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Research Article

Decision Making on Statistical Convergence of Sequences of Sets of Order (α, β)

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Abstract: In this paper we define Wijsman λ-statistical convergence of order (α, β) and studied some properties of this concept. We also make an effort to define Wijsman λ-statistical convergence of Musielak-Orlicz function of order (α, β) and examine some topological properties.

Keywords: statistical convergence, λ-statistical convergence, wijsman convergence, musielak-orlicz function

MSC: 40A35, 40G15, 40C05

1. Introduction and preliminaries

In the first edition of Zygmund [1] of celebrated monographs, the concept of statistical convergence first proposed by the name "almost convergence". Later on, the notion of statistical convergence was proposed by Schoenberg [2], Steinhaus [3], Fast [4], independently by Buck [5] and studied by number of authors [6–8]. Mursaleen [9], describe the idea of λ-statistical convergence for real sequence. Further information about λ-statistical convergence see [10–13] and references therein. Statistical convergence has gained significant attention in mathematical analysis, particularly for its utility in summability theory, functional analysis, and applications in real functions. One of the key developments has been the extension of statistical convergence to sequences of sets, which has proven relevant in fields such as fuzzy set theory, information theory, and multidimensional decision-making.

The concept of statistical convergence has been generalized in several directions. One important advancement is the study of generalized statistical convergence of sequences of sets of order α using the Orlicz function [14]. This work has contributed to a better understanding of how sets, rather than individual elements, converge statistically when the Orlicz function modulates the convergence. The Orlicz function provides a useful structure by introducing flexibility in the choice of moduli, capturing various forms of convergence behaviors. The current literature primarily addresses statistical

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convergence of sequences of sets of single order (e.g., order α) and focuses on function spaces governed by the traditional Orlicz function.

We extend this framework to handle sequences of sets governed by two orders, $(\alpha,\ \beta)$, where each order reflects a different level or dimension of the convergence process. Such extensions are vital for more complex systems, such as those found in multidimensional analysis, where multiple parameters influence convergence simultaneously.

Moreover, the classical Orlicz function, while useful, may not be sufficient to capture the full range of behaviors where the parameters governing convergence vary both with the position in the sequence and with the specific elements of the sets. Thus, we propose the use of the Musielak-Orlicz function, which generalizes the Orlicz function by allowing the moduli to vary based on position and elements.

This study introduces a new class of sequence spaces characterized by the Musielak-Orlicz function and develops the theory of statistical convergence of sequences of sets of order $(\alpha,\ \beta)$. By incorporating the Musielak-Orlicz function, we provide a more flexible framework for analyzing statistical convergence, particularly in situations where dynamic moduli are essential. This study introduces a new class of sequence spaces defined by the Musielak-Orlicz function and develops the theory of statistical convergence for sequences of sets of order $(\alpha,\ \beta)$. This extension fills a gap in the literature and provides a more generalized framework for multidimensional convergence analysis.

Recent developments in the field of statistical convergence have broadened the understanding of various types of convergence, with a particular focus on Wijsman convergence and summability techniques. The work of Gülle and Ulusu [15] on Wijsman deferred invariant statistical and p-deferred invariant equivalence of order α has made significant contributions to the study of statistical structures that involve deferred and invariant aspects. Their work highlights the utility of deferred invariant properties in statistical methods and their applications in diverse mathematical contexts. Similarly, Kişi and Gülle [16] have extended $\mathscr{I}$-Cesàro summability of sequences of random variables, emphasizing its role in probability theory and its impact on understanding convergence in probabilistic frameworks. Their study provides crucial insights into how summability methods can be adapted to handle sequences of random variables within the context of statistical convergence. Kişi's later work [17] explored $\mathscr{I}_\theta$-convergence in neutrosophic normed spaces, contributing to the broader understanding of convergence in spaces with more nuanced structures. This research illustrates the adaptability of convergence concepts to more abstract mathematical settings and underscores the importance of developing robust convergence theories for various types of spaces.

The generalization of statistical convergence was further studied in number theory, Fourier analysis and ergodic theory. Recent findings include generalisation of statistical convergence, structure of ideals of bounded continuous functions and integral summability, and Stone-Čech compactification of the natural numbers. For additional information, see [18–22] and references therein.

The sequence $\xi = (\xi_k)$ of real or complex number is known as statistical convergent to L if for every $\varepsilon > 0$,

$$\lim_m \frac{1}{m} |\{k \leq m: \ |\xi_k - L| \geq \varepsilon\}| = 0.$$

Specifically, denoted by $S - \lim \xi = L$ or $\xi_k \to L(S)$. S represents the set of all statistical convergent sequences.

The generalized de la Vallée-Poussin mean is given by

$$t_m(\xi) = \frac{1}{\lambda_m} \sum_{k \in C_m} \xi_k,$$

where $C_m = [m - \lambda_m + 1,\ m]$. The sequence $\xi = (\xi_k)$ is known to be $(V,\ \lambda)$-summable to L [23] if $t_m(\xi) \to L$ as $m \to \infty$. If $\lambda_m = m,$ then $(V,\ \lambda)$-summability brings to $(C,\ 1)$-summability.

