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Research Article

Decision-Making on Deferred Statistical Convergence of Measurable Functions of Two Variables

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Abstract: In this paper, we define and study strongly deferred Cesàro summable, strongly Cesàro summable, m-statistical convergence and m-deferred statistical convergence of real-valued Lebesgue measurable functions of two variables. Further, we present illustrative examples in support of our definitions. Also, we examine some properties and relations among these concepts under some restrictions. In addition, we present illustrative examples to show the essentiality of these restrictions.

Keywords: measurable functions in two variables, lebesgue measure, statistical convergence, deferred statistical convergence, strongly deferred cesàro mean

MSC: 28A20, 40G15

1. Introduction and preliminaries

The credit of introducing the idea of statistical convergence, which is a generalization of the usual notion of convergence, goes to Zygmund [1]. Steinhaus [2] and Fast [3] independently introduced the idea of statistical convergence and was later reintroduced by Schoenberg [4] independently. The idea of statistical convergence was later extended to double sequences by Mursaleen and Edely [5]. Over time, many researchers have contributed to the development of statistical convergence, providing significant results. For instance, Baliarsingh and Nayak [6] introduced the idea of statistical convergence of fuzzy sequences using fractional difference operators. Mursaleen and Mohiuddine [7] introduced the notion of statistical convergence in intuitionist fuzzy normed spaces, while Narrania and Raj [8] introduced the notion of deferred Cesàro statistical product measurable convergence for double sequences of measurable functions. Statistical convergence has broad applications, including its generalizations in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. It is also closely related to the

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concept of convergence in probability and has significant implications in numerical analysis, particularly in the design of fast methods for solving large linear systems of Toeplitz type. This connection extends to the analysis of eigenvalues and singular values of (preconditioned) matrix sequences (see [9, 10]), as well as the Cesàro operator and the Korovkin theory [11]. In approximation theory, statistical convergence has been extensively studied in the context of various operators. For instance, Turhan et al.[12] explored Kantorovich-Stancu type $(\alpha,\ \lambda,\ s)$-Bernstein operators and their approximation properties, Özger et al. [13] examined the rate of weighted statistical convergence for generalized blending-type Bernstein-Kantorovich operators. Additionally, Ansari et al. [14] provided numerical and theoretical approximation results for Schurer-Stancu operators with a shape parameter λ. Recently, Kumar et al. [15] explored asymptotically Wijsman lacunary sequences of order $(\alpha,\ \beta)$, Mursaleen et al. [16] studied invariant means and lacunary sequence spaces of order $(\alpha,\ \beta)$, Bilalov and Sadigova [17] investigated the convergence of functions with the help of m-density and m-statistical convergence, Et et al. [18] introduced the concepts of μ-deferred statistical convergence of real-valued measurable functions. For more applications, one may refer [19–26].

Throughout the paper, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the set of all real numbers, rational numbers and natural numbers, respectively.

A sequence $y = (y_{jk})$ is convergent in Pringsheim sense if for every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $|y_{jk} - L| < \varepsilon, \forall\ j,\ k > M$.

Let $Q \subseteq \mathbb{N}^2$ and $Q(r,\ s) = \{(i,\ j) \in Q : i \leq r,\ j \leq s\}$. The double natural density of Q is defined by

$$\delta^2(Q) = \lim_{r,\ s \to \infty} \frac{|Q(r,\ s)|}{rs},\ \text{if the limit exists.}$$

A double sequence $y = (y_{jk})$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{r,\ s \to \infty} \frac{1}{rs} \left| \{(j,\ k) : j \leq r,\ k \leq s : |y_{jk} - L| \geq \varepsilon\} \right| = 0.$$

A sequence $y = (y_{jk})$ is strongly Cesàro summable to l if

$$\lim_{r,\ s \to \infty} \frac{1}{rs} \sum_{j=1}^{r} \sum_{k=1}^{s} |y_{jk} - l| = 0.$$

In 1932, Agnew [27] gave the notion of deferred Cesàro mean of the sequence $y = (y_k)$ by

$$(D_{(a,\ b)}\ y)_s = \frac{1}{b(s) - a(s)} \sum_{k=a(s)+1}^{b(s)} y_k,$$

where $\{a(s)\}$ and $\{b(s)\}$ are sequences of positive natural numbers satisfying $a(s) < b(s)$ and $\lim_{s \to \infty} b(s) = \infty$.

In 2016, Dagadur and Sezgek [28] extended the notions of deferred Cesàro mean and deferred statistical convergence to double sequences.

Let $y = (y_{jk})$ be a double sequence and $a(s),\ b(s),\ c(r)$ and $d(r)$ be sequences of nonnegative integers satisfying the conditions $a(s) < b(s)$, $c(r) < d(r)$ and $\lim_{s \to \infty} b(s) = \infty$, $\lim_{r \to \infty} d(r) = \infty$. Then deferred Cesàro mean $(D_{(a,\ b,\ c,\ d)}\ y)_{r,\ s}$ of the double sequence y is defined by

$$(D_{(a,\ b,\ c,\ d)}y)_{r,\ s} = \frac{1}{(b(s)-a(s))(d(r)-c(r))} \sum_{k=c(r)+1}^{d(r)} \sum_{j=a(s)+1}^{b(s)} y_{jk}.$$

Let $M \subseteq \mathbb{N}^2$ and $M(r,\ s) = \{(j,\ k) \in M : a(s) < j \leq b(s),\ c(r) < k \leq d(r)\}$. The deferred double natural density of M is defined by

$$\delta_{\mathfrak{d}}(M) := \lim_{r,\ s \to \infty} \frac{|M(r,\ s)|}{(b(s)-a(s))(d(r)-c(r))},\ \text{if the limit exists.}$$

A double sequence $y = (y_{jk})$ is said to be deferred statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{r,\ s \to \infty} \frac{\left|\{(j,\ k) : a(s) < j \leq b(s),\ c(r) < k \leq d(r),\ |y_{jk} - L| \geq \varepsilon\}\right|}{(b(s)-a(s))(d(r)-c(r))} = 0.$$

In case $a(s) = 0,\ b(s) = s,\ c(r) = 0$ and $d(r) = r$, deferred statistical convergence coincides with usual statistical convergence.

