\}$. Label the legs by $L_1,\ L_2,\ \cdots,\ L_\ell$ each of length m. Let v^* denote the branch point of T and denote by $v_{i,\ j}$ the vertex en L_i of distance j from v^*.

Let ψ be the labeling defined as follows:

(i) $\psi(v^*) = 0$;

(ii) if i is any and j is odd,

$$\psi(v_{i,\ j}) = im - \frac{j-1}{2};$$

(iii) if i is any and j is even,

$$\psi(v_{i,\ j}) = (\ell - i)m + \frac{j}{2}.$$

To help compute the edge labels, we note that the local maxima of ψ occur at $v_{i,\ j}$ for which i y j have the same parity, that is,

$$i \equiv \begin{cases} \left[j + \dfrac{(-1)^{i+1}+1}{2}\right] (\mathrm{mod}\ 2), & \text{if } i \leq \left\lfloor \dfrac{\ell}{2} \right\rfloor \\ \left[j + \dfrac{(-1)^{i}+1}{2}\right] (\mathrm{mod}\ 2), & \text{if } i > \left\lfloor \dfrac{\ell}{2} \right\rfloor, \end{cases}$$

For such i and j, with $i \leq \left\lfloor \dfrac{\ell}{2} \right\rfloor$, we have

$$\psi(v_{i,\ j}) - \psi(v_{i,\ j+1}) = \ell m - 2mi + j > 0, \tag{1}$$

$$\psi(v_{i,\ j}) - \psi(v_{i,\ j-1}) = \ell m - 2mi + j - 1 > 0, \tag{2}$$

and for $i > \left\lfloor \dfrac{\ell}{2} \right\rfloor$ we have

$$\psi(v_{i,\ j}) - \psi(v_{i,\ j+1}) = 2mi - \ell m - j > 0, \tag{3}$$

$$\psi(v_{i,\ j}) - \psi(v_{i,\ j-1}) = 2mi - \ell m - j + 1 > 0. \tag{4}$$

Assume, for the sake of contradiction, that there exist two distinct edges with identical labels. By examining the indices of the vertices at the endpoints of these edges, we observe that it is possible to select distinct pairs of indices $(i,\ j)$ and $(i',\ j')$ such that i and j share the same parity, and similarly, i' and i' also have the same parity, and a edge incident on $v_{i,\ j}$ shares the same label as a different edge incident on $v_{i',\ j'}$, that is, one of these three cases occur:

$$\psi(v_{i,\ j}) - \psi(v_{i,\ j+1}) = \psi(v_{i',\ j'}) - \psi(v_{i',\ j'+1}), \tag{5}$$

$$\psi(v_{i,\ j}) - \psi(v_{i,\ j+1}) = \psi(v_{i',\ j'}) - \psi(v_{i',\ j'-1}), \tag{6}$$

$$\psi(v_{i,\ j}) - \psi(v_{i,\ j-1}) = \psi(v_{i',\ j'}) - \psi(v_{i',\ j'-1}). \tag{7}$$

When $i,\ i' \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ or $i,\ i' > \left\lfloor \frac{\ell}{2} \right\rfloor$ or $i \leq \left\lfloor \frac{\ell}{2} \right\rfloor, i' > \left\lfloor \frac{\ell}{2} \right\rfloor$.

Consider first the case where (5) and $i,\ i' \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ hold. From (1), we obtain $2m(i-i') + (j'-j) = 0$, which shows that $j \neq j'$, since otherwise $i = i'$ as well, contrary to the assumption that $(i,\ j) \neq (i',\ j')$. We therefore can write

$$2m = \frac{j - j'}{i - i'}.$$

Thus $|i - i'| \geq 1$ and $|j - j'| \leq m - 1$, and

$$2m = \frac{|j - j'|}{|i - i'|} \leq \frac{m-1}{1} = m - 1,$$

a contradiction.