λ-statistically convergent sequence were defined by Mursaleen [9] as follows, the sequence $\xi = (\xi_k)$ is known as λ-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{m\to\infty}\frac{1}{\lambda_m}|\{k\in C_m:\ |\xi_k-L|\geq\varepsilon\}|=0.$$

By S_λ we mean the set of all λ-statistically convergent sequences. If $\lambda_m = m,$ then S_λ is identical to S.

Consider a metric space $(Z,\ \mathscr{J})$. For any $\xi\in Z$ and A be any non-empty subset of Z, the distance from z to A is given by

$$\delta(z,\ A)=\inf_{y\in A}\mathscr{J}(z,\ y).$$

For A, A_k be any non-empty closed subsets of Z $(k\in\mathbb{N})$, the sequence (A_k) is known to be Wijsman convergent (see [24–27]) to A if $\lim_k\delta(z,\ A_k)=\delta(z,\ A)$ for each $z\in Z$, denoted by $W-\lim A_k=A$.

Nuray and Rhoades [28] proposed the concept of Wijsman statistical convergence and boundedness for the sequence (A_k) as follows, the sequence (A_k) is said to be Wijsman statistical convergent to A if the sequence $(\delta(z,\ A_k))$ is statistically convergent to $\delta(z,\ A)$, i.e., for $\varepsilon>0$ and for each $z\in Z$,

$$\lim_m\frac{1}{m}|\{k\leq m:\ |\delta(z,\ A_k)-\delta(z,\ A)|\geq\varepsilon\}|=0.$$

We can write it as $st-\lim_k A_k=A$ or $A_k\to A(S^W)$. We say that (A_k) is bounded if $\sup_k\delta(z,\ A_k)<\infty$ for each $z\in Z$. The set of all bounded sequences is represented by L_∞. For further information on Wijsman statistical convergent, see [26, 29–33] and references therein.

2. Outline

The article begins by introducing the primary objectives, which include defining and proving Wijsman λ-statistical convergence of order $(\alpha,\ \beta)$, and highlighting its importance within the broader mathematical framework. Following this, the necessary preliminary concepts, including definitions, notations, and a review of relevant literature, are presented to establish a foundation. The core sections of the paper focus on formally defining Wijsman λ-statistical convergence of order $(\alpha,\ \beta)$, presenting the main results, and examining key properties. This is then extended to include Wijsman λ-statistical convergence in the context of Musielak-Orlicz functions, where further results and topological properties are explored. The article concludes by summarizing the key findings and their implications, while suggesting potential directions for future research.

3. Wijsman λ-statistical convergence of order (α, β)

Gadjiev and Orhan introduced the idea of order statistical convergence [34] and after that α order statistical convergence was examined by Çolak [35], Çolak and Bektaş [36] studied α order λ-statistical convergence which is also investigated by many others see [37–41] and references therein.

In this paper, we define $S^W_{\lambda,\ (\alpha,\ \beta)}$-statistical convergence and establish the relationship between S^W_λ and w^W_λ. Additionally, we present the idea for real sequences of S^W_λ-statistical convergence of order $(\alpha,\ \beta)$ and established the inclusion relations between S^W-statistical convergence of order $(\alpha,\ \beta)$.

Definition 1 Suppose that $(Z,\ \mathscr{J})$ is a metric space, $0<\alpha\leq\beta\leq1$ and $\lambda=(\lambda_m)$ be a non-decreasing sequence of positive numbers such that $\lambda_{m+1}\leq\lambda_m+1,\ \lambda_1=1,\ \lambda_m\to\infty$ as $m\to\infty$ and $C_m=[m-\lambda_m+1,\ m]$. The family of sequences $\lambda = (\lambda_m)$ is denoted by Λ. For $A,\ A_k$ be any nonempty closed subset of $Z(k \in \mathbb{N})$, the sequence A_k is Wijsman λ-statistical convergent to A of order $(\alpha,\ \beta)$ or $S^W_{\lambda,\ (\alpha,\ \beta)}$-statistical to A, if for every $\varepsilon > 0$ and for each $z \in Z$,

$$\lim_m \frac{1}{\lambda_m^\alpha} |\{k \in C_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta = 0.$$

Specifically, we write $S^W_{\lambda,\ (\alpha,\ \beta)} - \lim A_k = A$ or $A_k \to A(S^W_{\lambda,\ (\alpha,\ \beta)})$.