Several authors studied the notion of deferred statistical convergence and gave interesting results (see [29–32]).

Motivated by the above mentioned work, we generalize deferred statistical convergence and strongly deferred Cesàro summability to measurable functions of two variables. Further, we explore their properties and investigate the relationship between them. This extension to measurable functions of two variables will allow us to handle complex data more effectively, improve methods in areas such as image processing, solving equations, and modeling economic systems.

2. Main results

Definition 1 A real-valued Lebesgue measurable function $f = f(x,\ y)$ of two variables in $J = [1,\ \infty) \times [1,\ \infty)$ is strongly Cesàro summable to any $L \in \mathbb{R}$ if

$$\lim_{z,\ w \to \infty} \frac{1}{zw} \int_1^w \int_1^z |f(x,\ y) - L|^p dxdy = 0\ (1 \leq p < \infty)\ \text{holds.} \tag{1}$$

The set of all strongly Cesàro summable functions will be denoted by $[D_p],\ 1 \leq p < \infty$. Throughout the paper, we have taken a pair of increasing functions $(\chi,\ \xi)$ where χ and ξ are functions from $[1,\ \infty)$ to $[1,\ \infty)$ satisfying

$$\chi(z) < \xi(z)\ \text{and}\ \lim_{z \to \infty} \xi(z) = \infty. \tag{2}$$

The set of all such pairs will be denoted by P. That is,

$$P := \{(\chi,\ \xi) : \chi,\ \xi\ \text{are increasing functions and satisfying 2}\}.$$

Definition 2 Let $(\chi,\ \xi),\ (\rho,\ \phi) \in P$. A Lebesgue measurable function $f: J \to \mathbb{R}$ is strongly deferred Cesàro summable (or, strongly $[D_{\chi,\ \rho}^{\xi,\ \phi}]$-summable) to $L \in \mathbb{R}$ if

$$\lim_{z,\ w\to\infty} \frac{1}{(\xi(z)-\chi(z))(\phi(w)-\rho(w))} \int_{\rho(w)}^{\phi(w)} \int_{\chi(z)}^{\xi(z)} |f(x,\ y)-L| dxdy = 0, \text{ holds.}$$

In this case, we write $[D_{\chi,\ \rho}^{\xi,\ \phi}] - \lim f(x,\ y) = L$.

The set of all strongly deferred Cesàro summable real valued measurable functions in two variables will be denoted by $[D_{\chi,\ \rho}^{\xi,\ \phi}]$.

Example 3 Define $f: J \to \mathbb{R}$ as follows:

$$f(x,\ y) := \begin{cases} 2;\ (x,\ y) \in \mathbb{Q} \times \mathbb{Q} \\ 1;\ (x,\ y) \in (\mathbb{Q} \times \mathbb{Q})^c. \end{cases}$$

Then for any $(\chi,\ \xi),\ (\rho,\ \phi) \in P$, $[D_{\chi,\ \rho}^{\xi,\ \phi}] - \lim f(x,\ y) = 1$.

Let $J = [1,\ \infty) \times [1,\ \infty)$ be a set, $\mathscr{M}$ be a sigma algebra of subsets of J and m be a sigma finite measure on $\mathscr{M}$ such that $m(J) = \infty$. Let $R \in \mathscr{M}$. Then measure of R will be denoted by $m(R)$.

Definition 4 Let R be a subset of J. Then the m-double density of R is denoted and defined as

$$\delta^m(R) :=\lim_{z,\ w\to\infty} \frac{m\big(R \cap (J_z \times J_w)\big)}{m\big(J_z \times J_w\big)},$$

where $J_z = [1,\ z]$ and $J_w = [1,\ w]$, provided the limit exists.

Definition 5 Let $(\chi,\ \xi),\ (\rho,\ \phi) \in P$ and R be a subset of J. Then the m-deferred double density of R is denoted and defined as

$$\delta_{\mathfrak{d}}^m(R) :=\lim_{z,\ w\to\infty} \frac{m\big(R \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))\big)}{m\big(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)\big)},$$

where $J_{\chi,\ \xi}(z) = [\chi(z),\ \xi(z)]$, $J_{\rho,\ \phi}(w) = [\rho(w),\ \phi(w)]$, provided the limit exists.

If we put $\chi(z) = 1,\ \xi(z) = z,\ \rho(w) = 1$ and $\phi(w) = w$, then the m-deferred double density of R coincides with m-double density of R.

Definition 6 Let $R = ([1,\ \infty)) \times 3$ be a subset of $J = [1,\ \infty) \times [1,\ \infty)$ and m be a Lebesgue measure, then for any $(\chi,\ \xi),\ (\rho,\ \phi) \in P$, $m\big(R \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))\big) = 0$ and hence the m-deferred double density of R is zero.

Definition 7 Let

$$R = (\cup_{m=1}^{\infty}[m^2,\ m^2+1]) \times (\cup_{n=1}^{\infty}[n^2,\ n^2+1])$$

be a subset of $J = [1,\ \infty) \times [1,\ \infty)$, $\chi(z) = 1,\ \xi(z) = z+1,\ \rho(w) = 1,\ \phi(w) = w+1$ and m be a Lebesgue measure.