Similar contradictions arise when (5), (6), (7) and $i,\ i' \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ or $i,\ i' > \left\lfloor \frac{\ell}{2} \right\rfloor$ or $i \leq \left\lfloor \frac{\ell}{2} \right\rfloor, i' > \left\lfloor \frac{\ell}{2} \right\rfloor$ hold. Thus, no two distinct edges bear the same label, and ψ is graceful. □

The labeling ψ place 0 at the center of the spider and notice that the difference between the labels at $v_{i,\ j}$ and $v_{i+1,\ j}$ are multiples of m. This is illustrated in the Figure 2, where $\ell = 5$ and $m = 4$.

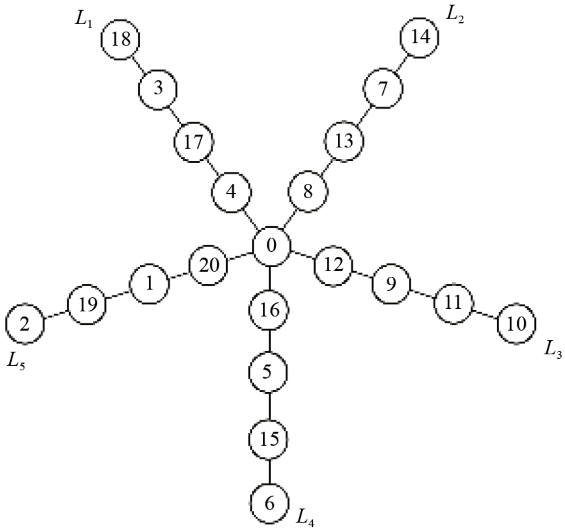

Figure 2. The labeling ψ for $\ell = 5$ and $m = 4$

The general idea of this proof was adopted from the paper [6], and a similar approach was addressed in [15].

The next theorem generalizes Theorem 4 of [9].

Theorem 2 Let T be a spider with $\ell(2k+1)-k$ legs, where ℓ of them have length $2m+1$, and $2k\ell-k$ have length $m+1$, for $\ell,\ k \geq 1$, then T is graceful.

Proof. Let $\ell(2k+1)-k$ be the number of legs of the spider T, where ℓ of them have length $2m+1$ and $\ell(2k+1)-k$ have length $m+1$. Note that T has $n+1 = \ell(2m+1)+(2k\ell-k)(m+1)+1$ vertices, to be labeled by the set $\{0,\ 1,\ 2,\ \cdots,\ n\}$. Let L_i, $i = 1,\ 2,\ \cdots,\ \ell(2k+1)-k$, denote the legs of T, of which $L_{N(p)}$ have length $2m+1$; $L_{B(q)}$ and $L_{H(r)}$ legs have length $m+1$, where:

$$\begin{cases} N(p) = (2k+1)p-k, & p = 1,\ 2,\ \cdots,\ \ell \\ B(q) = (k+1)\left\lfloor \dfrac{q+k-1}{k} \right\rfloor + q, & q = 1,\ 2,\ \cdots,\ k\ell-k \\ H(r) = (k+1)\left\lfloor \dfrac{r-1}{k} \right\rfloor + r, & r = 1,\ 2,\ \cdots,\ k\ell \end{cases}$$

Let v^* be the bifurcation point of T and let $v_{i,\ j}$ be the vertex in L_i at distance j from v^*.

Consider $\varphi:\ V(T) \longrightarrow \{0,\ 1,\ 2,\ \cdots,\ n\}$ the label given by:

(i) $\varphi(v^*) = 0$;

(ii) if $i = N(p)$ or $i = B(q)$, and j is odd.

$$\varphi(v_{i,\ j}) = \left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) m + i - \frac{j-1}{2}; \tag{8}$$

(iii) if $i = N(p)$ or $i = B(q)$, and j is even.

$$\varphi(v_{i,\ j}) = \ell(2m+1) + (2k\ell-k)(m+1) - \left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) m - i + \frac{j}{2}; \tag{9}$$

(iv) if $i = H(r)$ and j is odd.