If $\lambda_m = m$ for all $m \in \mathbb{N}$, we get the sequence space $S^W_{(\alpha,\ \beta)}$ such that

$$S^W_{(\alpha,\ \beta)} = \{(A_k) :\ \lim_m \frac{1}{m^\alpha} |\{k \in C_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta = 0\}$$

and if $\lambda_m = m$ for all $m \in \mathbb{N}$ and $\alpha = 1$ we get the sequence space S^W such that

$$S^W_{(1,\ \beta)} = \{(A_k) :\ \lim_m \frac{1}{m} |\{k \in C_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta = 0\}.$$

Definition 2 Consider a metric space $(Z,\ \mathscr{J})$, $0 < \alpha \leq \beta \leq 1$ and $\lambda = (\lambda_m) \in \Lambda$. For $A,\ A_k$ be the non empty closed subset of $Z(k \in \mathbb{N})$, the sequence (A_k) is Wijsman λ-Cesáro summable to A of order $(\alpha,\ \beta)$ or $w^W_{\lambda,\ (\alpha,\ \beta)}$-convergent to A, if for each $z \in Z$,

$$\lim_m \frac{1}{\lambda_m^\alpha} \sum_{k \in C_m} |\delta(z,\ A_k) - \delta(z,\ A)|^\beta = 0.$$

We can write it as $w^W_{\lambda,\ (\alpha,\ \beta)} - \lim A_k = A$ or $A_k \to A(w^W_{\lambda,\ (\alpha,\ \beta)})$. If $\lambda_m = m$ for all $m \in \mathbb{N}$, we have the sequence space $w^W_{(\alpha,\ \beta)}$ such that

$$w^W_{(\alpha,\ \beta)} = \{(A_k) :\ \lim_m \frac{1}{m^\alpha} \sum_{k \in C_m} |\delta(z,\ A_k) - \delta(z,\ A)|^\beta = 0\}$$

and if $\lambda_m = m$ for all $m \in \mathbb{N}$ and $\alpha = 1$, we have the sequence space w^W such that

$$w^W_{(1,\ \beta)} = \{(A_k) :\ \lim_m \frac{1}{m} \sum_{k \in C_m} |\delta(z,\ A_k) - \delta(z,\ A)|^\beta = 0\}.$$

Theorem 1 Suppose $(Z,\ \mathscr{J}))$ is a metric space, $0 < \alpha \leq \beta \leq 1$. If $S^W_{\lambda,\ (\alpha,\ \beta)} - \lim A_k = A$ then A is unique.

Proof. The proof is straight forward, so we omit the details. □

Theorem 2 Consider a metric space $(Z,\ \mathscr{J})$, $0 < \alpha \leq \beta \leq 1$ and $A,\ B,\ A_k,\ B_k$ be non-empty closed subsets of $Z(k \in \mathbb{N})$, then

(a) If $S^W_{\lambda,\ (\alpha,\ \beta)} - \lim A_k = A$ and $c \in \mathbb{C}$, then $S^W_{\lambda,\ (\alpha,\ \beta)} - \lim cA_k = cA$;

(b) If $S^W_{\lambda,\ (\alpha,\ \beta)} - \lim A_k = A$ and $S^W_{\lambda,\ \alpha,\ \beta} - \lim B_k = B$, then $S^W_{\lambda,\ \alpha,\ \beta} - \lim(A_k + B_k) = A + B$.

Proof. (a) The result is obvious for $c = 0$. Assume that $c \neq 0$, for every $\varepsilon > 0$ the result is derived from the following inequality,

$$\frac{1}{\lambda_m^{\alpha}}|\{k\in C_m:\ |\delta(cz,\ cA_k)-\delta(cz,\ cA)|\geq\varepsilon\}|^{\beta}=\frac{1}{\lambda_m^{\alpha}}\left|\left\{k\in C_m:\ |\delta(z,\ A_k)-\delta(z,\ A)|\geq\frac{\varepsilon}{|c|}\right\}\right|^{\beta}.$$

(b) For every $\varepsilon>0$. The result is derived from the following inequality,

$$\frac{1}{\lambda_m^{\alpha}}|\{k\in C_m:\ |\delta(z,\ A_k+B_k)-\delta(z,\ A+B)|\geq\varepsilon\}|^{\beta}$$

$$\leq\frac{1}{\lambda_m^{\alpha}}\left|\left\{k\in C_m:\ |\delta(z,\ A_k)-\delta(z,\ A)|\geq\frac{\varepsilon}{2}\right\}\right|^{\beta}$$

$$+\frac{1}{\lambda_m^{\alpha}}\left|\left\{k\in C_m:\ |\delta(z,\ B_k)-\delta(z,\ B)|\geq\frac{\varepsilon}{2}\right\}\right|^{\beta}.$$

□

If we take $\lambda_m=m$ in above theorem, then we have

Corollary 1 Consider a metric space $(Z,\ \mathscr{J})$, $\alpha,\ \beta\in(0,\ 1]$ and $A,\ B,\ A_k,\ B_k$ be non-empty closed subsets of Z $(k\in\mathbb{N})$, then

(a) If $S^W_{(\alpha,\ \beta)}-\lim A_k=A$ and $c\in\mathbb{C}$, then $S^W_{(\alpha,\ \beta)}-\lim cA_k=cA$;

(b) If $S^W_{(\alpha,\ \beta)}-\lim A_k=A$ and $S^W_{(\alpha,\ \beta)}-\lim B_k=B$, then $S^W_{(\alpha,\ \beta)}-\lim(A_k+B_k)=A+B$.

Theorem 3 If $0<\alpha\leq\beta<\gamma\leq1$, then $S^W_{\lambda,\ (\alpha,\ \beta)}\subset S^W_{\lambda,\ (\alpha,\ \gamma)}$ and the inclusion is strict.

