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Research Article

Decision-making on A Novel Stochastic Space-I of Solutions for Fuzzy Volterra-Type Non-linear Dynamical Economic Models

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Abstract: In this paper, we explain the sufficient conditions on a novel constructed stochastic space by weighted generalized Gamma matrix and variable exponent sequence spaces of fuzzy functions, for the Kannan contraction operator to have a unique fixed point. Finally, we discuss the numerous applications of solutions to Fuzzy Volterra-Type Non-linear Dynamical Economic Models and illustrate them with some examples.

***Keywords*:** nakano sequence space, generalized Gamma matrix, Kannan contraction mapping, dynamical economic models

MSC: 46B15, 46C05, 46E30

1. Introduction

Summable equations come up in many situations in critical point theory for non-smooth energy functionals, mathematical physics, control theory, bio-mathematics, difference variational inequalities, fuzzy set theory [1], probability theory [2] and traffic problems, to mention but a few. In particular, Volterra-type summable equations are fundamental in investigating dynamical systems [3] and stochastic processes [4, 5]. Some instances are in granular systems, sweeping processes, oscillation problems, control problems, decision-making problems [6], and so on. The solution of summable equations is contained in a specific sequence space. So there is a great interest in mathematics to construct new sequence spaces, see [7]. Mursaleen and Noman [8] examined some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞, and Mursaleen and Başar [9] constructed and investigated the domain of Cesàro mean of order one in some spaces of double sequences. Mustafa and Bakery [10] introduced the concept of private sequence space of fuzzy functions ($\mathfrak{pssff}$). Suppose $\mathscr{R}$ is the set of real numbers and $\mathbb{N}_0$ is the set of nonnegative integers. We have introduced the space, $({}_F\Omega_t(u,\ v))_\kappa$, which is the domain of the matrix $W_t = (\gamma^t_{ba}(x))$ in $\ell^F_{((v_a))}$, where

$$\gamma_{ba}^{t}(x) = \begin{cases} \dfrac{t(b!)\Gamma(a+t)u_a}{a!\Gamma(t+b+1)}, & 0 \le a \le b, \\ 0, & a > b, \end{cases} \tag{1}$$

for $t \ge 1$, $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ and $u_a \in (0,\ \infty)$, for all $a \in \mathbb{N}_0$.

In [11], Roopaei and Başar studied the Gamma spaces, containing ℓ_v, c_0 and ℓ_∞.

For any $0 < \varepsilon < 1$, Matloka [12] introduced the ε-level set of a fuzzy real x as follows:

$$x^\varepsilon = \{q \in \mathscr{R} :\ x(q) \ge \varepsilon\}.$$

The space $\mathscr{R}([0,\ 1])$ is the set of all x^ε is compact, normal, upper semi-continuous, and convex fuzzy numbers. $\overline{0}$ and $\overline{1}$ indicate the additive and multiplicative identity in $\mathscr{R}[0,\ 1]$, respectively. If $x \in \mathscr{R}([0,\ 1])$, then

$$\overline{x}(m) = \begin{cases} 1, & m = x \\ 0, & m \ne x. \end{cases}$$

Assume $\overline{x},\ \overline{y} \in \mathscr{R}([0,\ 1])$ and the ε-level sets are $[\overline{x}]^\varepsilon = [\overline{x}_1^\varepsilon,\ \overline{x}_2^\varepsilon],\ \ [\overline{y}]^\varepsilon = [\overline{y}_1^\varepsilon,\ \overline{y}_2^\varepsilon]$. A partial ordering for any $\overline{x},\ \overline{y} \in \mathscr{R}([0,\ 1])$ as follows: $\overline{x} \,\overline{\preceq}\, \overline{y}$ if and only if $\overline{x}^\varepsilon \le \overline{y}^\varepsilon$ if and only if $\overline{x_1}^\varepsilon \le \overline{y_1}^\varepsilon$ and $\overline{x_2}^\varepsilon \le \overline{y_2}^\varepsilon$.

If $\Pi : \mathbb{N}_0^2 \to \mathscr{R}$, $g : \mathbb{N}_0 \times \mathscr{R}([0,1]) \to \mathscr{R}([0,1])$, $\overline{J} : \mathbb{N}_0 \to \mathscr{R}([0,1])$, and $\overline{r} : \mathbb{N}_0 \to \mathscr{R}([0,1])$. For every $\overline{J} \in_F \Omega_t(u,v)$. Consider the Fuzzy Volterra-Type Non-linear Dynamical Economic Models [13]:

$$\overline{J_a} = \overline{r_a} + \sum_{q=0}^{\infty} \Pi(a,\ q) g(q,\ \overline{J_q}), \tag{2}$$

and presume $L : ({}_F\Omega_t(u,\ v))_\kappa \to ({}_F\Omega_t(u,\ v))_\kappa$, for certain functional κ, is defined as

$$L(\overline{J_a})_{a\in\mathbb{N}_0} = \left(\overline{r_a} + \sum_{q=0}^{\infty} \Pi(a,\ q) g(q,\ \overline{J_q})\right)_{a\in\mathbb{N}_0}. \tag{3}$$

Mustafa and Bakery [10], investigated the unique solution of fuzzy non-linear matrix system (2) of Kannan-type (3) in the operators' ideal generated by a weighted binomial matrix in the Nakano sequence space of extended s-fuzzy functions. Alsolmi et al. [14] examined the unique solution of nonlinear stochastic dynamical matrix systems Kannan-type in the operators' ideal generated by a weighted binomial matrix in the Nakano sequence space of extended s-soft functions. Bakery and Mohammed [15], explained Kannan nonexpansive operators on variable exponent Cesàro sequence space of fuzzy functions. Younis et al. [16] used numerical iterations to study the convergence of fixed points in graphical Bc-Kannan-contractions in extended b-metric spaces. They created novel fixed-point results using Bc-Kannan contraction and showed that every Kannan contraction is graphical but not the other way around. They used graphical analysis to demonstrate that their major findings are more general than the supporting research and that a fourth-order two-point boundary value problem representing elastic beam deformations may be solved. Some classes of Hammerstein integral equations and fractional differential equations have sufficient criteria for the existence of solutions discovered by Younis and Singh [17]. They extended the notion of Kannan mappings in view of F-contraction in the setting of b-metric like spaces. In the realm of stochastic differential equations, it is imperative to consider the following noteworthy publications to enhance the literature review: The study conducted by Li et al. [18] examined the presence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays. Shu et al. [19] investigated mild solutions and controllability for Riemann-Liouville fractional stochastic evolution equations with nonlocal conditions of order $1 < \alpha < 2$. The mild equation solutions were shown using the Laplace transform of the Riemann-Liouville derivative. They also estimated resolve operators with Riemann-Liouville fractional derivatives of the same order. They focused on the approximate controllability of nonlinear Riemann-Liouville fractional nonlocal stochastic systems of the same order, assuming the related linear system is controllable. Final findings were obtained utilizing the Lebesgue-dominated convergence theorem for approximation controllability. To find almost periodic solutions for fractional impulsive neutral stochastic differential equations with indefinite delay in Hilbert space, Ma et al. [20] utilized fractional calculus, operator semigroups, and the fixed point theorem. Finally, they provided an example to demonstrate the findings. The fuzzy function space, $({}_F\Omega_t(u,\ v))_\kappa$, has been provided with certain geometric and topological structures by us. In this space, the Kannan contraction operator is confirmed, and the operator has a fixed point. In the final part of this article, we discuss the myriad applications that may be found for solutions to Fuzzy Volterra-Type Non-linear Dynamical Economic Models and demonstrate how our discoveries might be employed.

2. The structure of $({}_F\Omega_t(u, v))_\kappa$

Some of the geometric and topological characteristics of the fuzzy function space $({}_F\Omega_t(u,\ v))_\kappa$ have been studied so far.

The set of all possible fuzzy real sequences is ${}_F\mho$. The space of all sequences of positive reals is denoted as $\mathscr{R}^{+\mathbb{N}_0}$.

Definition 1 $({}_F\Omega_t(u,\ v))_\kappa := \left\{\bar{r} = (\overline{r_m}) \in_F \mho : \kappa(\iota\bar{r}) < \infty,\ \text{for some } \iota > 0\right\}$, where $(v_m) \in \mathscr{R}^{+\mathbb{N}_0}$ and $\kappa(\bar{r}) = \sum_{m=0}^{\infty}\left(\frac{t(m!)\bar{\tau}\left(\sum_{n=0}^{m}\dfrac{\Gamma(n+t)}{\Gamma(n+1)}u_n\overline{r_n},\ \bar{0}\right)}{\Gamma(t+m+1)}\right)^{v_m}$.

Lemma 1 [21] Suppose $v_a > 0$ and $u_a,\ r_a \in \mathscr{R}$, for every $a \in \mathbb{N}_0$, and $\rho = \max\{1,\ \sup_a v_a\}$, then

$$|u_a + r_a|^{v_a} \leq 2^{\rho-1}\left(|u_a|^{v_a} + |r_a|^{v_a}\right). \tag{4}$$

Theorem 1 If $(v_a) \in \ell_\infty \cap \mathscr{R}^{+\mathbb{N}_0}$, then

$$({}_F\Omega_t(u,\ v))_\kappa = \left\{\bar{r} = (\overline{r_m}) \in_F \mho : \kappa(\iota\bar{r}) < \infty,\ \text{for every } \iota > 0\right\}.$$

Proof. Clearly, since (v_a) is bounded. □

Theorem 2 Suppose $(v_a) \in [1,\ \infty)^{\mathbb{N}_0} \cap \ell_\infty$, then $({}_F\Omega_t(u,\ v))_\kappa$ is a non-absolute type.