$$\varphi(v_{i,\ j}) = \left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) m + i + \frac{j-1}{2}; \tag{10}$$

(v) if $i = H(r)$ and j is even.

$$\varphi(v_{i,\ j}) = \ell(2m+1) + (2k\ell - k)(m+1) - \left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) m - i - \frac{j-2}{2}. \tag{11}$$

The proof of this theorem will be done following the steps of the proof of Theorem 1. Thus, to help compute the edge labels, note that the local maximum of j occurs at $v_{i,\ j}$ for which i and j have the same parity, that is,

$$i \equiv \begin{cases} \left(j + \dfrac{(-1)^{i+1}+1}{2}\right) (\text{mod } 2), & \text{if } i \leq \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor \\ \left(j + \dfrac{(-1)^{i}+1}{2}\right) (\text{mod } 2), & \text{if } i > \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor \end{cases} \tag{12}$$

Now let's calculate all the differences between the labels of vertex pairs $v_{i,\ j}$, $v_{i,\ j+1}$ and $v_{i,\ j}$, $v_{i,\ j-1}$ where $v_{i,\ j}$ is the local maximum of φ. Thus:

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j+1}) > 0 \textit{ and } \varphi(v_{i,\ j}) - \varphi(v_{i,\ j-1}) > 0 \tag{13}$$

Next, for i and j with the same parity, we list all possible cases where the equations in (13) hold for φ. We will only provide conditions for the "i" since "j" is determined by their parity.

For such i and j, with: $i = N(p)$ or $i = B(q)$ and $i \leq \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor$ we have:

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j+1}) = \ell(2m+1) + (2k\ell - k)(m+1) - 2m\left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) - 2i + j > 0, \tag{14}$$

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j-1}) = \ell(2m+1) + (2k\ell - k)(m+1) - 1 - 2m\left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) - 2i + j > 0. \tag{15}$$

For $i = N(p)$ or $i = B(q)$ and $i > \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor$ we have:

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j+1}) = -\ell(2m+1) - (2k\ell - k)(m+1) + 2m\left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) + 2i - j > 0, \tag{16}$$

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j-1}) = -\ell(2m+1) - (2k\ell - k)(m+1) + 1 + 2m\left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) + 2i - j > 0. \tag{17}$$

For $i = H(r)$ and $i \leq \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor$ we have:

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j+1}) = \ell(2m+1) + (2k\ell - k)(m+1) + 1 - 2m\left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) - 2i - j > 0, \tag{18}$$

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j-1}) = \ell(2m+1) + (2k\ell - k)(m+1) + 2 - 2m\left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) - 2i - j > 0. \tag{19}$$

For $i = H(r)$ and $i > \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor$ we have:

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j+1}) = -\ell(2m+1) - (2k\ell - k)(m+1) - 1 + 2m\left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) + 2i + j > 0, \tag{20}$$

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j-1}) = -\ell(2m+1) - (2k\ell - k)(m+1) - 2 + 2m\left(i + \left\lfloor \frac{i+k}{2k+1} \right\rfloor\right) + 2i + j > 0. \tag{21}$$

$\square$

Remark 2 The equations from (14) to (21) provides us the labels on the edges of graph T, which must not be repeated (definition of a graceful graph), and to prove this, we will follow the following reasoning.

Assume, for the sake of contradiction, that there exist two distinct edges with identical labels. By examining the indices of the vertices at the endpoints of these edges, we observe that it is possible to select distinct pairs of indices $(i,\ j)$ and $(i',\ j')$ such that i and j share the same parity, and similarly, i' and i' also have the same parity, and a edge incident on $v_{i,\ j}$ shares the same label as a different edge incident on $v_{i',\ j'}$, that is, one of these three cases occur:

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j+1}) = \varphi(v_{i',\ j'}) - \varphi(v_{i',\ j'+1}), \tag{22}$$

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j+1}) = \varphi(v_{i',\ j'}) - \varphi(v_{i',\ j'-1}), \tag{23}$$

$$\varphi(v_{i,\ j}) - \varphi(v_{i,\ j-1}) = \varphi(v_{i',\ j'}) - \varphi(v_{i',\ j'-1}), \tag{24}$$

when $i,\ i' = N(p)$ or $i,\ i' = B(q)$ or $i,\ i' = H(r)$ and $i \leq \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor$ or $\left(i > \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor\right)$.