Proof. The following inequality is used for the proof

$$\frac{1}{\lambda_m^{\gamma}}|\{k\in C_m:\ |\delta(z,\ A_k)-\delta(z,\ A)|\geq\varepsilon\}|^{\beta}\geq\frac{1}{\lambda_m^{\alpha}}|\{k\in C_m:\ |\delta(z,\ A_k)-\delta(z,\ A)|\geq\varepsilon\}|^{\beta}.$$

To establish that the inclusion is strict, let λ be given and consider the sequence (A_k) of non-empty closed subsets of $(Z,\ \mathscr{J})$, which is defined as

$$A_k=\begin{cases}\{(k,\ k)\}, & \text{if } m-[\sqrt{\lambda_m}]+1\leq k\leq m;\\ \{(0,\ 0)\}, & \text{otherwise.}\end{cases}$$

$$=\frac{1}{\lambda_m^{\gamma}}|\{k\in C_m:\ m-[\sqrt{\lambda_m}]+1\leq k\leq m\}|\leq\frac{\sqrt{\lambda_m}}{\lambda_m^{\gamma}}.$$

Then we have $(A_k)\in S^W_{\lambda,\ (\gamma,\ \beta)}$, for $\frac{1}{2}<\gamma\leq1$ but $(A_k)\notin S^W_{\lambda,\ (\alpha,\ \beta)}$, for $0<\alpha\leq\frac{1}{2}$. □

Theorem 4 For non-empty closed subsets of metric space $(Z,\ \mathscr{J})$ and (A_k) be the sequence which is $S^W_{\lambda,\ (\alpha,\ \beta)}$-convergent to A, then it is S^W_{λ}-convergent to A for $0<\alpha\leq\beta\leq1$.

Proof. The proof is straight forward, so we omit the details. □

Theorem 5 Consider a metric space $(Z,\ \mathscr{J})$ and $0<\alpha\leq\beta\leq1$. Then $S^W_{(\alpha,\ \beta)}\subset S^W_{\lambda,\ (\alpha,\ \beta)}$ if $\lim_{m\to\infty}\inf\frac{\lambda_m^{\alpha}}{m^{\alpha}}>0$.

Proof. If $A_k \to A(S^W_{(\alpha,\,\beta)})$, then for every $\varepsilon > 0$ and for sufficiently large m we have

$$\frac{1}{m^\alpha}|\{k \le m:\ |\delta(z,\,A_k) - \delta(z,\,A)| \ge \varepsilon\}|^\beta$$

$$\ge \frac{1}{m^\alpha}|\{k \in C_m:\ |\delta(z,\,A_k) - \delta(z,\,A)| \ge \varepsilon\}|^\beta$$

$$\ge \frac{\lambda_m^\alpha}{m^\alpha}\frac{1}{\lambda_m^\alpha}|\{k \in C_m:\ |\delta(z,\,A_k) - \delta(z,\,A)| \ge \varepsilon\}|^\beta.$$

By using the given condition and taking limit $m \to \infty$, we have

$$A_k \to A(S^W_{\lambda,\,(\alpha,\,\beta)}).$$

□

Corollary 2 Suppose $(Z,\ \mathscr{J})$ is a metric space, $\lambda = (\lambda_m) \in \Lambda$ and $0 < \alpha \le \beta \le 1$, then

$$S^W_{\lambda,\,(\alpha,\,\beta)} \subset S^W.$$

Proof. Suppose that $A_k \in S^W_{\lambda,\,(\alpha,\,\beta)}$. We have to show that $A_k \in S^W$. Since $A_k \in S^W_{\lambda,\,(\alpha,\,\beta)}$ and from Theorem 5 $\lim_{m\to\infty}\inf\frac{\lambda_m^\alpha}{m^\alpha} > 0$, we have

$$\frac{1}{\lambda_m^\alpha}|\{k \in C_m:\ |\delta(z,\,A_k) - \delta(z,\,A)| \ge \varepsilon\}|^\beta \le \frac{1}{\lambda_m^\alpha}|\{k \in C_m:\ |\delta(z,\,A_k) - \delta(z,\,A)| \ge \varepsilon\}|^\beta$$

$$\le \frac{1}{m^\alpha}|\{k \in C_m:\ |\delta(z,\,A_k) - \delta(z,\,A)| \ge \varepsilon\}|^\beta$$

$$\le \frac{1}{m}|\{k \le m:\ |\delta(z,\,A_k) - \delta(z,\,A)| \ge \varepsilon\}|$$

Therefore, $A_k \in S^W$. This completes the result. □

Theorem 6 Let $\lambda = (\lambda_m) \in \Lambda$ and $0 < \alpha \le \beta \le 1$. Then $S^W \subset S^W_{\lambda,\,(\alpha,\,\beta)}$ if and only if

$$\lim_m \frac{\lambda_m^\alpha}{m} > 0. \tag{1}$$

Proof. Suppose that condition (1) holds and $(A_k) \in S^W$. Given $\varepsilon > 0$ we get

$$\{k \le m:\ |\delta(z,\,A_k) - \delta(z,\,A)| \ge \varepsilon\} \supset \{k \in C_m:\ |\delta(z,\,A_k) - \delta(z,\,A) \ge \varepsilon\}.$$