Proof. Obviously, as

$$\kappa\Big(\overline{1},\, -\overline{1},\, \overline{0},\, \overline{0},\, \overline{0},\, \dots\Big) = (u_0)^{v_0} + \left(\frac{|u_0 - tu_1|}{1+t}\right)^{v_1} + \left(\frac{2\,|u_0 - tu_1|}{(t+2)(t+1)}\right)^{v_2} + \cdots$$

$$\neq (u_0)^{v_0} + \left(\frac{u_0 + tu_1}{1+t}\right)^{v_1} + \left(\frac{2(u_0 + tu_1)}{(t+2)(t+1)}\right)^{v_2} + \cdots$$

$$= \kappa\Big(\overline{1},\, \overline{1},\, \overline{0},\, \overline{0},\, \overline{0},\, \dots\Big).$$

□

Definition 2 Assume that $v_m \geq 1$, for every $m \in \mathbb{N}_0$. The absolute type space $(|_F\Omega_t|(u,\, v))_{\wp}$ is defined as: $(|_F\Omega_t|(u,\, v))_{\wp} := \Big\{\overline{r} = (\overline{r_m}) \in_F \mho : \wp(\iota\overline{r}) < \infty, \text{ for some } \iota > 0\Big\}$, where

$$\wp(\overline{r}) = \sum_{m=0}^{\infty}\left(\frac{t(m!)\overline{\tau}\left(\sum_{n=0}^{m}\frac{\Gamma(n+t)}{\Gamma(n+1)}u_n|\overline{r_n}|,\, \overline{0}\right)}{\Gamma(t+m+1)}\right)^{v_m}.$$

Theorem 3 $(|_F\Omega_t|(u,\, v))_{\wp} \subsetneqq (_F\Omega_t(u,\, v))_{\kappa}$, if $(v_a) \in (1,\, \infty)^{\mathbb{N}_0} \cap \ell_\infty$ with $\left(\frac{(a+1)!\Gamma(t+1)}{\Gamma(t+a+1)}\right) \notin \ell_{(v_a)}$.

Proof. Suppose $\overline{j} \in (|_F\Omega_t|(u,\, v))_{\wp}$, since

$$\sum_{b=0}^{\infty}\left(\frac{t(b!)\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\overline{j_a},\, \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b} \leq \sum_{b=0}^{\infty}\left(\frac{t(b!)\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{j_a}|,\, \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b} < \infty.$$

Hence $\overline{j} \in (_F\Omega_t(u,\, v))_{\kappa}$. When we put $\overline{i} = \left(\frac{(-\overline{1})^a a!\Gamma(t)}{\Gamma(a+t)u_a}\right)_{a\in\mathbb{N}_0}$, we have $\overline{i} \in (_F\Omega_t(u,\, v))_{\kappa}$ and $\overline{i} \notin (|_F\Omega_t|(u,\, v))_{\wp}$.□

Assume $_F\mathscr{Q}$ is a linear space of sequences of fuzzy functions, $\overline{e_m} = (\overline{0},\, \overline{0},\, ...,\, \overline{1},\, \overline{0},\, \overline{0},\, \cdots)$, while $\overline{1}$ locates at the m^{th} position and $[m]$ marks an integral part of $m \in \mathscr{R}$.

Definition 3 [10] The space $_F\mathscr{Q}$ is called a $\mathfrak{pssff}$, if it verifies the following conditions:

(i) If $m \in \mathbb{N}_0$, then $\overline{e_m} \in_F \mathscr{Q}$,

(ii) Suppose $\overline{r} = (\overline{r_m}) \in_F \mho$, $|\overline{w}| = (|\overline{w_m}|) \in_F \mathscr{Q}$ and $|\overline{r_m}| \leq |\overline{w_m}|$, for all $m \in \mathbb{N}_0$, then $|\overline{r}| \in_F \mathscr{Q}$,

(iii) $\left(\left|\overline{r_{[\frac{a}{2}]}}\right|\right)_{a=0}^{\infty} \in_F \mathscr{Q}$, if $(|\overline{r_a}|)_{a=0}^{\infty} \in_F \mathscr{Q}$.

Notations 1 [22]

(1) $\mathscr{F}$ is the space of finite sequences of fuzzy numbers.

(2) $\overline{\theta} = (\overline{0},\, \overline{0},\, \overline{0},\, \dots)$.

(3) MI and MD indicate the space of all monotonic increasing and decreasing sequences of positive reals, respectively.

Definition 4 [23] A subspace of the $\mathfrak{pssff}$ is said to be a pre-modular $\mathfrak{pssff}$ (p-m-pssff), when one has a function $\kappa :_F \mathscr{Q} \to [0,\, \infty)$ verifies the following conditions:

(a) Assume $\overline{r} \in_F \mathscr{Q}$, $\kappa(|\overline{r}|) = 0 \Longleftrightarrow \overline{r} = \overline{\theta}$, and $\kappa(\overline{r}) \geq 0$,

(b) Suppose $\overline{r} \in_F \mathscr{Q}$ and $\iota \in \mathscr{R}$, then $E_0 \geq 1$ so that $\kappa(\iota\overline{r}) \leq |\iota|E_0\kappa(\overline{r})$,

(c) there are $G_0 \geq 1$ such that $\kappa(\overline{r}+\overline{m}) \leq G_0(\kappa(\overline{r}) + \kappa(\overline{m}))$, for all $\overline{r},\, \overline{m} \in_F \mathscr{Q}$,

(d) $\kappa(|\overline{i_m}|) \leq \kappa(|\overline{j_m}|)$, whenever $|\overline{i_m}| \leq |\overline{j_m}|$, for every $m \in \mathbb{N}_0$,
(e) there are $D_0 \geq 1$ with $\kappa(|\overline{r}|) \leq \kappa(|\overline{r_{[.]}}|) \leq D_0 \kappa(|\overline{r}|)$,
(f) the closure of $\mathscr{F} =_F \mathscr{Q}_\kappa$,
(g) there are $\delta > 0$ such that $\kappa(\overline{\mu},\ \overline{0}, \overline{0},\ \overline{0},\ ...) \geq \delta|\mu|\kappa(\overline{1},\ \overline{0},\ \overline{0},\ \overline{0},\ ...)$.

Definition 5 [23] The $\mathfrak{pssff}$ ${}_F\mathscr{Q}_\kappa$ is called a pre-quasi normed $\mathfrak{pssff}$ (p-qN-pssff), when κ satisfies the conditions (a)-(c) of Definition 2The space ${}_F\mathscr{Q}_\kappa$ is said to be a pre-quasi Banach $\mathfrak{pssff}$ (p-qB-pssff), if ${}_F\mathscr{Q}$ is complete equipped with κ.

Theorem 4 [10] If the space is p-m-pssff, then it is p-qN-pssff.

Theorem 5 Assuming that

(h1) $(v_a) \in MI \cap \ell_\infty$ with $v_0 > 1$,

(h2) $\left(\dfrac{\Gamma(a+t)}{\Gamma(a+1)} u_a\right)_{a=0}^{\infty} \in MD$ or, $\left(\dfrac{\Gamma(a+t)}{\Gamma(a+1)} u_a\right)_{a=0}^{\infty} \in MI \cap \ell_\infty$ and one has $\lambda \geq 1$ with

$$\frac{\Gamma(2a+t+1)}{\Gamma(2a+2)} u_{2a+1} \leq \lambda \frac{\Gamma(a+t)}{\Gamma(a+1)} u_a,$$

then $({}_F\Omega_t(u,\ v))_\kappa$ is a p-qB-pssff.

Proof. First, we have to prove that $({}_F\Omega_t(u,\ v))_\kappa$ is a **p-m-pssff**.

(a) Clearly, $\kappa(|\overline{r}|) = 0 \Leftrightarrow \overline{r} = \overline{\theta}$ and $\kappa(\overline{r}) \geq 0$.

The conditions (i1) and (c): When $\overline{i},\ \overline{j} \in ({}_F\Omega_t(u,\ v))_\kappa$, one has

$$\kappa(\overline{r}+\overline{m}) = \sum_{q=0}^{\infty} \left(\frac{q!t\overline{\tau}\left(\sum_{w=0}^{q} \dfrac{\Gamma(w+t)}{\Gamma(w+1)} u_w \left(\overline{r_w}+\overline{m_w}\right),\ \overline{0}\right)}{\Gamma(t+q+1)} \right)^{v_q}$$

$$\leq 2^{\rho-1} \left(\sum_{q=0}^{\infty} \left(\frac{q!t\overline{\tau}\left(\sum_{w=0}^{q} \dfrac{\Gamma(w+t)}{\Gamma(w+1)} u_w \overline{r_w},\ \overline{0}\right)}{\Gamma(t+q+1)} \right)^{v_q} + \sum_{q=0}^{\infty} \left(\frac{q!t\overline{\tau}\left(\sum_{w=0}^{q} \dfrac{\Gamma(w+t)}{\Gamma(w+1)} u_w \overline{m_w},\ \overline{0}\right)}{\Gamma(t+q+1)} \right)^{v_q} \right)$$

$$= 2^{\rho-1}(\kappa(\overline{r}) + \kappa(\overline{m})) < \infty,$$

therefore, $\overline{r}+\overline{m} \in ({}_F\Omega_t(u,\ v))_\kappa$.