Then,

$$\lim_{z,\ w\to\infty} \frac{m\big(R\cap([1,\ z+1]\times[1,\ w+1])\big)}{zw} \leq \lim_{z,\ w\to\infty} \frac{\sqrt{zw}}{zw} = 0.$$

Therefore, m-deferred double density of R is zero.

Definition 8 Define

$$A_k := \bigcup_{s\in[k,\ k+1]} \bigcup_{n=k}^{\infty} \Big([s+n-2k+1,\ n-k+2]\times s\Big),\ \text{where } k\in\mathbb{N}.$$

Let

$$A = \bigcup_{k=1}^{\infty} A_k$$

be a subset of J, $\chi(z)=1,\ \xi(z)=z+1,\ \rho(w)=1,\ \phi(w)=w+1$ and m be a Lebesgue measure. Then

$$\lim_{z,\ w\to\infty} \frac{m\big(A\cap(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))\big)}{m\big(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\big)} = \lim_{z,\ w\to\infty} \frac{m\big(A\cap([1,\ z+1]\times[1,\ w+1])\big)}{zw}$$

$$\leq \lim_{z,\ w\to\infty} \frac{(\frac{z}{2})w}{zw} = \frac{1}{2}.$$

Hence, m-deferred double density of A is $\frac{1}{2}$.

Let $f: J\to\mathbb{R}$ and $L\in\mathbb{R}$. For any $\varepsilon>0$, define

$$R_\varepsilon^f := \{(x,\ y)\in J : |f(x,\ y)-L|\geq\varepsilon\} \text{ and } T_\varepsilon^f := \{(x,\ y)\in J : |f(x,\ y)-L|<\varepsilon\}.$$

Clearly, $J=R_\varepsilon^f\cup T_\varepsilon^f$ and $R_\varepsilon^f\cap T_\varepsilon^f=\phi$.

Definition 9 A function $f: J\to\mathbb{R}$ is m-statistically convergent to a real number L if for every $\varepsilon>0$,

$$\lim_{z,\ w\to\infty} \frac{m\big(R_\varepsilon^f\cap(J_z\times J_w)\big)}{m\big(J_z\times J_w\big)} = 0$$

holds, where $J_z=[1,\ z]$ and $J_w=[1,\ w]$. In this case, we write $[S]-\lim f(x,\ y)=L$.

The set of all m-statistical convergent functions of two variable is denoted by $[S]$.

Definition 10 Let $(\chi,\ \xi),(\rho,\ \phi)\in P$. A function $f: J\to\mathbb{R}$ is called m-deferred statistically convergent to a real number L if for every $\varepsilon>0$,

$$\lim_{z,\, w\to\infty} \frac{m\big(R_\varepsilon^f \cap (J_{\chi,\, \xi}(z) \times J_{\rho,\, \phi}(w))\big)}{m\big(J_{\chi,\, \xi}(z) \times J_{\rho,\, \phi}(w)} = 0 \left(or \lim_{z,\, w\to\infty} \frac{m\big(T_\varepsilon^f \cap (J_{\chi,\, \xi}(z) \times J_{\rho,\, \phi}(w))\big)}{m\big(J_{\chi,\, \xi}(z) \times J_{\rho,\, \phi}(w)} = 1 \right)$$

holds. It is denoted by $[S_{\chi,\, \rho}^{\xi,\, \phi}] - \lim f(x,\, y) = L$. Let us denote the set of all m-deferred statistically convergent functions of two variables by $[S_{\chi,\, \rho}^{\xi,\, \phi}]$. If we put $\chi(z) = 1,\ \xi(z) = z,\ \rho(w) = 1$ and $\phi(w) = w$, then the m-deferred statistical convergence coincides with m-statistical convergence.

Consider the following examples:

Example 11 Let $A = (3,\, 4) \times (4,\, 5)$ be a subset of $J = [1,\, \infty) \times [1,\, \infty)$, m be a Lebesgue measure and $\chi(z) = z,\ \xi(z) = z+1,\ \rho(w) = w,\ \phi(w) = w+1$.

Define $f : J \to \mathbb{R}$ as

$$f(x,\, y) := \begin{cases} 1;\ (x,\, y) \in A \\ 0;\ \text{otherwise.} \end{cases}$$

Then $[S_{\chi,\, \rho}^{\xi,\, \phi}] - \lim f(x,\, y) = 0$.

Example 12 Let $\chi(z) = 1,\ \xi(z) = z+1,\ \rho(w) = 1,\ \phi(w) = w+1$ and m be a Lebesgue measure.

Define $f : J \to \mathbb{R}$ as

$$f(x,\, y) := \begin{cases} n;\ (x,\, y) \in (\cup_{m=1}^{\infty} [m^3,\, m^3+1]) \times (\cup_{n=1}^{\infty} [n^3,\, n^3+1]) \\ -1;\ \text{otherwise.} \end{cases}$$

Then f is m-deferred statistically convergent to -1.

Example 13 Let $\chi(z) = z,\ \xi(z) = z+1,\ \rho(w) = w,\ \phi(w) = w+1$ and m be a Lebesgue measure.

Define $f : J \to \mathbb{R}$ as follows:

$$f(x,\, y) := \begin{cases} n;\ (x,\, y) \in (\cup_{m=1}^{\infty} [m^3,\, m^3+1]) \times (\cup_{n=1}^{\infty} [n^3,\, n^3+1]) \\ -1;\ \text{otherwise.} \end{cases}$$

Then

$$\lim_{z,\, w\to\infty} \frac{m\big(\{(x,\, y) : (x,\, y) \in [z,\, z+1]) \times [w,\, w+1] \text{ and } |f(x,\, y) - (-1)| \geq \varepsilon\}\big)}{m([z,\, z+1]) \times [w,\, w+1])}$$

does not exists. Hence, f is not m-deferred statistically convergent to -1.