Then, we have

$$\frac{1}{m}|\{k \le m: \ |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\}|$$

$$\ge \frac{1}{m}|\{k \in C_m: \ |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\}|$$

$$= \frac{\lambda_m^\alpha}{m}\frac{1}{\lambda_m^\alpha}|\{k \in C_m: \ |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\}|^\beta.$$

By taking limit as $m \to \infty$ from the condition (1), we get $A_k \to A(S^W)$ implies

$$A_k \to A(S^W_{\lambda, (\alpha, \beta)}).$$

Next, we assume that $\lim_{m\to\infty} \inf \frac{\lambda_m^\alpha}{m} = 0$. Consider a subsequence (m_i) such that $\frac{\lambda_{m_i}^\alpha}{m_i} < \frac{1}{i}$. We establish a sequence (A_k) as

$$A_k = \begin{cases} \{(1, 1)\}, & \text{if } k \in C_{m_i} \text{ for some } i; \\ \{(0, 0)\}, & \text{otherwise.} \end{cases}$$

Thus clearly $(A_k) \in S^W$ but $(A_k) \notin S^W_{\lambda, (\alpha, \beta)}$. Since $S^W_{\lambda, (\alpha, \beta)} \subset S^W_\lambda$, we have $(A_k) \notin S^W_{\lambda, (\alpha, \beta)}$, which contradicts the given condition. Therefore condition (1) holds. □

Theorem 7 Consider a metric space $(Z, \mathscr{J})$, $0 < \alpha \le \beta \le \gamma \le 1$ and two sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \le \nu_m$ for all $m \in \mathbb{N}$. If

$$\lim_{m\to\infty} \inf \frac{\lambda_m^\alpha}{\lambda_m^\gamma} \tag{2}$$

then $S^W_{\lambda, (\gamma, \beta)} \subseteq S^W_{\lambda, (\alpha, \beta)}$.

Proof. Suppose $\lambda_m \le \nu_m$ for all $m \in \mathbb{N}$ and the relation (2) holds. Thus $C_m \subset J_m$, where $J_m = [m - \nu_m + 1, \ m]$ and for $\varepsilon > 0$, we can write

$$\{k \in J_m: \ |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\} \supset \{k \in C_m: \ |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\}.$$

Thus, we have

$$\frac{1}{\nu_m^\gamma}|\{k \in J_m: \ |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\}|^\beta \ge \frac{\lambda_m^\alpha}{\nu_m^\gamma}\frac{1}{\lambda_m^\alpha}|\{k \in C_m: \ |\delta(z, A_k) - \delta(z, A)| \ge \varepsilon\}|^\beta,$$

for all $m \in \mathbb{N}$.

As limit $m \to \infty$ in above inequality and from condition (2), we have

$$S^W_{\lambda,\ (\gamma,\ \beta)} \subseteq S^W_{\lambda,\ (\alpha,\ \beta)}.$$

$\square$

Corollary 3 Consider a metric space $(Z, \mathscr{J})$ and the sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$. If condition (2) holds, then

(a) $S^W_{\nu,\ (\alpha,\ \beta)} \subseteq S^W_{\lambda,\ (\alpha,\ \beta)}$, for $\alpha,\ \beta \in (0,\ 1]$,

(b) $S^W_{\nu} \subseteq S^W_{\lambda,\ (\alpha,\ \beta)}$, for $\alpha,\ \beta \in (0,\ 1]$,

(c) $S^W_{\nu} \subseteq S^W_{\lambda}$.

Theorem 8 Consider a metric space $(Z, \mathscr{J})$, $0 < \alpha \leq \beta \leq \gamma \leq 1$ and the sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$. If

$$\lim_{m\to\infty} \inf \frac{\nu_m}{\lambda^\gamma_m} = 1 \tag{3}$$

then $S^W_{\lambda,\ (\alpha,\ \beta)} \subseteq S^W_{\nu,\ (\gamma,\ \beta)}$.

Proof. Let $S^W_{\lambda,\ (\alpha,\ \beta)} - \lim A_k = A$ and condition (3) is satisfied. Since $C_m \subset J_m$, for $\varepsilon > 0$ we can write

$$\frac{1}{\nu^\gamma_m}|\{k \in J_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta$$

$$= \frac{1}{\nu^\gamma_m}|\{m - \nu_m + 1 \leq k \leq m - \lambda_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta$$

$$+ \frac{1}{\nu^\gamma_m}|\{k \in C_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta$$

$$\leq \frac{\nu_m - \lambda_m}{\nu^\gamma_m} + \frac{1}{\nu^\gamma_m}|\{k \in C_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta$$

$$\leq \frac{\nu_m - \lambda^\gamma_m}{\lambda^\gamma_m} + \frac{1}{\nu^\gamma_m}|\{k \in C_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta$$

$$\leq \left(\frac{\nu_m}{\lambda^\gamma_m} - 1\right) + \frac{\lambda^\alpha_m}{\nu^\gamma_m}\frac{1}{\lambda^\alpha_m}|\{k \in C_m :\ |\delta(z,\ A_k) - \delta(z,\ A)| \geq \varepsilon\}|^\beta.$$

Using the condition (3) and $S^W_{\lambda,\ (\alpha,\ \beta)} - \lim A_k = A$ the above inequality on right side goes to zero as $m \to \infty$. Hence, $S^W_{\lambda,\ (\alpha,\ \beta)} \subseteq S^W_{\nu,\ (\gamma,\ \beta)}$. $\square$

Corollary 4 Suppose $(Z, \mathscr{J})$ is a metric space and the sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$. If condition (3) holds, then

(a) $S^W_{\lambda,\ (\alpha,\ \beta)} \subseteq S^W_{\nu,\ (\alpha,\ \beta)}$ for $0 < \alpha \leq \beta \leq 1$.