The condition (b): Assume $\delta \in \mathscr{R}$, $\overline{r} \in ({}_F\Omega_t(u,\ v))_\kappa$ and since $(v_q) \in MI \cap \ell_\infty$, one has

$$\kappa(\delta\overline{r}) = \sum_{q=0}^{\infty} \left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{q} \dfrac{\Gamma(a+t)}{\Gamma(a+1)} u_a \delta \overline{r_a},\ \overline{0}\right)}{\Gamma(t+q+1)} \right)^{v_q} \leq \sup_q |\delta|^{v_q} \sum_{q=0}^{\infty} \left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{q} \dfrac{\Gamma(a+t)}{\Gamma(a+1)} u_a \overline{r_a},\ \overline{0}\right)}{\Gamma(t+q+1)} \right)^{v_q}$$

$$\leq E_0|\delta|\kappa(\overline{r}) < \infty,$$

where $E_0 = \max\left\{1,\ \sup_b |\delta|^{v_b-1}\right\} \geq 1.$ Therefore, $\delta\overline{r} \in ({}_F\Omega_t(u,\ v))_\kappa$.

Since $(v_q) \in MI \cap \ell_\infty$ and $v_0 > 1$, we have

$$\sum_{q=0}^{\infty}\left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\overline{(e_b)_a},\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$=\sum_{q=0}^{b-1}\left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\overline{(e_b)_a},\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}+\sum_{q=b}^{\infty}\left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\overline{(e_b)_a},\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}=0+\sum_{q=b}^{\infty}\left(\frac{q!tu_b\Gamma(t+b)}{b!\Gamma(t+q+1)}\right)^{v_q}$$

$$\leq\sup_{q=b}^{\infty}\left(\frac{t\Gamma(t+b)u_b}{b!}\right)^{v_q}\sum_{q=b}^{\infty}\left(\frac{q!}{\Gamma(t+q+1)}\right)^{v_q}\leq\sup_{q=b}^{\infty}\left(\frac{t\Gamma(t+b)u_b}{b!}\right)^{v_q}\sum_{q=b}^{\infty}\left(\frac{1}{q+1}\right)^{v_q}<\infty.$$

Therefore, $\overline{e_b}\in({}_F\Omega_t(u,\ v))_{\kappa}$, for every $b\in\mathbb{N}_0$.

The conditions (i2) and (d): Suppose $|\overline{i_a}|\leq|\overline{r_a}|$, for every $a\in\mathbb{N}_0$ and $|\overline{r}|\in({}_F\Omega_t(u,\ v))_{\kappa}$, one can see

$$\kappa(|\bar{i}|)=\sum_{m=0}^{\infty}\left(\frac{m!t\overline{\tau}\left(\sum_{a=0}^{m}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{i_a}|,\ \overline{0}\right)}{\Gamma(t+m+1)}\right)^{v_m}\leq\sum_{m=0}^{\infty}\left(\frac{m!t\overline{\tau}\left(\sum_{a=0}^{m}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_a}|,\ \overline{0}\right)}{\Gamma(t+m+1)}\right)^{v_m}=\kappa(|\overline{r}|)<\infty,$$

therefore $|\bar{i}|\in({}_F\Omega_t(u,\ v))_{\kappa}$.

The conditions (i3) and (e): If $(|\overline{r_a}|)\in({}_F\Omega_t(u,\ v))_{\kappa}$, so that $(v_q)\in\mathbf{MI}\cap\ell_{\infty}$ and $\left(\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right)_{a=0}^{\infty}\in\mathbf{MD}$, one has

$$\kappa(|\overline{r_{[\frac{a}{2}]}}|)=\sum_{q=0}^{\infty}\left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_{[\frac{a}{2}]}}|,\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$=\sum_{q=0}^{\infty}\left(\frac{2q!t\overline{\tau}\left(\sum_{a=0}^{2q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_{[\frac{a}{2}]}}|,\ \overline{0}\right)}{\Gamma(t+2q+1)}\right)^{v_{2q}}+\sum_{q=0}^{\infty}\left(\frac{(2q+1)!t\overline{\tau}\left(\sum_{a=0}^{2q+1}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_{[\frac{a}{2}]}}|,\ \overline{0}\right)}{\Gamma(t+2q+2)}\right)^{v_{2q+1}}$$

$$\leq\sum_{q=0}^{\infty}\left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{2q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_{[\frac{a}{2}]}}|,\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}+\sum_{q=0}^{\infty}\left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{2q+1}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_{[\frac{a}{2}]}}|,\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$\leq\sum_{q=0}^{\infty}\left(\frac{q!t\overline{\tau}\left(\frac{\Gamma(2q+t)}{\Gamma(2q+1)}u_{2q}|\overline{r_q}|+\sum_{a=0}^{q}\left(\frac{\Gamma(2a+t)}{\Gamma(2a+1)}u_{2a}+\frac{\Gamma(2a+t+1)}{\Gamma(2a+2)}u_{2a+1}\right)|\overline{r_a}|,\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$+\sum_{q=0}^{\infty}\left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{q}\left(\frac{\Gamma(2a+t)}{\Gamma(2a+1)}u_{2a}+\frac{\Gamma(2a+t+1)}{\Gamma(2a+2)}u_{2a+1}\right)|\overline{r_a}|,\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$\leq 2^{\rho-1}\left(\sum_{q=0}^{\infty}\left(\frac{q!t\overline{\tau}\left(\sum_{a=0}^{q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_a}|,\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}+\sum_{q=0}^{\infty}\left(\frac{2(q!)t\overline{\tau}\left(\sum_{a=0}^{q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_a}|,\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}\right)$$

$$+\sum_{q=0}^{\infty}\left(\frac{2(q!)t\overline{\tau}\left(\sum_{a=0}^{q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a|\overline{r_a}|,\ \overline{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}\leq D_0\kappa(|\overline{r}|)<\infty,$$

where $D_0\geq(2^{2\rho-1}+2^{\rho-1}+2^{\rho})\geq 1$. Therefore, $(|\overline{r_{[\frac{q}{2}]}}|)\in({}_F\Omega_t(u,\ v))_\kappa$.

The condition (f): Clearly, the closure of $\mathscr{F}={}_F\Omega_t(u,\ v)$.

The condition (g): One has $0<\delta\leq\sup_l|\mu|^{v_b-1}$ with $\kappa(\overline{\mu},\ \overline{0},\ \overline{0},\ \overline{0},\ ...)\geq\delta|\mu|\kappa(\overline{1},\overline{0},\ \overline{0},\ \overline{0},\ ...)$, for all $\mu\neq 0$ and $\delta>0$, if $\mu=0$.

By Theorem 4, the space $({}_F\Omega_t(u,\ v))_\kappa$ is a p-qN-pssff. Second, to prove that $({}_F\Omega_t(u,\ v))_\kappa$ is a Banach space, assume $\overline{A^m}=(\overline{A_a^m})_{a=0}^{\infty}$ is a Cauchy sequence in $({}_F\Omega_t(u,\ v))_\kappa$, one has for every $\gamma\in(0,\ 1)$, we have $m_0\in\mathbb{N}_0$ so that $m,\ n\geq m_0$, hence

$$\kappa(\overline{A^m}-\overline{A^n})=\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\left(\overline{A_a^m}-\overline{A_a^n}\right),\overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}<\gamma^{\rho}.$$

So $\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\left(\overline{A_a^m}-\overline{A_a^n}\right),\ \overline{0}\right)<\gamma$. Since $(\mathscr{R}([0,\ 1]),\ \overline{\tau})$ is a complete metric space. So $(\overline{A_a^n})$ is a Cauchy sequence in $\mathscr{R}([0,\ 1])$, for fixed $a\in\mathbb{N}_0$. Hence it is convergent to $\overline{A_a^0}\in\mathscr{R}([0,\ 1])$. Therefore, $\kappa(\overline{A^m}-\overline{A^0})<\gamma^{\rho}$, for all $m\geq m_0$. Obviously, from setup (c) that $\overline{A^0}\in({}_F\Omega_t(u,\ v))_\kappa$. □

Remark 1 The importance of this space is that we can construct a family of probability density functions as follows:

$$f(b)=\frac{1}{\kappa(\overline{j})}\left(\frac{t(b!)\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\overline{j_a},\ \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}.$$

(1) Figure 1 explains the pdf when $t=1$, $u_a=1$, $v_a=\frac{3(a+1)}{a+2}$, and $\overline{j_a}=\overline{\frac{1}{a+1}}$ and the pdf when $t=1$, $u_a=\frac{1}{a+1}$, $v_a=\frac{3(a+1)}{a+2}$, and $\overline{j_a}=\overline{\frac{1}{a+1}}$.

(2) Figure 2 explains the pdf when $t = 1.5$, $u_a = \dfrac{1.5\Gamma(a+1)\Gamma(1.5)}{\Gamma(a+1.5)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{\dfrac{1}{a+1}}$ and the pdf when $t = 1.5$, $u_a = \dfrac{1.5\Gamma(a+1)\Gamma(1.5)}{(a+1.5)\Gamma(a+1.5)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{\dfrac{1}{a+1}}$.