Consider the case where (22) and $i,\ i' = N(p)$ hold. From (14), we obtain

$$2m\left[(i - i') + \left(\left\lfloor \frac{i+k}{2k+1} \right\rfloor - \left\lfloor \frac{i'+k}{2k+1} \right\rfloor\right)\right] + 2(i - i') + (j' - j) = 0,$$

which shows that $j \neq j'$, since it is contrary to the assumption that $(i,\ j) \neq (i',\ j')$. Consequently, we can write

$$2m\left(1+\frac{\left\lfloor\dfrac{i+k}{2k+1}\right\rfloor-\left\lfloor\dfrac{i'+k}{2k+1}\right\rfloor}{i-i'}\right)+2=\frac{j-j'}{i-i'}.$$

Thus $|i-i'|\geq 1$ and $|j-j'|\leq 2m$, and $0\leq\dfrac{\left\lfloor\dfrac{i+k}{2k+1}\right\rfloor-\left\lfloor\dfrac{i'+k}{2k+1}\right\rfloor}{i-i'}<1$, and

$$2m+2\leq\left|2m\left(1+\frac{\left\lfloor\dfrac{i+k}{2k+1}\right\rfloor-\left\lfloor\dfrac{i'+k}{2k+1}\right\rfloor}{i-i'}\right)+2\right|=\left|\frac{j'-j}{i-i'}\right|\leq 2m,$$

a contradiction.

Similar contradictions arise when (22), (23), (24) and $i,\ i'=N(p)$ or $i,\ i'=B(q)$ or $i,\ i'=H(r)$ and and $i\leq\left\lfloor\dfrac{\ell(2k+1)-k}{2}\right\rfloor$ $\left(\text{or } i>\left\lfloor\dfrac{\ell(2k+1)-k}{2}\right\rfloor\right)$ hold. Thus, no two distinct edges bear the same labels, and φ is graceful.

Example 1 Replacing $k=2$, $\ell=2$ and $m=2$ in Theorem 2 we will construct a graceful label for a spider T of 8 legs, of which 2 legs have length 5 and 6 legs have length 3.

Following the proof of the theorem, let us denote by $L_i,\ i=1,\ 2,\ \ldots,\ 8$ the eight legs of T, where these legs are divided between types:

$$L_{N(p)}=L_{5p-2},\qquad p=1,\ 2\qquad \Rightarrow N(p)=3,\ 8$$

$$L_{B(q)}=L_{3\left\lfloor\frac{q+1}{2}\right\rfloor+q},\qquad q=1,\ 2\qquad \Rightarrow B(q)=4,\ 5$$

$$L_{H(r)}=L_{3\left\lfloor\frac{r-1}{2}\right\rfloor+r},\qquad r=1,\ 2,\ 3,\ 4\qquad \Rightarrow H(r)=1,\ 2,\ 6,\ 7.$$

Thus, $L_{N(p)}$: L_3, L_8 have length 5; $L_{B(q)}$: L_4, L_5 and $L_{H(r)}$: L_1, L_2, L_6, L_7 have length 3. Now, let's calculate the labels $\varphi(v_{i,\ j})$ for the vertices of T, to do this, let's find the pairs $(i,\ j)$ that verify the conditions in (II) of the labeled φ in the proof, and then we evaluate them at $\varphi(v_{i,\ j})$.

Replacing the values of $N(p)$ and $B(q)$ in $II)$ we obtain that: $i=3,\ 8$, or $i=4,\ 5$ and j is odd.