(b) $S_{\lambda}^{W} \subseteq S_{\nu,\ (\alpha,\ \beta)}^{W}$ for $0 < \alpha \leq \beta \leq 1$.
(c) $S_{\lambda}^{W} \subseteq S_{\nu}^{W}$.

4. Wijsman λ-statistical convergence of Musielak-Orlicz function of order (α, β)

An Orlicz function M is a function $M:\ [0,\ \infty) \to [0,\ \infty)$ which is continuous, convex and nondecreasing define for $z > 0$ such that $M(0) = 0,\ M(z) > 0$ and $M(z) \to \infty$ as $z \to \infty$. When the convexity of Orlicz function is changed to $M(z+y) \leq M(z) + M(y)$ then it is known as modulus function and characterised by Ruckle [42]. An Orlicz function M is known to be satisfy Δ_2-condition for all values t, if there exists $Q \geq 1$ such that

$$M(2t) \leq QM(t),\ t \geq 0. \tag{4}$$

The Δ_2-condition is a critical growth condition applied to Orlicz functions in the theory of functional spaces, particularly Orlicz spaces. The inequality (4) imposes a restriction on the rate at which $M(t)$ can grow as t increases. Specifically, it ensures that when the argument t is doubled, the value of the function $M(2t)$ is controlled by a multiple Q of the value $M(t)$. The constant Q serves as a uniform bound across the entire domain $t \geq 0$, effectively limiting the rapid growth of the function.

A sequence $\mathscr{M} = (\mathfrak{F}_k)$ of Orlicz functions is known as Musielak-Orlicz function [43]. A sequence $\mathscr{N} = (\mathfrak{N}_k)$ is given by

$$\mathfrak{N}_k(v) = \sup\{|v|t - \mathfrak{F}_k(t):\ t \geq 0\},\ k = 1,\ 2,\ \ldots$$

is known as complementary function of a Musielak-Orlicz function $\mathscr{M}$, the sequence space generated by Musielak-Orlicz function $t_{\mathscr{M}}$ and its subspace $h_{\mathscr{M}}$ are defined as follows

$$t_{\mathscr{M}} = \{z \in w:\ I_{\mathscr{M}}(cz) < \infty \text{ for some } c > 0\}$$

$$h_{\mathscr{M}} = \{z \in w:\ I_{\mathscr{M}}(cz) < \infty \text{ for all } c > 0\},$$

where $I_{\mathscr{M}}$ is a convex modular defined by

$$I_{\mathscr{M}} = \sum_{k=1}^{\infty} M_k(z_k),\ z = (z_k) \in t_{\mathscr{M}}.$$

Definition 3 Consider a metric space $(Z,\ \mathscr{J})$, $\alpha,\ \beta \in (0,\ 1]$ and $\mathscr{M}$ be a Musielak-Orlicz function and a sequence of strictly positive real numbers $p = (p_k),\ \lambda = (\lambda_m)$ be a sequence of positive reals, and for $\rho > 0$, we define

$$w_{\lambda,\ (\alpha,\ \beta)}^{W}[\mathscr{M},\ p] = \left\{(A_k) \in Z:\ \lim_{m\to\infty} \frac{1}{\lambda_m^{\alpha}} \sum_{k\in C_m} \left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\ A_k) - \delta(z,\ A)|}{\rho}\right)\right]^{p_k}\right]^{\beta} = 0\right\},$$

for some A and for $z \in Z$.

If $\mathscr{M}(z) = z$ and $p_k = p$ for all $k \in \mathbb{N}$ then we have $w^W_{\lambda,\,(\alpha,\,\beta)}[\mathscr{M},\, p] = w^W_{\lambda,\,(\alpha,\,\beta)}(p)$ and if $\mathscr{M}(z) = z$ then we have

$$w^W_{\lambda,\,(\alpha,\,\beta)}[\mathscr{M},\, p] = w^W_{\lambda,\,(\alpha,\,\beta)}[p].$$

Theorem 9 Suppose that $\mathscr{M}$ be a Musielak-Orlicz function, $0 < \alpha \leq \beta \leq \gamma \leq 1$ and a bounded sequence (p_k), $0 < \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$ and a sequence of positive reals $\lambda = (\lambda_m)$, then $w^W_{\lambda,\,(\alpha,\,\beta)}[\mathscr{M},\, p] \subset S^W_{\lambda,\,(\gamma,\,\beta)}$.