(3) Figure 3 explains the pdf when $t = 2$, $u_a = \dfrac{2}{a+1}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{\dfrac{1}{a+1}}$ and the pdf when $t = 2$, $u_a = \dfrac{2}{(a+1)(a+2)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{\dfrac{1}{a+1}}$.

(4) Figure 4 explains the pdf when $t = 1$, $u_a = 1$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$ and the pdf when $t = 1$, $u_a = \dfrac{1}{a+1}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$.

(5) Figure 5 explains the pdf when $t = 1.5$, $u_a = \dfrac{1.5\Gamma(a+1)\Gamma(1.5)}{\Gamma(a+1.5)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$ and the pdf when $t = 1.5$, $u_a = \dfrac{1.5\Gamma(a+1)\Gamma(1.5)}{(a+1.5)\Gamma(a+1.5)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$.

(6) Figure 6 explains the pdf when $t = 2$, $u_a = \dfrac{2}{a+1}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$ and the pdf when $t = 2$, $u_a = \dfrac{2}{(a+1)(a+2)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$.

3. Kannan's contraction fixed points

This section is devoted to discussing the existence of fixed points of Kannan contraction operators acting on this new space under the setups of Theorem 5. Several numerical examples are offered to explain our results.

Many mathematicians used the Banach Fixed Point Theorem [24] to generalize some contraction operators, for instance, the Kannan contraction operator [25], Kannan operators in modular vector spaces [26], Kannan p-qN contraction operator [15], and Kannan p-qN non-expansive operator [15].

Definition 6 [15] A p-qN-pssff κ on ${}_F\mathscr{Q}$ verifies the Fatou property, whenever for every $\{\overline{i^{(a)}}\} \subseteq_F \mathscr{Q}_\kappa$ such that $\lim_{a\to\infty} \kappa(\overline{i^{(a)}} - \overline{i}) = \overline{0}$ and $\overline{j} \in_F \mathscr{Q}_\kappa$, one has $\kappa(\overline{j} - \overline{i}) \leq \sup_q \inf_{a\geq q} \kappa(\overline{j} - \overline{i^{(a)}})$.

We will use the following notations:

$$\kappa_1(\overline{j}) = \left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\overline{j_a},\ \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]^{\frac{1}{\rho}} \text{ and } \kappa_2(\overline{j}) = \sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\overline{j_a},\ \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b},$$

for every $\overline{j} \in_F \Omega_t(u,\ v)$.

Theorem 6 The function κ_1 verifies the Fatou property.

Proof. If $\{\overline{j^{(d)}}\} \subseteq ({}_F\Omega_t(u,\ v))_{\kappa_1}$ with $\lim_{d\to\infty} \kappa_1(\overline{j^{(d)}} - \overline{j}) = 0$. Obviously, $\overline{j} \in ({}_F\Omega_t(u,\ v))_{\kappa_1}$. For all $\overline{i} \in ({}_F\Omega_t(u,\ v))_{\kappa_1}$, then

$$\kappa_1(\bar{i}-\bar{j}) = \left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a(\overline{i_a}-\overline{j_a}),\ \bar{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]^{\frac{1}{\rho}}$$

$$\leq \left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a(\overline{i_a}-\overline{j_a^{(d)}}),\ \bar{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]^{\frac{1}{\rho}} + \left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a(\overline{j_a^{(d)}}-\overline{j_a}),\ \bar{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]^{\frac{1}{\rho}}$$

$$\leq \sup_q \inf_{d\geq q} \kappa_1(\bar{i}-\overline{j^{(d)}}).$$

□

Theorem 7 The function κ_2 does not verify the Fatou property.

Proof. Suppose $\{\overline{j^{(d)}}\} \subseteq ({}_F\Omega_t(u,\ v))_{\kappa_2}$ with $\lim_{b\to\infty}\kappa_2(\overline{j^{(d)}}-\bar{j}) = 0$. Evidently, $\bar{j} \in ({}_F\Omega_t(u,\ v))_{\kappa_2}$. For all $\bar{i} \in ({}_F\Omega_t(u,\ v))_{\kappa_2}$, we have

$$\kappa_2(\bar{i}-\bar{j}) = \sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a(\overline{i_a}-\overline{j_a}),\ \bar{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}$$

$$\leq 2^{\rho-1}\left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a(\overline{i_a}-\overline{j_a^{(d)}}),\ \bar{0}\right)}{\Gamma(t+b+1)}\right)^{v_b} + \sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a(\overline{j_a^{(d)}}-\overline{j_a}),\ \bar{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]$$

$$\leq 2^{\rho-1}\sup_q \inf_{d\geq q} \kappa_2(\bar{i}-\overline{j^{(d)}}).$$

Therefore, κ_2 does not verify the Fatou property. □

Definition 7 [15] An operator $G :_F \mathscr{Q}_\kappa \to_F \mathscr{Q}_\kappa$ is said to be a Kannan κ-contraction, when one has $\varepsilon \in [0,\ \frac{1}{2})$, with $\kappa(G\bar{j}-G\bar{k}) \leq \varepsilon(\kappa(G\bar{j}-\bar{j})+\kappa(G\bar{k}-\bar{k}))$, for every $\bar{j},\ \bar{k} \in_F \mathscr{Q}_\kappa$. If $G(\bar{j}) = \bar{j}$, then $\bar{j} \in_F \mathscr{Q}_\kappa$ is said to be a fixed point of G.

Theorem 8 Assume $G : ({}_F\Omega_t(u,\ v))_{\kappa_1} \to ({}_F\Omega_t(u,\ v))_{\kappa_1}$ is Kannan κ_1-contraction operator, then G has a unique fixed point.

Proof. Suppose $\bar{k} \in_F \Omega_t(u,\ v)$, then $G^b\bar{k} \in_F \Omega_t(u,\ v)$. Since G is a Kannan κ_1-contraction, we have

$$\kappa_1(G^{b+1}\overline{k}-G^b\overline{k})\leq \varepsilon\left(\kappa_1(G^{b+1}\overline{k}-G^b\overline{k})+\kappa_1(G^b\overline{k}-G^{b-1}\overline{k})\right)\Rightarrow$$

$$\kappa_1(G^{b+1}\overline{k}-G^b\overline{k})\leq \frac{\varepsilon}{1-\varepsilon}\kappa_1(G^b\overline{k}-G^{b-1}\overline{k})\leq\left(\frac{\varepsilon}{1-\varepsilon}\right)^2\kappa_1(G^{b-1}\overline{k}-G^{b-2}\overline{k})\leq\ldots$$

$$\leq\left(\frac{\varepsilon}{1-\varepsilon}\right)^b\kappa_1(G\overline{k}-\overline{k}).$$

Therefore, for every $a,\ b\in\mathbb{N}_0$ with $a>b$ we have

$$\kappa_1(G^b\overline{k}-G^a\overline{k})\leq\varepsilon\left(\kappa_1(G^b\overline{k}-G^{b-1}\overline{k})+\kappa_1(G^a\overline{k}-G^{a-1}\overline{k})\right)$$

$$\leq\varepsilon\left(\left(\frac{\varepsilon}{1-\varepsilon}\right)^{b-1}+\left(\frac{\varepsilon}{1-\varepsilon}\right)^{a-1}\right)\kappa_1(G\overline{k}-\overline{k}).$$

Hence $\{G^b\overline{k}\}$ is a Cauchy sequence in $({}_F\Omega_t(u,\ v))_{\kappa_1}$. Since $({}_F\Omega_t(u,\ v))_{\kappa_1}$ is p-qB. Then $\overline{q}\in({}_F\Omega_t(u,\ v))_{\kappa_1}$ such that $\lim_{b\to\infty}G^b\overline{k}=\overline{q}$. To prove that $G(\overline{q})=\overline{q}$. As κ_1 verifies the Fatou property, we have

$$\kappa_1(G\overline{q}-\overline{q})\leq\sup_i\inf_{b\geq i}\kappa_1(G^{b+1}\overline{k}-G^b\overline{k})\leq\sup_i\inf_{b\geq i}\left(\frac{\varepsilon}{1-\varepsilon}\right)^b\kappa_1(G\overline{k}-\overline{k})=0,$$

so $G(\overline{q})=\overline{q}$. Hence, $\overline{q}$ is a fixed point of G. To prove the uniqueness of the fixed point. For two different fixed points $\overline{i},\ \overline{q}\in({}_F\Omega_t(u,\ v))_{\kappa_1}$ of G. Then

$$\kappa_1(\overline{i}-\overline{q})\leq\kappa_1(G\overline{i}-G\overline{q})\leq\varepsilon\left(\kappa_1(G\overline{i}-\overline{i})+\kappa_1(G\overline{q}-\overline{q})\right)=0.$$

So $\overline{i}=\overline{q}$. □

Corollary 1 Assume $G:({}_F\Omega_t(u,\ v))_{\kappa_1}\to({}_F\Omega_t(u,\ v))_{\kappa_1}$ is Kannan κ_1-contraction, then G has a unique fixed point $\overline{q}$ with $\kappa_1(G^b\overline{k}-\overline{q})\leq\varepsilon\left(\frac{\varepsilon}{1-\varepsilon}\right)^{b-1}\kappa_1(G\overline{k}-\overline{k})$.