Example 14 Consider the set A as defined in example 8, $\chi(z) = 1,\ \xi(z) = z+1,\ \rho(w) = 1,\ \phi(w) = w+1$ and m be a lebesgue measure. Define

$$f(x,\, y) := \begin{cases} 1;\ (x,\, y) \in A \\ 2;\ \text{otherwise.} \end{cases}$$

Then f is not m-deferred statistically convergent to 2.

Definition 15 Let $(\chi,\ \xi),\ (\rho,\ \phi) \in P$ and $f(x,\ y)$ and $g(x,\ y)$ be any functions. Then $f(x,\ y)$ and $g(x,\ y)$ are called equivalent functions with respect to $(\chi,\ \xi),\ (\rho,\ \phi)$ if the set $N := \{(x,\ y) \in J : f(x,\ y) \neq g(x,\ y)\}$ has m-deferred double density zero. It is denoted by $f(x,\ y) \sim g(x,\ y)$ (w.r.t $(\chi,\ \xi), (\rho,\ \phi)$).

Theorem 16 Let $(\chi,\ \xi),\ (\rho,\ \phi) \in P$ and $f(x,\ y) \sim g(x,\ y)$ (w.r.t $(\chi,\ \xi),\ (\rho,\ \phi)$). If $f \in [S_{\chi,\ \rho}^{\xi,\ \phi}]$, then $g \in [S_{\chi,\ \rho}^{\xi,\ \phi}]$ and vice versa.

Proof. Assume that $f \in [S_{\chi,\ \rho}^{\xi,\ \phi}]$. For any $\varepsilon > 0$,

$$R_\varepsilon^g \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)) = (R_\varepsilon^g \cap N_{\chi,\ \rho}^{\xi,\ \phi}) \cup (R_\varepsilon^g \cap (N_{\chi,\ \rho}^{\xi,\ \phi})^c),$$

where $N_{\chi,\ \rho}^{\xi,\ \phi} := (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)) \cap N$ and $(N_{\chi,\ \rho}^{\xi,\ \phi})^c := (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)) \cap N^c$. From this equality, we have

$$R_\varepsilon^g \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)) \subset (R_\varepsilon^g \cap N_{\chi,\ \rho}^{\xi,\ \phi}) \cup \left(R_\varepsilon^f \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))\right)$$

and

$$\frac{m\Big(R_\varepsilon^g \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))\Big)}{m\Big(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)\Big)} \leq \frac{m\Big(R_\varepsilon^g \cap N_{\chi,\ \rho}^{\xi,\ \phi}\Big)}{m\Big(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)\Big)} + \frac{m\Big(R_\varepsilon^f \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)\Big)}{m\Big(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)\Big)}.$$

If we take limit as $z,\ w \to \infty$, then $g \in [S_{\chi,\ \rho}^{\xi,\ \phi}]$. The converse can be proved by following the same steps.

Theorem 17 Let $(\chi,\ \xi),\ (\rho,\ \phi) \in P$ and $f(x,\ y)$ be a real-valued Lebesgue measurable function in two variables. Then the following statements hold:

(i) If $[S_{\chi,\ \rho}^{\xi,\ \phi}] - \lim f(x,\ y) = L$ and $s \in \mathbb{R}$, then $[S_{\chi,\ \rho}^{\xi,\ \phi}] - \lim sf(x,\ y) = sL$.

(ii) If $[S_{\chi,\ \rho}^{\xi,\ \phi}] - \lim f(x,\ y) = L_1$ and $[S_{\chi,\ \rho}^{\xi,\ \phi}] - \lim g(x,\ y) = L_2$, then $[S_{\chi,\ \rho}^{\xi,\ \phi}] - \lim(f(x,\ y) + g(x,\ y)) = L_1 + L_2$.

(iii) If $[D_{\chi,\ \rho}^{\xi,\ \phi}] - \lim f(x,\ y) = L$ and $s \in \mathbb{R}$, then $[D_{\chi,\ \rho}^{\xi,\ \phi}] - \lim sf(x,\ y) = sL$.

(iv) If $[D_{\chi,\ \rho}^{\xi,\ \phi}] - \lim f(x,\ y) = L_1$ and $[D_{\chi,\ \rho}^{\xi,\ \phi}] - \lim g(x,\ y) = L_2$, then $[D_{\chi,\ \rho}^{\xi,\ \phi}] - \lim(f(x,\ y) + g(x,\ y)) = L_1 + L_2$.

From the Theorem 17, it can be concluded that for any pair functions $(\chi,\ \xi),\ (\rho,\ \phi) \in P$ the function sets $[D_{\chi,\ \rho}^{\xi,\ \phi}]$ and $[S_{\chi,\ \rho}^{\xi,\ \phi}]$ are vector spaces over real numbers.

Theorem 18 Let $(\chi,\ \xi),\ (\rho,\ \phi) \in P$. If f is strongly deferred Cesàro summable, then it is m-deferred statistically convergent, that is, $[D_{\chi,\ \rho}^{\xi,\ \phi}] \subset [S_{\chi,\ \rho}^{\xi,\ \phi}]$ and the inclusion is strict.