Therefore, we can calculate the labels of the vertices: $v_{3,\ 1}$, $v_{3,\ 3}$, $v_{3,\ 5}$ in L_3; $v_{8,\ 1}$, $v_{8,\ 3}$, $v_{8,\ 5}$ in L_8; $v_{4,\ 1}$, $v_{4,\ 3}$ in L_4; $v_{5,\ 1}$, $v_{5,\ 3}$ in L_5. Replacing these vertices into equation (8), we have $\varphi(v_{3,\ 1})=11$, $\varphi(v_{3,\ 10})=4$; $\varphi(v_{3,\ 5})=9$, $\varphi(v_{8,\ 1})=11$, $\varphi(v_{8,\ 3})=27$; $\varphi(v_{8,\ 5})=26$, $\varphi(v_{4,\ 1})=14$, $\varphi(v_{4,\ 3})=13$; $\varphi(v_{5,\ 1})=17$, $\varphi(v_{5,\ 3})=16$, we can see these labels highlighted in dark in Figure 3.

Thus, by calculating the remaining labels for the vertices of T. We can successfully construct a graceful labeling for T as shown in Figure 3.

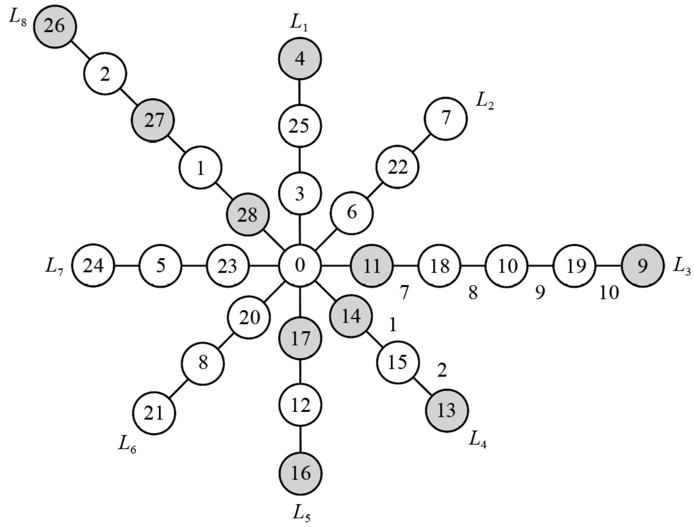

Figure 3. The labeling φ for $k = 2$, $\ell = 2$ and $m = 2$

Now, according Observation 2, let's calculate the differences that occur in the equations (14) to (21). Thus, to calculate the differences in equations (14) and (15), we must first calculate the pairs $(i,\ j)$, where i and j must have the same parity, that is, verify the equation (12) and "i" must satisfy:

$i = N(p) = 3,\ 8$ or $i = B(q) = 4,\ 5$ and $i \leq \left\lfloor \dfrac{\ell(2k+1)-k}{2} \right\rfloor = 4$ since $k = \ell = 2$. Therefore $i = 3,\ 4$. For $i = 3$: Let's calculate the j such that $i,\ j$ have the same parity, for that let's replace $i = 3$ in the equation (12), we have

$$3 = (j+1) \bmod 2,$$

then $j = 2,\ 4$, therefore the pairs $(3,\ 2)$ and $(3,\ 4)$ have the same parity.

For $i = 4$: Let us calculate j (similarly as for $i = 3$), so $j = 2$, therefore the pair $(4,\ 2)$ have the same parity. Finally let us substitute these $(i,\ j)$ into the equations (14) and (15), thus we obtain:

$$\varphi(v_{3,\ 2}) - \varphi(v_{3,\ 3}) = 8 \text{ and } \varphi(v_{3,\ 2}) - \varphi(v_{3,\ 1}) = 7$$

$$\varphi(v_{3,\ 4}) - \varphi(v_{3,\ 5}) = 10 \text{ and } \varphi(v_{3,\ 4}) - \varphi(v_{3,\ 3}) = 9$$

$$\varphi(v_{4,\ 2}) - \varphi(v_{4,\ 3}) = 2 \text{ and } \varphi(v_{4,\ 2}) - \varphi(v_{4,\ 1}) = 1,$$

these differences represent the labels on the edges between the corresponding vertices, so that T is a graceful graph, see these labels in Figure 3. The other differences are calculated using the equations from (16) to (21).