Proof. Assume that $(A_k) \in w^W_{\lambda,\,(\alpha,\,\beta)}[\mathscr{M},\, p]$. For $\varepsilon > 0$ be given. As $\lambda_m^\alpha \leq \lambda_m^\gamma$ for each m we can write

$$\frac{1}{\lambda_m^\alpha} \sum_{k \in C_m} \left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\, A_k) - \delta(z,\, A)|}{\rho}\right)\right]^{p_k}\right]^\beta$$

$$= \frac{1}{\lambda_m^\alpha} \left\{ \left[\sum_{\substack{k \in C_m \\ \Psi \geq \varepsilon}} \left[\mathfrak{F}_k\left(\frac{|\delta(z,\, A_k) - \delta(z,\, A)|}{\rho}\right)^{p_k}\right]\right]^\beta \right.$$

$$\left. + \left[\sum_{\substack{k \in C_m \\ \Psi < \varepsilon}} \left[\mathfrak{F}_k\left(\frac{|\delta(z,\, A_k) - \delta(z,\, A)|}{\rho}\right)\right]^{p_k}\right]^\beta \right\}, \text{ where } \Psi = |\delta(z,\, A_k) - \delta(z,\, A)|$$

$$\geq \frac{1}{\lambda_m^\gamma} \left\{ \left[\sum_{\substack{k \in C_m \\ \Psi \geq \varepsilon}} \left[\mathfrak{F}_k\left(\frac{|\delta(z,\, A_k) - \delta(z,\, A)|}{\rho}\right)\right]^{p_k}\right]^\beta \right.$$

$$\left. + \left[\sum_{\substack{k \in C_m \\ \Psi < \varepsilon}} \left[\mathfrak{F}_k\left(\frac{|\delta(z,\, A_k) - \delta(z,\, A)|}{\rho}\right)\right]^{p_k}\right]^\beta \right\}$$

$$\geq \frac{1}{\lambda_m^\gamma} \left[\sum_{\substack{k \in C_m \\ \Psi \geq \varepsilon}} \left[\mathfrak{F}_k\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k}\right]^\beta$$

$$\geq \frac{1}{\lambda_m^\gamma} \left[\sum_{\substack{k \in C_m \\ \Psi \geq \varepsilon}} \min\Big([\mathfrak{F}_k(\varepsilon_1)]^h,\, [\mathfrak{F}_k(\varepsilon_2)]^H\Big)\right]^\beta$$

$$\geq \frac{1}{\lambda_m^\gamma} \Big[|\{k \in C_m :\ |\delta(z,\, A_k) - \delta(z,\, A)| \geq \varepsilon\}| \min\Big([\mathfrak{F}_k(\varepsilon_1)]^h,\, [\mathfrak{F}_k(\varepsilon_2)]^H\Big)\Big]^\beta,$$

where $\varepsilon_1 = \dfrac{\varepsilon}{\rho}$. Therefore from above inequality we get $(A_k) \in S^W_{\lambda,\,(\gamma,\,\beta)}$. □

Corollary 5 Suppose that $\mathscr{M}$ be a Musielak-Orlicz function, $0 < \alpha \leq \beta \leq \gamma \leq 1$, and $\lambda = (\lambda_m) \in \Lambda$, then

$$w^W_{\lambda,\ (\alpha,\ \beta)}[\mathscr{M},\ p] \subset S^W_{\lambda,\ (\alpha,\ \beta)}.$$

Theorem 10 Suppose that $\mathscr{M}$ be a Musielak-Orlicz function, $\lambda = (\lambda_m) \in \Lambda$ and (A_k) be a sequence in L_∞. If $\lim_{m\to\infty} \dfrac{\lambda_m}{\lambda_m^\alpha} = 1$, then

$$S^W_{\lambda,\ (\alpha,\ \beta)} \subset w^W_{\lambda,\ (\alpha,\ \beta)}[\mathscr{M},\ p].$$

Proof. Suppose that $(A_k) \in L_\infty$ and $S^W_{\lambda,\ \alpha,\ \beta} - \lim A_k = A$. Since $(A_k) \in L_\infty$ then there exist $k > 0$ such that $|\delta(z,\ A_k) - \delta(z,\ A)| \leq k$ for all k. For given $\varepsilon > 0$, we get

$$\frac{1}{\lambda_m^\alpha}\sum_{k\in C_m}\left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\ A_k)-\delta(z,\ A)|}{\rho}\right)\right]^{p_k}\right]^\beta$$

$$= \frac{1}{\lambda_m^\alpha}\left[\sum_{\substack{k\in C_m\\ \Psi\geq\varepsilon}}\left[\mathfrak{F}_k\left(\frac{|\delta(z,\ A_k)-\delta(z,\ A)|}{\rho}\right)\right]^{p_k}\right]^\beta$$

$$+ \frac{1}{\lambda_m^\alpha}\left[\sum_{\substack{k\in C_m\\ \Psi<\varepsilon}}\left[\mathfrak{F}_k\left(\frac{|\delta(z,\ A_k)-\delta(z,\ A)|}{\rho}\right)\right]^{p_k}\right]^\beta, \text{ where } \Psi = |\delta(z,\ A_k)-\delta(z,\ A)|$$