Proof. By Theorem 8, there is a unique fixed point $\overline{q}$ of G. Then

$$\kappa_1(G^b\overline{k}-\overline{q})=\kappa_1(G^b\overline{k}-G\overline{q})\leq\varepsilon\left(\kappa_1(G^b\overline{k}-G^{b-1}\overline{k})+\kappa_1(G\overline{q}-\overline{q})\right)=\varepsilon\left(\frac{\varepsilon}{1-\varepsilon}\right)^{b-1}\kappa_1(G\overline{k}-\overline{k}).$$

□

Definition 8 [15] Let ${}_F\mathscr{Q}_\kappa$ be a p-qN-pssff, $G:{}_F\mathscr{Q}_\kappa\to{}_F\mathscr{Q}_\kappa$ and $\overline{j}\in{}_F\mathscr{Q}_\kappa$. The operator G is called κ-sequentially continuous at $\overline{j}$, if and only if, assume $\lim_{i\to\infty}\kappa(\overline{g_i}-\overline{j})=0$, then $\lim_{i\to\infty}\kappa(G\overline{g_i}-G\overline{j})=0$.

In the next theorem, we explain how Kannan-type contractions are different from Banach contractions, taking into account the continuity of the mappings.

Theorem 9 Supposing that $G : ({}_F\Omega_t(u,\ v))_{\kappa_2} \to ({}_F\Omega_t(u,\ v))_{\kappa_2}$. The element $\overline{k} \in ({}_F\Omega_t(u,\ v))_{\kappa_2}$ is the unique fixed point of G, when the next setups are satisfied:

(c1) G is Kannan κ_2-contraction,

(c2) G is κ_2-sequentially continuous at $\overline{k} \in ({}_F\Omega_t(u,\ v))_{\kappa_2}$,

(c3) there is $\overline{u} \in ({}_F\Omega_t(u,\ v))_{\kappa_2}$ with $\{G^q\overline{u}\}$ has $\{G^{q_i}\overline{u}\}$ converges to $\overline{k}$.

Proof. If $\overline{k}$ is not a fixed point of G, we get $G\overline{k} \neq \overline{k}$. By the setups (c2) and (c3), we have

$$\lim_{q_i\to\infty} \kappa_2(G^{q_i}\overline{u} - \overline{k}) = 0 \text{ and } \lim_{q_i\to\infty} \kappa_2(G^{q_i+1}\overline{u} - G\overline{k}) = 0.$$

According to the proofs of theorem 5 and theorem 8, since G is Kannan κ_2-contraction, we have

$$\begin{aligned} 0 < \kappa_2(G\overline{k} - \overline{k}) &= \kappa_2\left((G\overline{k} - G^{q_i+1}\overline{u}) + (G^{q_i}\overline{u} - \overline{k}) + (G^{q_i+1}\overline{u} - G^{q_i}\overline{u})\right) \\ &\leq 2^{2\rho-2}\kappa_2\left(G^{q_i+1}\overline{u} - G\overline{k}\right) + 2^{2\rho-2}\kappa_2\left(G^{q_i}\overline{u} - \overline{k}\right) + 2^{\rho-1}\varepsilon\left(\frac{\varepsilon}{1-\varepsilon}\right)^{q_i-1}\kappa_2(G\overline{u} - \overline{u}). \end{aligned}$$

Let $q_i \to \infty$, there is a contradiction. Hence, $\overline{k}$ is a fixed point of G. For the uniqueness of $\overline{k}$, let we have two different fixed points $\overline{k},\ \overline{r} \in ({}_F\Omega_t(u,\ v))_{\kappa_2}$ of G. Hence

$$\kappa_2(\overline{k} - \overline{r}) \leq \kappa_2(G\overline{k} - G\overline{r}) \leq \varepsilon\left(\kappa_2(G\overline{k} - \overline{k}) + \kappa_2(G\overline{r} - \overline{r})\right) = 0.$$

Therefore, $\overline{k} = \overline{r}$. $\square$

Example 1 Suppose

$$A : \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty},\ \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_1} \to \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty},\ \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_1}$$

and

$$A(\overline{h}) = \begin{cases} \dfrac{\overline{h}}{4}, & \kappa_1(\overline{h}) \in [0,\ 1), \\ \\ \dfrac{\overline{h}}{5}, & \kappa_1(\overline{h}) \in [1,\ \infty). \end{cases}$$

For every $\overline{h},\ \overline{r} \in \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty},\ \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_1}$. If $\kappa_1(\overline{h}),\ \kappa_1(\overline{r}) \in [0,\ 1)$, one has

$$\kappa_1(A\overline{h} - A\overline{r}) = \kappa_1(\frac{\overline{h}}{4} - \frac{\overline{r}}{4}) \leq \frac{1}{\sqrt[4]{27}}\left(\kappa_1(\frac{3\overline{h}}{4}) + \kappa_1(\frac{3\overline{r}}{4})\right) = \frac{1}{\sqrt[4]{27}}\left(\kappa_1(A\overline{h} - \overline{h}) + \kappa_1(A\overline{r} - \overline{r})\right).$$

For all $\kappa_1(\overline{h}),\ \kappa_1(\overline{r}) \in [1,\ \infty)$, one has

$$\kappa_1(A\overline{h}-A\overline{r}) = \kappa_1(\frac{\overline{h}}{5}-\frac{\overline{r}}{5}) \leq \frac{1}{\sqrt[4]{64}}\Big(\kappa_1(\frac{4\overline{h}}{5})+\kappa_1(\frac{4\overline{r}}{5})\Big) = \frac{1}{\sqrt[4]{64}}\Big(\kappa_1(A\overline{h}-\overline{h})+\kappa_1(A\overline{r}-\overline{r})\Big).$$

For all $\kappa_1(\overline{h}) \in [0,\ 1)$ and $\kappa_1(\overline{r}) \in [1,\ \infty)$, we get

$$\kappa_1(A\overline{h}-A\overline{r}) = \kappa_1(\frac{\overline{h}}{4}-\frac{\overline{r}}{5}) \leq \frac{1}{\sqrt[4]{27}}\kappa_1(\frac{3\overline{h}}{4})+\frac{1}{\sqrt[4]{64}}\kappa_1(\frac{4\overline{r}}{5}) \leq \frac{1}{\sqrt[4]{27}}\Big(\kappa_1(\frac{3\overline{h}}{4})+\kappa_1(\frac{4\overline{r}}{5})\Big)$$

$$= \frac{1}{\sqrt[4]{27}}\Big(\kappa_1(A\overline{h}-\overline{h})+\kappa_1(A\overline{r}-\overline{r})\Big).$$

Then A is Kannan κ_1-contraction. Since κ_1 verifies the Fatou property. From Theorem 9, A has a unique fixed point $\overline{\theta}$. If $\{\overline{h^{(b)}}\} \subseteq \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty},\ \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_1}$ with $\lim_{b\to\infty}\kappa_1(\overline{h^{(b)}}-\overline{h^{(0)}}) = 0$, where $\overline{h^{(0)}} \in \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty},\ \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_1}$ so that $\kappa_1(\overline{h^{(0)}}) = 1$. Since κ_1 is continuous, we obtain

$$\lim_{b\to\infty}\kappa_1(A\overline{h^{(b)}}-A\overline{h^{(0)}}) = \lim_{b\to\infty}\kappa_1\Big(\frac{\overline{h^{(b)}}}{4}-\frac{\overline{h^{(0)}}}{5}\Big) = \kappa_1\Big(\frac{\overline{h^{(0)}}}{20}\Big) > 0.$$

Hence A is not κ_1-sequentially continuous at $\overline{h^{(0)}}$. This gives A is not continuous at $\overline{h^{(0)}}$.
For all $\overline{h},\ \overline{r} \in \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty},\ \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_2}$. Suppose $\kappa_2(\overline{h}),\ \kappa_2(\overline{r}) \in [0,\ 1)$, we have

$$\kappa_2(A\overline{h}-A\overline{r}) = \kappa_2(\frac{\overline{h}}{4}-\frac{\overline{r}}{4}) \leq \frac{2}{\sqrt{27}}\Big(\kappa_2(\frac{3\overline{h}}{4})+\kappa_2(\frac{3\overline{r}}{4})\Big) = \frac{2}{\sqrt{27}}\Big(\kappa_2(A\overline{h}-\overline{h})+\kappa_2(A\overline{r}-\overline{r})\Big).$$

Suppose $\kappa_2(\overline{h}),\ \kappa_2(\overline{r}) \in [1,\ \infty)$, we have

$$\kappa_2(A\overline{h}-A\overline{r}) = \kappa_2(\frac{\overline{h}}{5}-\frac{\overline{r}}{5}) \leq \frac{1}{4}\Big(\kappa_2(\frac{4\overline{h}}{5})+\kappa_2(\frac{4\overline{r}}{5})\Big) = \frac{1}{4}\Big(\kappa_2(A\overline{h}-\overline{h})+\kappa_2(A\overline{r}-\overline{r})\Big).$$

For all $\kappa_2(\overline{h}) \in [0,\ 1)$ and $\kappa_2(\overline{r}) \in [1,\ \infty)$, we have

$$\kappa_2(A\overline{h}-A\overline{r}) = \kappa_2(\frac{\overline{h}}{4}-\frac{\overline{r}}{5}) \leq \frac{2}{\sqrt{27}}\kappa_2(\frac{3\overline{h}}{4})+\frac{1}{4}\kappa_2(\frac{4\overline{r}}{5}) \leq \frac{2}{\sqrt{27}}\Big(\kappa_2(\frac{3\overline{h}}{4})+\kappa_2(\frac{4\overline{r}}{5})\Big)$$

$$= \frac{2}{\sqrt{27}}\Big(\kappa_2(A\overline{h}-\overline{h})+\kappa_2(A\overline{r}-\overline{r})\Big).$$

Therefore, A is Kannan κ_2-contraction and $A^q(\overline{h}) = \begin{cases} \frac{\overline{h}}{4^q}, & \kappa_2(\overline{h}) \in [0,\ 1), \\ \frac{\overline{h}}{5^q}, & \kappa_2(\overline{h}) \in [1,\ \infty). \end{cases}$

Clearly, A is κ_2-sequentially continuous at $\overline{\theta}$ and $\{A^q\overline{h}\}$ has a subsequence $\{A^{q_j}\overline{h}\}$ converges to $\overline{\theta}$. By Theorem 9, $\overline{\theta}$ is the unique fixed point of A.