The next theorem generalizes Theorem 2 of [9].

Theorem 3 Let T be a spider with $\ell(2k+1)-k+1$ legs, where ℓ of them have length $2m+1$, and $2k\ell-k+1$ have length $m+1$, for $\ell,\ m,\ k \geq 1$, then T is graceful.

Proof. Let $\ell(2k+1)-k+1$ be the number of legs of the spider T, where ℓ of them have length $2m+1$ and $2k\ell-k+1$ have length $m+1$. Note that T has $n+1 = \ell(2m+1)+(2k\ell-k+1)(m+1)+1$ vertices, to be labeled by the set

$\{0,\ 1,\ 2,\ \cdots,\ n\}$. Label the legs by $L_1,\ L_2,\ \cdots,\ L_{\ell(2k+1)-k+1}$, where the legs $L_{M(p)}$ and $L_{N(q)}$ have length $2m+1$; the rest legs $L_{A(r)}, L_{B(s)}, L_{H(t)}, L_{V(u)}$ and $L_{W(v)}$ have length $m+1$, where:

$$\begin{cases} M(p) = (2k+1)p-k, & p = 1,\ 2,\ \cdots,\ \left\lfloor \dfrac{\ell}{2} \right\rfloor \\ N(q) = \ell(2k+1)-k+1-(2k+1)(q-1), & p = 1,\ 2,\ \cdots,\ \left\lceil \dfrac{\ell}{2} \right\rceil \\ A(r) = (k+1)\left\lfloor \dfrac{r-1}{k} \right\rfloor + r + (k+1), & r = 1,\ 2,\ \cdots,\ k\left\lfloor \dfrac{\ell}{2} \right\rfloor \\ B(s) = (k+1)\left\lfloor \dfrac{s-1}{k} \right\rfloor + s, & s = 1,\ 2,\ \cdots,\ k\left\lfloor \dfrac{\ell}{2} \right\rfloor \\ H(t) = \ell(2k+1)-k+1-\left[(k+1)\left\lfloor \dfrac{t-1}{k} \right\rfloor + t\right], & t = 1,\ 2,\ \cdots,\ k\left\lceil \dfrac{\ell}{2} \right\rceil \\ V(u) = \ell(2k+1)-k+1-\left[(k+1)\left\lfloor \dfrac{u-1}{k} \right\rfloor + u\right]-k, & u = 1,\ 2,\ \cdots,\ k\left(\left\lceil \dfrac{\ell}{2} \right\rceil - 1\right) \\ W(v) = (2k+1)\left\lfloor \dfrac{\ell}{2} \right\rfloor v + 1, & v = 1. \end{cases}$$

Let v^* denote the branch point of T and denote $v_{i,\ j}$ the vertex in L_i of distance j from v^*.
Let ψ the labeling defined as follows:
(i) $\psi(v^*) = 0$;
(ii) if $i = N(q)$ or $i = A(r)$ or $i = V(u)$ or $i = \begin{cases} W(v), & \text{if } \ell \text{ is even} \\ \emptyset, & \text{if } \ell \text{ is odd,} \end{cases}$ and j in odd.

$$\psi(v_{i,\ j}) = \left(\left\lfloor \frac{i+k-1}{2k+1} \right\rfloor\right) m + i - \frac{j-1}{2};$$

(iii) if $i = N(q)$ or $i = A(r)$ or $i = V(u)$ or $i = \begin{cases} W(v), & \text{if } \ell \text{ is even} \\ \emptyset, & \text{if } \ell \text{ is odd,} \end{cases}$ and j in even.