Proof. Let $f(x,\ y) \in [D_{\chi,\ \rho}^{\xi,\ \phi}]$ and suppose on the contrary that $f(x,\ y) \notin [S_{\chi,\ \rho}^{\xi,\ \phi}]$. Therefore, there exists a fixed $\varepsilon_0 > 0$ such that m-deferred double density of $R_{\varepsilon_0}^f \neq 0$. That is,

$$\lim_{z,\ w \to \infty} \frac{m\big(R_{\varepsilon_0}^f \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))\big)}{m\big(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)} > 0, \text{ holds.} \tag{3}$$

Now,

$$\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy \geq \int_{\substack{\rho(w)\\ |f(x,\ y)-L|\geq\varepsilon_0}}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy$$

$$\geq m\big(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\big)\varepsilon_0$$

$$\geq m\big(R^f_{\varepsilon_0}\cap(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\big)\varepsilon_0$$

and it implies that

$$\frac{1}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))}\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy$$

$$\geq\frac{1}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))}m\big(R^f_{\varepsilon_0}\cap(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\big)\varepsilon_0.$$

On taking limit $z,\ w\to\infty$ on both sides of this inequality and using (3), we get $f(x,\ y)\notin\ [D^{\xi,\ \phi}_{\chi,\ \rho}]$, which is a contradiction. Therefore, our assumption is wrong and hence, $f(x,\ y)\in[S^{\xi,\ \phi}_{\chi,\ \rho}]$.

To show the inclusion is strict, let $(\chi,\ \xi),\ (\rho,\ \phi)\in P$ and define a function $f=f(x,\ y)$ as

$$f(x,\ y)=\begin{cases} xy;\ \xi(z)-\sqrt{\xi(z)}\leq x\leq\xi(z),\ \phi(w)-\sqrt{\phi(w)}\leq y\leq\phi(w)\\ 0\ \text{ otherwise.}\end{cases}$$

If we consider $[D^{\xi,\ \phi}_{\chi,\ \rho}]$ for the sequence $\chi(z)$ and $\rho(w)$ satisfying $0<\chi(z)\leq\xi(z)-\sqrt{\xi(z)}$ and $0<\rho(w)\leq\phi(w)-\sqrt{\phi(w)}$.

So, for an arbitrary $\varepsilon>0$, we have

$$\lim_{z,\ w\to\infty}\frac{m\big(R^f_\varepsilon\cap(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\big)}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))}=\lim_{z,\ w\to\infty}\frac{\sqrt{\xi(z)\phi(w)}}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))}=0.$$

Hence, $[S^{\xi,\ \phi}_{\chi,\ \rho}]-\lim f(x,\ y)=0$, i.e., $f(x,\ y)\in[S^{\xi,\ \phi}_{\chi,\ \rho}]$.

On the other hand,

$$\lim_{z,w\to\infty} \frac{1}{m(J_{\chi,\xi}(z)\times J_{\rho,\phi}(w))} \int_{\rho(w)}^{\phi(w)} \int_{\chi(z)}^{\xi(z)} |f(x,y)-L|dxdy$$

$$= \lim_{z,w\to\infty} \frac{1}{m(J_{\chi,\xi}(z)\times J_{\rho,\phi}(w))} \int_{\phi(w)-\sqrt{\phi(w)}}^{\phi(w)} \int_{\xi(z)-\sqrt{\xi(z)}}^{\xi(z)} |xy|dxdy$$

$$= \lim_{z,w\to\infty} \frac{(2\phi^{3/2}(w)-\phi(w))(2\xi^{3/2}(z)-\xi(z))}{(\xi(z)-\chi(z))(\phi(w)-\rho(w))} = \infty.$$

So, $f(x,\ y) \notin [D_{\chi,\ \rho}^{\xi,\ \phi}]$. Hence, the proof of the theorem is completed. □

The converse of the above theorem holds if $f(x,\ y)$ is a bounded function which can be proved in the following theorem.

Theorem 19 Let $(\chi,\ \xi),\ (\rho,\ \phi) \in P$. If a bounded function f is m-deferred statistically convergent, then it is strongly deferred Cesàro summable, that is, $[S_{\chi,\ \rho}^{\xi,\ \phi}] \subset [D_{\chi,\ \rho}^{\xi,\ \phi}]$.

Proof. Suppose that $[S_{\chi,\ \rho}^{\xi,\ \phi}] - \lim f(x,\ y) = L$. Since, f is bounded, there exists $M > 0$ such that $|f(x,\ y) - L| \leq M$, for all $(x,\ y) \in J$.

For an arbitrary $\varepsilon > 0$, we have

$$\frac{1}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))} \int_{\rho(w)}^{\phi(w)} \int_{\chi(z)}^{\xi(z)} |f(x,\ y)-L|dxdy$$

$$= \frac{1}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))} \left[\underset{(x,\ y)\in R_\varepsilon^f}{\int_{\rho(w)}^{\phi(w)} \int_{\chi(z)}^{\xi(z)}} |f(x,\ y)-L|dxdy + \underset{(x,\ y)\in T_\varepsilon^f}{\int_{\rho(w)}^{\phi(w)} \int_{\chi(z)}^{\xi(z)}} |f(x,\ y)-L|dxdy \right]$$

$$\leq M\ \frac{m\big(R_\varepsilon^f \cap (J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\big)}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))} + \varepsilon \frac{m\big(T_\varepsilon^f \cap (J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\big)}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))}.$$

On taking limit $z,\ w \to \infty$, the desired result is obtained.

The next theorem tell us about the relationship between statistical convergence and m-deferred statistical convergence.

Theorem 20 Let $(\chi,\ \xi),\ (\rho,\ \phi) \in P$ such that $\liminf_z \dfrac{\chi(z)}{\xi(z)} \neq 1$ and $\liminf_w \dfrac{\rho(w)}{\phi(w)} \neq 1$, then $[S] \subset [S_{\chi,\ \rho}^{\xi,\ \phi}]$.