Example 2 Assume

$$A : \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty}, \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_2} \to \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty}, \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_2}$$

and

$$A(\overline{h}) = \begin{cases} \frac{1}{4}(\overline{e_1}+\overline{h}), & \overline{h_0}(m) \in [0,\ \frac{1}{3}), \\ \frac{1}{3}\overline{e_1}, & \overline{h_0}(m) = \frac{1}{3}, \\ \frac{1}{4}\overline{e_1}, & \overline{h_0}(m) \in (\frac{1}{3},\ 1]. \end{cases}$$

Since $\overline{h_0},\ \overline{r_0} \in [0,\ \frac{1}{3})$, one has

$$\kappa_2(A\overline{h}-A\overline{r}) = \kappa_2(\frac{1}{4}(\overline{h_0}-\overline{r_0},\ \overline{h_1}-\overline{r_1},\ \overline{h_2}-\overline{r_2},\ \ldots)) \leq \frac{2}{\sqrt{27}}\left(\kappa_2(\frac{3\overline{h}}{4})+\kappa_2(\frac{3\overline{r}}{4})\right)$$

$$\leq \frac{2}{\sqrt{27}}\left(\kappa_2(A\overline{h}-\overline{h})+\kappa_2(A\overline{r}-\overline{r})\right).$$

For every $\overline{h_0},\ \overline{r_0} \in (\frac{1}{3},\ 1]$, then for every $\varepsilon > 0$, one has

$$\kappa_2(A\overline{h}-A\overline{r}) = 0 \leq \varepsilon\left(\kappa_2(A\overline{h}-\overline{h})+\kappa_2(A\overline{r}-\overline{r})\right).$$

For every $\overline{h_0} \in [0,\ \frac{1}{3})$ and $\overline{r_0} \in (\frac{1}{3},\ 1]$, we have

$$\kappa_2(A\overline{h}-A\overline{r}) = \kappa_2(\frac{\overline{h}}{4}) \leq \frac{1}{\sqrt{27}}\kappa_2(\frac{3\overline{h}}{4}) = \frac{1}{\sqrt{27}}\kappa_2(A\overline{h}-\overline{h}) \leq \frac{1}{\sqrt{27}}\left(\kappa_2(A\overline{h}-\overline{h})+\kappa_2(A\overline{r}-\overline{r})\right).$$

Therefore, A is Kannan κ_2-contraction. Evidently, A is κ_2-sequentially continuous at $\frac{1}{3}\overline{e_1}$ and there is $\overline{h} = (\overline{h_0},\ \overline{h_1},\ \overline{h_2},\ \cdots) \in \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty}, \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_2}$ with $\overline{h_0} \in [0,\ \frac{1}{3})$ so that the sequence of iterates

$\{A^q\overline{h}\} = \left\{\sum_{a=1}^{q} \frac{1}{4^a}\overline{e_1} + \frac{1}{4^q}\overline{h}\right\}$ has a subsequence $\{A^{q_j}\overline{h}\} = \left\{\sum_{a=1}^{q_j} \frac{1}{4^a}\overline{e_1} + \frac{1}{4^{q_j}}\overline{h}\right\}$ converges to $\frac{1}{3}\overline{e_1}$. By Theorem 9, the operator A has one fixed point $\frac{1}{3}\overline{e_1}$. Recall that A is not continuous at $\frac{1}{3}\overline{e_1}$.

For all $\overline{h}, \overline{r} \in \left(\Omega_p^F\left(\left(\frac{a!}{(a+5)\Gamma(a+t)}\right)_{a=0}^{\infty}, \left(\frac{2a+3}{a+2}\right)_{a=0}^{\infty}\right)\right)_{\kappa_1}$. If $\overline{h_0}, \overline{r_0} \in [0, \frac{1}{3})$, we have

$$\kappa_1(A\overline{h} - A\overline{r}) = \kappa_1(\frac{1}{4}(\overline{h_0} - \overline{r_0}, \overline{h_1} - \overline{r_1}, \overline{h_2} - \overline{r_2}, \ldots)) \leq \frac{1}{\sqrt[4]{27}}\left(\kappa_1(\frac{3\overline{h}}{4}) + \kappa_1(\frac{3\overline{r}}{4})\right)$$

$$\leq \frac{1}{\sqrt[4]{27}}\left(\kappa_1(A\overline{h} - \overline{h}) + \kappa_1(A\overline{r} - \overline{r})\right).$$

If $\overline{h_0}, \overline{r_0} \in (\frac{1}{3}, 1]$, then for every $\varepsilon > 0$, we have

$$\kappa_1(A\overline{h} - A\overline{r}) = 0 \leq \varepsilon\left(\kappa_1(A\overline{h} - \overline{h}) + \kappa_1(A\overline{r} - \overline{r})\right).$$

Assume $\overline{h_0} \in [0, \frac{1}{3})$ and $\overline{r_0} \in (\frac{1}{3}, 1]$, one obtains

$$\kappa_1(A\overline{h} - A\overline{r}) = \kappa_1(\frac{\overline{h}}{4}) \leq \frac{1}{\sqrt[4]{27}}\kappa_1(\frac{3\overline{h}}{4}) = \frac{1}{\sqrt[4]{27}}\kappa_1(A\overline{h} - \overline{h}) \leq \frac{1}{\sqrt[4]{27}}\left(\kappa_1(A\overline{h} - \overline{h}) + \kappa_1(A\overline{r} - \overline{r})\right).$$

Therefore, A is Kannan κ_1-contraction mapping. Since κ_1 satisfies the Fatou property. By Theorem 8, the operator A has a unique fixed point $\frac{1}{3}\overline{e_1}$.

4. Applications

In this section, we have introduced a solution in $({}_F\Omega_t(u, v))_{\kappa_1}$ to Volterra-type summable equation of fuzzy functions (2) with the setups of Theorem 5.

Theorem 10 The Fuzzy Volterra-Type Non-linear Dynamical Economic Model (2) has one and only one solution in $({}_F\Omega_t(u, v))_{\kappa_1}$, when $\overline{\eta} : \mathbb{N}_0 \to \mathscr{R}([0, 1])$, one has $\varepsilon \in \mathscr{R}$ with $\sup_b |\varepsilon|^{\frac{v_b}{p}} \in [0, \frac{1}{2})$ and for all $b \in \mathbb{N}_0$, then

$$\left|\sum_{a=0}^{b}\left(\sum_{q\in\mathbb{N}_0}\Pi(a, q)[g(q, \overline{J_q}) - g(q, \overline{\eta_q})]\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|$$

$$\leq|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{r_a} - \overline{J_a} + \sum_{q=0}^{\infty}\Pi(a, q)g(q, \overline{J_q})\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|$$

$$+|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{r_a} - \overline{\eta_a} + \sum_{q=0}^{\infty}\Pi(a, q)g(q, \overline{\eta_q})\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|.$$

Proof. Suppose the setups are confirmed. If the operator $L: ({}_F\Omega_t(u,\ v))_{\kappa_1} \to ({}_F\Omega_t(u,\ v))_{\kappa_1}$ is defined by equation (3). So

$$\kappa_1(L\overline{J}-L\overline{\eta}) = \left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a(L\overline{J_a}-L\overline{\eta_a}),\ \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]^{\frac{1}{\rho}}$$

$$= \left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\left(\sum_{q\in\mathbb{N}_0}\Pi(a,\ q)[g(q,\ \overline{J_q})-g(q,\ \overline{\eta_q})]\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a,\ \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]^{\frac{1}{\rho}}$$

$$\leq \sup_b|\varepsilon|^{\frac{v_b}{\rho}}\left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\left(\overline{r_a}-\overline{J_a}+\sum_{q=0}^{\infty}\Pi(a,\ q)g(q,\ \overline{J_q})\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a,\ \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]^{\frac{1}{\rho}}$$

$$+\sup_b|\varepsilon|^{\frac{v_b}{\rho}}\left[\sum_{b=0}^{\infty}\left(\frac{b!t\overline{\tau}\left(\sum_{a=0}^{b}\left(\overline{r_a}-\overline{\eta_a}+\sum_{q=0}^{\infty}\Pi(a,\ q)g(q,\ \overline{\eta_q})\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a,\ \overline{0}\right)}{\Gamma(t+b+1)}\right)^{v_b}\right]^{\frac{1}{\rho}}$$

$$= \sup_b|\varepsilon|^{\frac{v_b}{\rho}}\left(\kappa_1(L\overline{J}-\overline{J})+\kappa_1(L\overline{\eta}-\overline{\eta})\right).$$

□

From Theorem 8, one has a unique solution of (2) in $({}_F\Omega_t(u,\ v))_{\kappa_1}$.