$$\psi(v_{i,\ j}) = \ell(2m+1) + (2k\ell-k+1)(m+1) - \left(\left\lfloor \frac{i+k-1}{2k+1} \right\rfloor\right) m + i - \frac{j}{2};$$

(iv) if $i = M(p)$ or $i = B(s)$ or $i = H(t)$ or $i = \begin{cases} \emptyset, & \text{if } \ell \text{ is even} \\ W(v), & \text{if } \ell \text{ is odd,} \end{cases}$ and j in odd.

$$\psi(v_{i,\ j}) = \left(\left\lfloor \frac{i+k-1}{2k+1} \right\rfloor\right) m + i + \frac{j-1}{2};$$

(v) if $i = M(p)$ or $i = B(s)$ or $i = H(t)$ or $i = \begin{cases} \emptyset, & \text{if } \ell \text{ is even} \\ W(v), & \text{if } \ell \text{ is odd,} \end{cases}$ and j in even.

$$\psi(v_{i,\ j}) = \ell(2m+1) + (2k\ell-k+1)(m+1) - \left(\left\lfloor \frac{i+k-1}{2k+1} \right\rfloor\right) m - i - \frac{j-2}{2}.$$

the rest of the proof follows the same technique of the proofs of the Theorems 1 and 2. An example is illustrated in the Figure 4, where $k = 2$, $\ell = 1$ and $m = 2$.

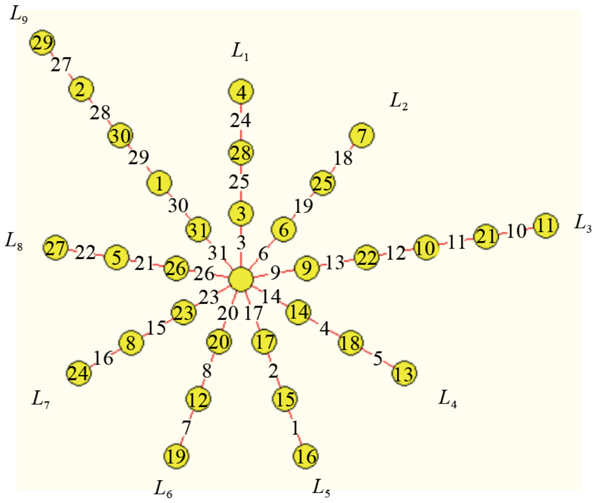

Figure 4. The labeling ψ for $k = 2$, $\ell = 2$ and $m = 2$

□

The next theorem generalizes Theorem 3 of [9].

Theorem 4 Let T be a spider with $\ell(2k+1)+k+1$ legs,where one leg has length $2m+1$, ℓ legs have length $2m+2$, and $2k\ell+k$ legs have length $m+1$, for $\ell,\ m,\ k \geq 1$, then T is graceful.

Proof. Let $\ell(2k+1)+k+1$ be the number of legs of T, where one leg has length $2m+1$, ℓ legs have length $2m+2$ and $2k\ell+k$ legs have length $m+1$. Note that T has $n+1 = (2k\ell+k)(m+1)+\ell(2m+2)+2m+2$ vertices, to be labeled by the set $\{0,\ 1,\ 2,\ \cdots,\ n\}$. Label the legs by $L_1,\ L_2,\ \cdots,\ L_{\ell(2k+1)+k+1}$, where the legs $L_{N(p)}$ have length $m+1$, the legs $L_{B(q)}$ have length $2m+2$ and the leg $L_{\ell(2k+1)+k+1}$ have length $2m+1$, with:

$$\begin{cases} N(P) = p + \left\lfloor \dfrac{p-1}{2k} \right\rfloor, & p = 1,\ 2,\ \cdots,\ 2k\ell+k \\ B(q) = (2k+1)q, & q = 1,\ 2,\ \cdots,\ \ell \end{cases}$$

Let v^* denote the branch point of T and denote $v_{i,\ j}$ the vertex in L_i of distance j from v^*.