Proof. Suppose that $f(x,\ y) \in [S]$. So, we have

$$\lim_{z,\ w\to\infty} \frac{m\big(R_\varepsilon^f \cap (J_z \times J_w\big)}{m(J_z \times J_w)} = 0.$$

Since, $\lim_{z\to\infty} \xi(z) = \infty$ and $\lim_{w\to\infty} \phi(w) = \infty$, we have

$$\lim_{z,\ w\to\infty} \frac{m\big(R_\varepsilon^f \cap (J_\xi(z) \times J_\phi(w)\big)}{m(J_\xi(z) \times J_\phi(w))} = 0.$$

In this case, $J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\subset J_{\xi}(z)\times J_{\phi}(w)$. This implies that $m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))\le m(J_{\xi}(z)\times J_{\phi}(w))$. Hence, we have

$$\frac{m\left(R_{\varepsilon}^{f}\cap(J_{\xi}(z)\times J_{\phi}(w)\right)}{m(J_{\xi}(z)\times J_{\phi}(w))}\ge\frac{m\left(R_{\varepsilon}^{f}\cap(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\right)}{m(J_{\xi}(z)\times J_{\phi}(w))}$$

$$=\frac{m\left(R_{\varepsilon}^{f}\cap(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w)\right)}{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))}\ \frac{m(J_{\chi,\ \xi}(z)\times J_{\rho,\ \phi}(w))}{m(J_{\xi}(z)\times J_{\phi}(w))}.$$

Taking limit $z,\ w\to\infty$ on both sides of this inequality and using the assumptions of the theorem, we get the desired result. □

Remark The conditions $\liminf_{z}\dfrac{\chi(z)}{\xi(z)}\ne 1$ and $\liminf_{w}\dfrac{\rho(w)}{\phi(w)}\ne 1$ in Theorem 20 are essential.

Consider the example

Let $A=(\cup_{m=1}^{\infty}[m^{2},\ m^{2}+1])\times(\cup_{n=1}^{\infty}[n^{2},\ n^{2}+1])$ be a subset of $J=[1,\ \infty)\times[1,\ \infty)$ and $\chi(z)=z,\ \xi(z)=z+1,\ \rho(w)=w,\ \phi(w)=w+1$.

Define $f:J\to\mathbb{R}$ as follows:

$$f(x,\ y)=\begin{cases}1;\ (x,\ y)\in A\\ -1;\ \text{otherwise.}\end{cases}$$

Clearly, f is m-statistically convergent to -1 but f is not m-deferred statistically convergent to -1 as

$$\lim_{z,\ w\to\infty}\frac{m\left(\{(x,\ y):(x,\ y)\in[z,\ z+1])\times[w,\ w+1]\text{ and }|f(x,\ y)-(-1)|\ge\varepsilon\}\right)}{m([z,\ z+1])\times[w,\ w+1])}$$

does not exists.

The next theorem tell us about the relationship between strongly Cesàro summable and strongly deferred Cesàro summable.

Theorem 21 Let $(\chi,\ \xi),\ (\rho,\ \phi)\in P$ such that $\liminf_{z}\dfrac{\chi(z)}{\xi(z)}\ne 1$ and $\liminf_{w}\dfrac{\rho(w)}{\phi(w)}\ne 1$, then $[D]\subset[D_{\chi,\ \rho}^{\xi,\ \phi}]$.

Proof. Suppose that $f(x,\ y)\in[D]$. Then $\lim\limits_{z,\ w\to\infty}\dfrac{1}{zw}\displaystyle\int_{1}^{w}\int_{1}^{z}|f(x,\ y)-L|dxdy=0$.

Since, $\lim_{z\to\infty}\xi(z)=\infty$ and $\lim_{w\to\infty}\phi(w)=\infty$, we have

$$\lim_{z,\ w\to\infty}\frac{1}{\xi(z)\phi(w)}\int_{1}^{\phi(w)}\int_{1}^{\xi(z)}|f(x,\ y)-L|dxdy=0.$$

From the monotonicity property of integrals, we have

$$\int_{1}^{\phi(w)}\int_{1}^{\xi(z)}|f(x,\ y)-L|dxdy\ge\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy.$$

This implies that

$$\frac{1}{\xi(z)\phi(w)}\int_1^{\phi(w)}\int_1^{\xi(z)}|f(x,\ y)-L|dxdy$$

$$\geq\frac{(\xi(z)-\chi(z))(\phi(w)-\rho(w))}{\xi(z)\phi(w)}\times\frac{1}{(\xi(z)-\chi(z))(\phi(w)-\rho(w))}\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy.$$

After taking limit $z,\ w\to\infty$, we obtain

$$\lim_{z,\ w\to\infty}\frac{1}{(\xi(z)-\chi(z))(\phi(w)-\rho(w))}\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy=0.$$

This proves our assertion. □

Remark: The conditions $\liminf_z\frac{\chi(z)}{\xi(z)}\neq 1$ and $\liminf_w\frac{\rho(w)}{\phi(w)}\neq 1$ in Theorem 21 are essential.

Consider the function $f(x,\ y)=\frac{1}{2\sqrt{x}}$, for all $(x,\ y)\in J$. Let $\chi(z)=z,\ \xi(z)=z+1,\ \rho(w)=w$ and $\phi(w)=w+1$. Then $\liminf_z\frac{\chi(z)}{\xi(z)}=1$ and $\liminf_w\frac{\rho(w)}{\phi(w)}=1$.