Example 3 Let $\left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\ \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_1}$.

Assume the Fuzzy Volterra-Type Non-linear Dynamical Economic Models:

$$\overline{j_a} = \overline{\cos(2a+1)} + \sum_{q=0}^{\infty}2^{a+q}\frac{\overline{j_{a-2}^r}}{\overline{j_{a-1}^d}+\overline{q^2+1}}, \tag{5}$$

so that $r,\ d>0$ and $\overline{j_{-2}}(t),\ \overline{j_{-1}}(t)>0$, for every $t\in\mathscr{R}$ and assume

$$L: \left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\ \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_1} \to \left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\ \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_1},$$

is defined by

$$L(\overline{j_a})_{a=0}^{\infty} = \left(\overline{\cos(2a+1)} + \sum_{q=0}^{\infty} 2^{a+q} \frac{\overline{j_{a-2}^{r}}}{\overline{j_{a-1}^{d}} + \overline{q^2+1}}\right)_{a=0}^{\infty}. \tag{6}$$

Clearly, we get $\varepsilon \in \mathscr{R}$ with $\sup_b |\varepsilon|^{\frac{2b+3}{2b+4}} \in [0, \frac{1}{2})$ and for all $b \in \mathbb{N}_0$, one has

$$\left|\sum_{a=0}^{b}\left(\sum_{q=0}^{\infty} 2^{a} \frac{\overline{j_{a-2}^{r}}}{\overline{j_{a-1}^{d}} + \overline{q^2+1}}\left(2^q - 2^q\right)\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|$$

$$\preceq |\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{\cos(2a+1)} - \overline{j_a} + \sum_{q=0}^{\infty} 2^{a+q} \frac{\overline{j_{a-2}^{r}}}{\overline{j_{a-1}^{d}} + \overline{q^2+1}}\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right| +$$

$$|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{\cos(2a+1)} - \overline{\eta_a} + \sum_{q=0}^{\infty} 2^{a+q} \frac{\overline{\eta_{a-2}^{r}}}{\overline{\eta_{a-1}^{d}} + \overline{q^2+1}}\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|.$$

By Theorem 11, the system (5) has a unique solution in $\left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty}, \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_1}$.

Theorem 11 Suppose $L : ({}_F\Omega_t(u, v))_{\kappa_2} \to ({}_F\Omega_t(u, v))_{\kappa_2}$ is defined by (3) and $v_0 > 1$. The Fuzzy Volterra-Type Non-linear Dynamical Economic Model (2) has a unique solution $\bar{l} \in ({}_F\Omega_t(u, v))_{\kappa_2}$, if the next setups are verified:

(1) If $\Pi : \mathbb{N}_0^2 \to \mathscr{R},\ g : \mathbb{N}_0 \times \mathscr{R}([0, 1]) \to \mathscr{R}([0, 1]),\ \bar{j} : \mathbb{N}_0 \to \mathscr{R}([0, 1]),\ \bar{r} : \mathbb{N}_0 \to \mathscr{R}([0, 1]),\ \bar{k} : \mathbb{N}_0 \to \mathscr{R}([0, 1])$, assume one has $\varepsilon \in \mathscr{R}$ with $2^{\rho-1}\sup_b |\varepsilon|^{v_b} \in [0, \frac{1}{2})$ and for every $b \in \mathbb{N}_0$, then

$$\left|\sum_{a=0}^{b}\left(\sum_{q\in\mathbb{N}_0}\Pi(a, q)[g(q, \overline{j_q}) - g(q, \overline{k_q})]\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|$$

$$\preceq |\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{r_a} - \overline{j_a} + \sum_{q=0}^{\infty}\Pi(a, q)g(q, \overline{j_q})\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right| +$$

$$|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{r_a} - \overline{k_a} + \sum_{q=0}^{\infty}\Pi(a, q)g(q, \overline{k_q})\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|.$$

(2) L is κ_2-sequentially continuous at $\bar{l} \in ({}_F\Omega_t(u, v))_{\kappa_2}$,

(3) there is $\bar{i} \in ({}_F\Omega_t(u, v))_{\kappa_2}$ with $\{W^q\bar{i}\}$ has $\{W^{q_j}\bar{i}\}$ converging to $\bar{l}$.

Proof. We have

$$\kappa_2(L\bar{j}-L\bar{k})=\sum_{q=0}^{\infty}\left(\frac{q!t\bar{\tau}\left(\sum_{a=0}^{q}\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a(L\overline{j_a}-L\overline{k_a}),\ \bar{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$=\sum_{q=0}^{\infty}\left(\frac{q!t\bar{\tau}\left(\sum_{a=0}^{q}\left(\sum_{q\in\mathbb{N}_0}\Pi(a,\ q)[g(q,\ \overline{j_q})-g(q,\ \overline{k_q})]\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a,\ \bar{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$\leq 2^{\rho-1}\sup_q|\varepsilon|^{v_q}\sum_{q=0}^{\infty}\left(\frac{q!t\bar{\tau}\left(\sum_{a=0}^{q}\left(\overline{r_a}-\overline{j_a}+\sum_{q=0}^{\infty}\Pi(a,\ q)g(q,\ \overline{j_q})\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a,\ \bar{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$+2^{\rho-1}\sup_q|\varepsilon|^{v_q}\sum_{q=0}^{\infty}\left(\frac{q!t\bar{\tau}\left(\sum_{a=0}^{q}\left(\overline{r_a}-\overline{k_a}+\sum_{q=0}^{\infty}\Pi(a,\ q)g(q,\ \overline{k_q})\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a,\ \bar{0}\right)}{\Gamma(t+q+1)}\right)^{v_q}$$

$$=2^{\rho-1}\sup_q|\varepsilon|^{v_q}\left(\kappa_2(L\bar{j}-\bar{j})+\kappa_2(L\bar{k}-\bar{k})\right).$$

By Theorem 9, we obtain a unique solution $\bar{l}\in({}_F\Omega_t(u,\ v))_{\kappa_2}$ of equation (2). □

Example 4 Consider $\left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\ \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_2}$.

Suppose the summable equations (5).

Assume $L:\left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\ \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_2}\to\left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\ \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_2}$ defined by (6). If L is κ_2-sequentially continuous at $\bar{l}\in\left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\ \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_2}$, and there is $\bar{i}\in\left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\ \left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_2}$ with $\{L^q\bar{i}\}$ has $\{L^{q_j}\bar{i}\}$ converging to $\bar{l}$. Obviously, one has $\varepsilon\in\mathscr{R}$ with $2^{\rho-1}\sup_b|\varepsilon|^{\frac{2b+3}{b+2}}\in[0,\ \frac{1}{2})$ and for every $b\in\mathbb{N}_0$, we have

$$\left|\sum_{a=0}^{b}\left(\sum_{q=0}^{\infty}2^{a}\frac{\overline{j_{a-2}^{b}}}{\overline{j_{a-1}^{d}}+\overline{q^2+1}}\left(2^{q}-2^{q}\right)\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|$$

$$\leq|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{\cos(2a+1)}-\overline{j_a}+\sum_{q=0}^{\infty}2^{a+m}\frac{\overline{j_{a-2}^{b}}}{\overline{j_{a-1}^{d}}+\overline{q^2+1}}\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|$$

$$+|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{\cos(2a+1)}-\overline{k_a}+\sum_{q=0}^{\infty}2^{a+m}\frac{\overline{k_{a-2}^{b}}}{\overline{k_{a-1}^{d}}+\overline{q^2+1}}\right)\frac{\Gamma(a+t)}{\Gamma(a+1)}u_a\right|.$$

According to Theorem 11, the Volterra-type summable equation of fuzzy functions (5) has a unique solution $\bar{l}\in\left(\Omega_p^F\left(\left(\frac{b!}{(b+1)\Gamma(b+t)}\right)_{b=0}^{\infty},\left(\frac{2b+3}{b+2}\right)_{b=0}^{\infty}\right)\right)_{\kappa_2}$.

5. Conclusion

In this article, we offered some topological and geometric properties of $\left({}_F\Omega_t(u,v)\right)_\kappa$. The existence of a fixed point in the Kannan contraction operator on this space is discussed. Many numerical experiments were conducted to verify our hypotheses. Fuzzy Volterra-Type Non-linear Dynamical Economic Models are also studied. All contraction operators in this new fuzzy function space are examined for their fixed points, and a novel generic solution space for multiple stochastic nonlinear dynamical systems is presented.