Let ϕ the labeling defined as follows:

(i) $\phi(v^*) = 0$;

(ii) if $i < \ell(2k+1)+k+1$ and j in odd,

$$\psi(v_{i,\ j}) = \left(i \left\lfloor \frac{i+k}{2k+1} \right\rfloor \right)(m+1) + \frac{j-1}{2};$$

(iii) if $i < \ell(2k+1)+k+1$ and j in even,

$$\psi(v_{i,\ j}) = (2k\ell+k)(m+1)+\ell(2m+2)+2m+1 - \left(\left\lfloor \frac{i+k}{2k+1} \right\rfloor \right)(m+1) - \frac{j-2}{2};$$

(iv) if $i = \ell(2k+1)+k+1$ and j in odd,

$$\psi(v_{i,\ j}) = \left(i\left\lfloor \frac{i+k}{2k+1} \right\rfloor\right)(m+1) - \frac{j-1}{2} - 1;$$

(v) if $i = \ell(2k+1)+k+1$ and j in even,

$$\psi(v_{i,\ j}) = (2k\ell+k)(m+1) + \ell(2m+2) + 2m+1 - \left(\left\lfloor \frac{i+k}{2k+1} \right\rfloor\right)(m+1) + \frac{j}{2} + 1$$

the rest of the proof follows the same technique of the proofs of the Theorems 1 and 2. An example is illustrated in the Figure 5, where $k = 2$, $\ell = 1$ and $m = 2$.

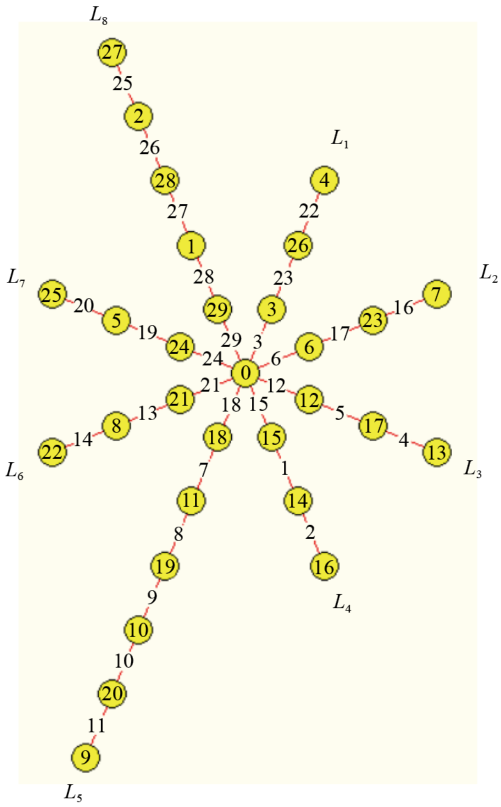

Figure 5. The labeling ϕ for $k = 2$, $\ell = 2$ and $m = 2$

□

3. Conclusions

In this paper, the graceful of families of spider graphs with $\ell(2k+1)-k$ legs, where ℓ and $2k\ell-k$ have lengths $2m+1$ and $m+1$ respectively, was demonstrated. Additionally, the graceful of families with $\ell(2k+1)-k+1$ legs, where ℓ and $2k\ell - k + 1$ legs have lengths $2m+1$ and $m+1$ respectively. Lastly, graceful was established for the family with $\ell(2k+1)+k+1$ legs, where 1, ℓ and $2\ell-1$ have lengths $2m+1$, $2m+2$ and $m+1$ respectively. These families represent new results in the literature, contributing to the efforts to prove the conjecture that spider graphs are graceful. The basic strategy employed for constructing a graceful spider (or construct the graceful labeling function), is to start with a graceful path, from which we remove certain edges along with their labels. Next, we reconstruct these edges within the same network in such a way as to obtain an elegant spider. The question asked in the Remark 1, is still open, i.e., the authors believe that there are more families of elegant spiders that can be constructed with this method.

Conflict of interest

The authors declare that they have no competing interests.

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