Now,

$$\lim_{z,\ w\to\infty}\frac{1}{zw}\int_1^w\int_1^z|f(x,\ y)-L|dxdy=\lim_{z,\ w\to\infty}\frac{1}{zw}\int_1^w\int_1^z\left|\frac{1}{2\sqrt{x}}-0\right|dxdy$$

$$=\lim_{z,\ w\to\infty}\frac{(\sqrt{z}-1)(w-1)}{zw}$$

$$=0.$$

Therefore, f is strongly Cesàro summable to 0. But,

$$\lim_{z,\ w\to\infty}\frac{1}{(\xi(z)-\chi(z))(\phi(w)-\rho(w))}\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy$$

$$=\lim_{z,\ w\to\infty}\int_w^{w+1}\int_z^{z+1}|f(x,\ y)-L|dxdy$$

$$=\lim_{z,\ w\to\infty}\int_w^{w+1}\int_z^{z+1}\left|\frac{1}{2\sqrt{x}}-0\right|dxdy$$

$$=\lim_{z,\ w\to\infty}(\sqrt{z+1}-z)=\infty.$$

This implies that f is not strongly deferred Cesàro summable to 0.

Theorem 22 Let $(\chi,\ \xi),\ (\rho,\ \phi),\ (\chi',\ \xi'),\ (\rho',\ \phi') \in P$ with $\chi(z) \leq \chi'(z) < \xi'(z) \leq \xi(z)$ and $\rho(w) \leq \rho'(w) < \phi'(w) \leq \phi(w)$ such that $\dfrac{m(J_{\chi',\ \xi'}(z) \times J_{\rho',\ \phi'}(w))}{m(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))} > 0$. Then the followings are true.

(i) $[S_{\chi,\ \rho}^{\xi,\ \phi}] \subset [S_{\chi',\ \rho'}^{\xi',\ \phi'}]$.

(ii)$[D_{\chi,\ \rho}^{\xi,\ \phi}] \subset [D_{\chi',\ \rho'}^{\xi',\ \phi'}]$.

Proof. Let $f(x,\ y) \in [S_{\chi,\ \rho}^{\xi,\ \phi}]$. Then we have

$$\lim_{z,\ w\to\infty} \frac{m\left(R_\varepsilon^f \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))\right)}{m(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))} = 0.$$

For given $\varepsilon > 0$, we have

$$R_\varepsilon^f \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)) \supset R_\varepsilon^f \cap (J_{\chi',\ \xi'}(z) \times J_{\rho',\ \phi'}(w)).$$

This implies that

$$m\left(R_\varepsilon^f \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))\right) \geq m\left(R_\varepsilon^f \cap (J_{\chi',\ \xi'}(z) \times J_{\rho',\ \phi'}(w))\right).$$

So, we get the following inequality

$$\frac{m\left(R_\varepsilon^f \cap (J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w))\right)}{m\left(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)\right)} \geq \frac{m\left(J_{\chi',\ \xi'}(z) \times J_{\rho',\ \phi'}(w)\right)}{m\left(J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)\right)} \ \frac{m\left(R_\varepsilon^f \cap (J_{\chi',\ \xi'}(z) \times J_{\rho',\ \phi'}(w))\right)}{m\left(J_{\chi',\ \xi'}(z) \times J_{\rho',\ \phi'}(w)\right)}.$$

After taking limit $z,\ w \to \infty$, we get the desired result, i.e., $f(x,\ y) \in [S_{\chi',\ \rho'}^{\xi',\ \phi'}]$.

(ii) Since, $J_{\chi',\ \xi'}(z) \times J_{\rho',\ \phi'}(w) \subset J_{\chi,\ \xi}(z) \times J_{\rho,\ \phi}(w)$, we have

$$\int_{\rho'(w)}^{\phi'(w)} \int_{\chi'(z)}^{\xi'(z)} |f(x,\ y) - L| dxdy \leq \int_{\rho(w)}^{\phi(w)} \int_{\chi(z)}^{\xi(z)} |f(x,\ y) - L| dxdy.$$

This implies that

$$\frac{1}{(\xi'(z) - \chi'(z))(\phi' - \rho')} \int_{\rho'(w)}^{\phi'(w)} \int_{\chi'(z)}^{\xi'(z)} |f(x,\ y) - L| dxdy$$

$$\leq \frac{(\xi(z) - \chi(z))(\phi(w) - \rho(w))}{(\xi'(z) - \chi'(z))(\phi'(w) - \rho'(w))} \times \frac{1}{(\xi(z) - \chi(z))(\phi(w) - \rho(w))} \int_{\rho(w)}^{\phi(w)} \int_{\chi(z)}^{\xi(z)} |f(x,\ y) - L| dxdy.$$

On taking limit $z,\ w \to \infty$ on both sides of this inequality, we get the desired result. □

Theorem 23 Let $(\chi,\ \xi),\ (\rho,\ \phi), (\chi',\ \xi') \in P$ with $\chi(z) \leq \chi'(z) < \xi'(z) \leq \xi(z)$ and $f(x,\ y) \in [S_{\chi,\ \rho}^{\chi',\ \phi}] \cap [S_{\xi',\ \rho}^{\xi,\ \phi}]$. If $f(x,\ y) \in [S_{\chi',\ \rho}^{\xi',\ \phi}]$ then $f(x,\ y) \in [S_{\chi,\ \rho}^{\xi,\ \phi}]$.