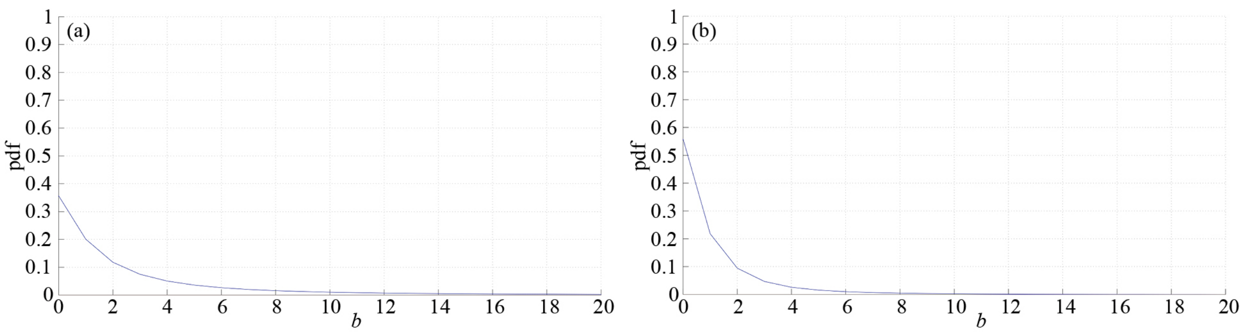

Figure 1. (a) The pdf when $t=1$, $u_a=1$, $v_a=\frac{3(a+1)}{a+2}$, and $\overline{j_a}=\overline{\frac{1}{a+1}}$ and (b) the pdf when $t=1$, $u_a=\frac{1}{a+1}$, $v_a=\frac{3(a+1)}{a+2}$, and $\overline{j_a}=\overline{\frac{1}{a+1}}$

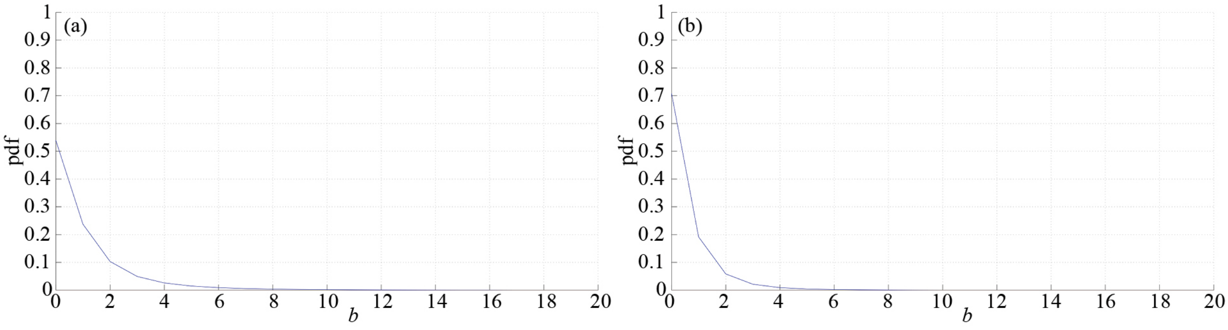

Figure 2. (a) The pdf when $t = 1.5$, $u_a = \dfrac{1.5\Gamma(a+1)\Gamma(1.5)}{\Gamma(a+1.5)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{\dfrac{1}{a+1}}$ and (b) the pdf when $t = 1.5$, $u_a = \dfrac{1.5\Gamma(a+1)\Gamma(1.5)}{(a+1.5)\Gamma(a+1.5)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{\dfrac{1}{a+1}}$

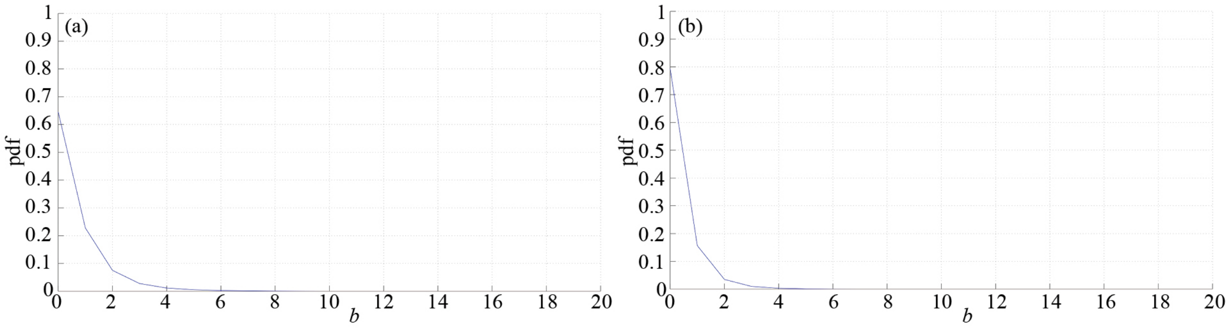

Figure 3. (a) The pdf when $t = 2$, $u_a = \dfrac{2}{a+1}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{\dfrac{1}{a+1}}$ and (b) the pdf when $t = 2$, $u_a = \dfrac{2}{(a+1)(a+2)}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{\dfrac{1}{a+1}}$

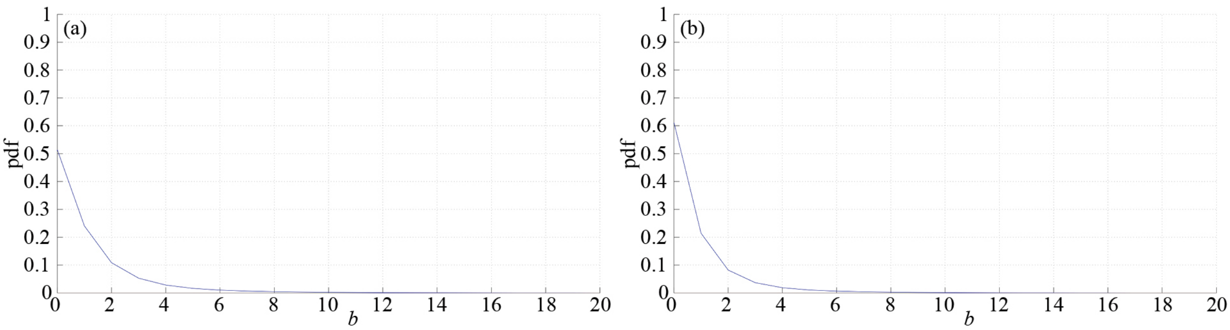

Figure 4. (a) The pdf when $t = 1$, $u_a = 1$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$ and (b) the pdf when $t = 1$, $u_a = \dfrac{1}{a+1}$, $v_a = \dfrac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$

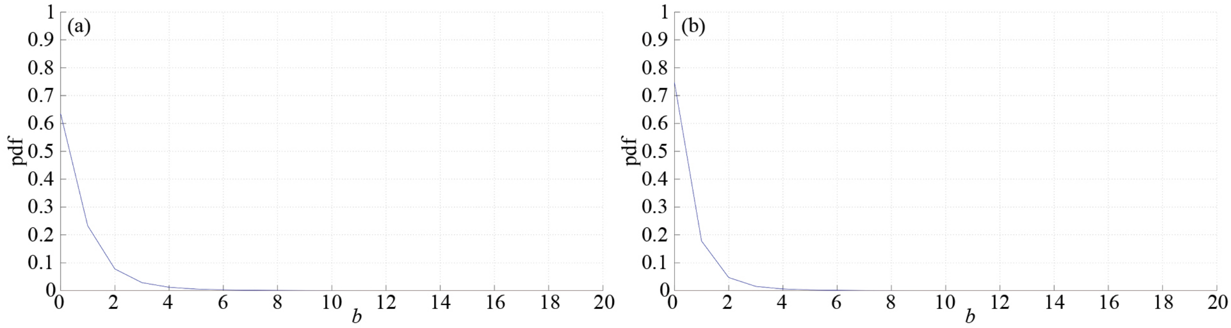

Figure 5. (a) The pdf when $t = 1.5$, $u_a = \frac{1.5\Gamma(a+1)\Gamma(1.5)}{\Gamma(a+1.5)}$, $v_a = \frac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$ and (b) the pdf when $t = 1.5$, $u_a = \frac{1.5\Gamma(a+1)\Gamma(1.5)}{(a+1.5)\Gamma(a+1.5)}$, $v_a = \frac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$

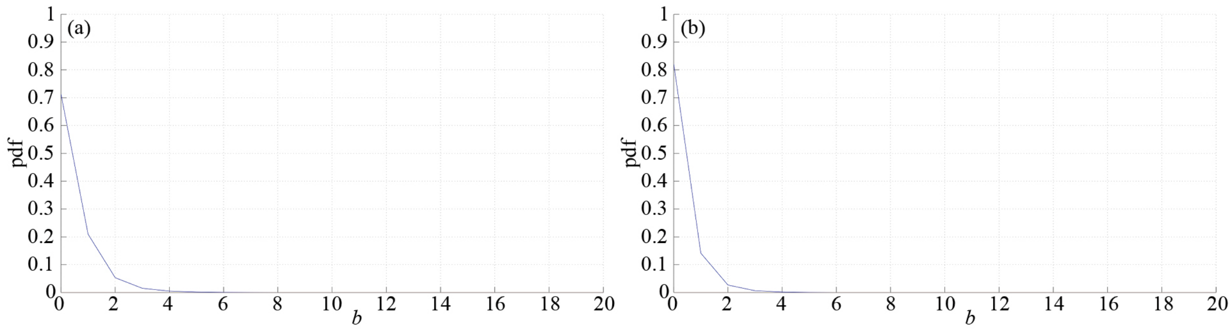

Figure 6. (a) The pdf when $t = 2$, $u_a = \frac{2}{a+1}$, $v_a = \frac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$ and (b) the pdf when $t = 2$, $u_a = \frac{2}{(a+1)(a+2)}$, $v_a = \frac{3(a+1)}{a+2}$, and $\overline{j_a} = \overline{e^{-a}}$

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Conflict of interest

The authors declare no competing financial interest.

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