Proof. Let $[S_{\chi', \rho}^{\xi', \phi}] - \lim f(x, y) = L$. For any $\varepsilon > 0$, we have

$$R_\varepsilon^f \cap (J_{\chi, \xi}(z) \times J_{\rho, \phi}(w)) = \left(R_\varepsilon^f \cap (J_{\chi, \chi'}(r) \times J_{\rho, \phi}(w))\right) \cup \left(R_\varepsilon^f \cap (J_{\chi', \xi'}(z) \times J_{\rho, \phi}(w))\right)$$

$$\cup \left(R_\varepsilon^f \cap (J_{\xi', \xi}(z) \times J_{\rho, \phi}(w))\right).$$

This implies that

$$m\left(R_\varepsilon^f \cap (J_{\chi, \xi}(z) \times J_{\rho, \phi}(w))\right) = m\left(R_\varepsilon^f \cap (J_{\chi, \chi'}(z) \times J_{\rho, \phi}(w))\right) + m\left(R_\varepsilon^f \cap (J_{\chi', \xi'}(z) \times J_{\rho, \phi}(w))\right)$$

$$+ m\left(R_\varepsilon^f \cap (J_{\xi', \xi}(z) \times J_{\rho, \phi}(w))\right).$$

From this, we get the following inequality

$$\frac{m\left(R_\varepsilon^f \cap (J_{\chi, \xi}(z) \times J_{\rho, \phi}(w))\right)}{m\left(J_{\chi, \xi}(z) \times J_{\rho, \phi}(w)\right)} \leq \frac{m\left(R_\varepsilon^f \cap (J_{\chi, \chi'}(z) \times J_{\rho, \phi}(w))\right)}{m\left(J_{\chi, \chi'}(z) \times J_{\rho, \phi}(w)\right)} + \frac{m\left(R_\varepsilon^f \cap (J_{\chi', \xi'}(z) \times J_{\rho, \phi}(w))\right)}{m\left(J_{\chi', \xi'}(z) \times J_{\rho, \phi}(w)\right)}$$

$$+ \frac{m\left(R_\varepsilon^f \cap (J_{\xi', \xi}(z) \times J_{\rho, \phi}(w))\right)}{m\left(J_{\xi', \xi}(z) \times J_{\rho, \phi}(w)\right)}$$

because $m\left(J_{\chi, \xi}(z) \times J_{\rho, \phi}(w)\right) \geq m\left(J_{\chi, \chi'}(z) \times J_{\rho, \phi}(w)\right)$, $m\left(J_{\chi, \xi}(z) \times J_{\rho, \phi}(w)\right) \geq m\left(J_{\chi', \xi'}(z) \times J_{\rho, \phi}(w)\right)$ and $m\left(J_{\chi, \xi}(z) \times J_{\rho, \phi}(w)\right) \geq m\left(J_{\xi', \xi}(z) \times J_{\rho, \phi}(w)\right)$.

On taking limit $z, w \to \infty$ and using the given assumptions, we get $f(x, y) \in [S_{\chi, \rho}^{\xi, \phi}]$. □

Theorem 24 Let $(\chi, \xi), (\rho, \phi), (\chi', \xi') \in P$ with $\chi(z) \leq \chi'(z) < \xi'(z) \leq \xi(z)$ and $f(x, y) \in [D_{\chi, \rho}^{\chi', \phi}] \cap [D_{\xi', \rho}^{\xi, \phi}]$. If $f(x, y) \in [D_{\chi', \rho}^{\xi', \phi}]$ then $f(x, y) \in [D_{\chi, \rho}^{\xi, \phi}]$.

Proof. Let $[D_{\chi', \rho}^{\xi', \phi}] - \lim f(x, y) = L$.

Since,

$$J_{\chi, \xi}(z) \times J_{\rho, \phi}(w)) = (J_{\chi, \chi'}(z) \times J_{\rho, \phi}(w)) \cup (J_{\chi', \xi'}(z) \times J_{\rho, \phi}(w)) \cup (J_{\xi', \xi}(z) \times J_{\rho, \phi}(w)),$$

we have the following equality

$$\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy=\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\chi'(z)}|f(x,\ y)-L|dxdy$$

$$+\int_{\rho(w)}^{\phi(w)}\int_{\chi'(z)}^{\xi'(z)}|f(x,\ y)-L|dxdy$$

$$+\int_{\rho(w)}^{\phi(w)}\int_{\xi'(z)}^{\xi(z)}|f(x,\ y)-L|dxdy.$$

From this, we have

$$\frac{1}{(\xi(z)-\chi(z))(\phi(w)-\rho(w))}\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\xi(z)}|f(x,\ y)-L|dxdy$$

$$\leq\frac{1}{(\chi'(z)-\chi(z))(\phi(w)-\rho(w))}\int_{\rho(w)}^{\phi(w)}\int_{\chi(z)}^{\chi'(z)}|f(x,\ y)-L|dxdy$$

$$+\frac{1}{(\xi'(z)-\chi'(z))(\phi(w)-\rho(w))}\int_{\rho(w)}^{\phi(w)}\int_{\chi'(z)}^{\xi'(z)}|f(x,\ y)-L|dxdy$$

$$+\frac{1}{(\xi(z)-\xi'(z))(\phi(w)-\rho(w))}\int_{\rho(w)}^{\phi(w)}\int_{\xi'(z)}^{\xi(z)}|f(x,\ y)-L|dxdy,$$

because $(\xi(z)-\chi(z))\geq(\chi'(z)-\chi(z))$, $(\xi(z)-\chi(z))\geq(\xi'(z)-\chi'(z))$ and $(\xi(z)-\chi(z))\geq(\xi(z)-\xi'(z))$.

Hence, we get $f(x,\ y)\in[D_{\chi,\ \rho}^{\xi,\ \phi}]$ when $z,\ w\to\infty$ and using the given assumptions. □

3. Conclusion

In the present paper, we explore the properties of m-deferred statistical convergence and strongly deferred Cesàro summability of real valued Lebesgue measurable functions of two variables. We show that strongly deferred Cesàro summable function is m-deferred statistically convergent. But the converse is true for bounded measurable functions. Besides this, we also prove that a m-statistically convergent function is m-deferred statistically convergent and strongly Cesàro summable function is strongly deferred Cesàro summable under some restrictions. As a future work, one can obtain the corresponding results using deferred nörlund summability to functions of d variables of order $(\alpha_1,\ \ldots,\ \alpha_d)$, $d\geq 3$.

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Conflict of interest

The authors declare that they have no competing interests.

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