$$\leq \frac{1}{\lambda_m^\alpha}\left[\sum_{\substack{k\in C_m\\ \Psi\geq\varepsilon}}\max\left\{\left[\mathfrak{F}_k\left(\frac{k}{\rho}\right)\right]^h,\ \left[\mathfrak{F}_k\left(\frac{k}{\rho}\right)\right]^H\right\}\right]^\beta + \frac{1}{\lambda_m^\alpha}\left[\sum_{\substack{k\in C_m\\ \Psi\geq\varepsilon}}\left[\mathfrak{F}_k\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k}\right]^\beta$$

$$\leq \max\left\{\left[\mathfrak{F}_k(\frac{k}{\rho})\right]^h,\ \left[\mathfrak{F}_k\left(\frac{k}{\rho}\right)\right]^H\right\}\frac{1}{\lambda_m^\alpha}|\delta(z,\ A_k)-\delta(z,\ A)| \geq \varepsilon|^\beta$$

$$+ \frac{\lambda_m}{\lambda_m^\alpha}\left[\max\left\{\left[\mathfrak{F}_k\left(\frac{\varepsilon}{\rho}\right)\right]^h,\ \left[\mathfrak{F}_k\left(\frac{\varepsilon}{\rho}\right)\right]^H\right\}\right]^\beta.$$

Thus, we have $(A_k) \in w^W_{\lambda,\ (\alpha,\ \beta)}[\mathscr{M},\ p]$. □

Theorem 11 Let $\lambda = (\lambda_m) \in \Lambda,\ 0 < \alpha \leq \beta \leq \gamma \leq 1$ and p be a positive real number, then

$$w^W_{\lambda,\ (\alpha,\ \beta)}(p) \subseteq w^W_{\gamma,\ (\alpha,\ \beta)}(p).$$

Proof. The proof is straight forward, so we omit the details. □

Corollary 6 Let $\lambda = (\lambda_m) \in \Lambda$ and a positive real number p, then

$$w^W_{\lambda,\,(\alpha,\,\beta)}(p) \subseteq w^W_{\lambda}(p).$$

Theorem 12 Let $\lambda = (\lambda_m) \in \Lambda,\ 0 < \alpha \leq \beta \leq \gamma \leq 1$ and a positive real number p, then

$$w^W_{\lambda,\,(\alpha,\,\beta)}(p) \subseteq S^W_{\lambda,\,(\gamma,\,\beta)}.$$

Proof. Suppose that $(A_k) \in w^W_{\lambda,\,\alpha,\,\beta}(p)$ and for $\varepsilon > 0$, we have

$$\lim_m \frac{1}{\lambda_m^{\alpha}} \sum_{k\in C_m} \left[|\delta(z,\,A_k) - \delta(z,\,A)|^p\right]^{\beta}$$

$$= \lim_m \frac{1}{\lambda_m^{\alpha}} \sum_{\substack{k\in C_m\\ \Psi\geq\varepsilon}} \left[|\delta(z,\,A_k) - \delta(z,\,A)|^p\right]^{\beta}$$

$$+ \lim_m \frac{1}{\lambda_m^{\alpha}} \sum_{\substack{k\in C_m\\ \Psi<\varepsilon}} \left[|\delta(z,\,A_k) - \delta(z,\,A)|^p\right]^{\beta},\ \text{where } \Psi = |\delta(z,\,A_k) - \delta(z,\,A)|$$

$$\geq \lim_m \frac{1}{\lambda_m^{\gamma}} \sum_{\substack{k\in C_m\\ \Psi\geq\varepsilon}} \left[|\delta(z,\,A_k) - \delta(z,\,A)|^p\right]^{\beta}$$

$$\geq \lim_m \frac{1}{\lambda_m^{\gamma}} |\{k \in C_m :\ \delta(z,\,A_k) - \delta(z,\,A)| \geq \varepsilon\}|^{\beta}.\varepsilon^p.$$

Thus, we have

$$\frac{1}{\lambda_m^{\alpha}} \sum_{k\in C_m} \left[|\delta(z,\,A_k) - \delta(z,\,A)|^p\right]^{\beta} \geq \frac{1}{\lambda_m^{\gamma}} |\{k \in C_m :\ \delta(z,\,A_k) - \delta(z,\,A)| \geq \varepsilon\}|^{\beta}.\varepsilon^p.$$

Therefore the last inequality implies that $(A_k) \in S^W_{\lambda,\,(\gamma,\,\beta)}$ if $(A_k) \in w^W_{\lambda,\,(\alpha,\,\beta)}(p)$. This completes the proof. □

Theorem 13 For $0 < \alpha \leq \beta \leq \gamma \leq 1$, let $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ be two sequences in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$ and if condition (3) holds, then

$$w^W_{\nu,\,(\gamma,\,\beta)}(p) \subseteq w^W_{\lambda,\,(\alpha,\,\beta)}(p).$$

Proof. The proof is straight forward, so we omit the details. □

Corollary 7 Suppose that the sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$. If condition (3) holds, then

(a) $w^W_{\nu,\,(\alpha,\,\beta)}(p) \subseteq w^W_{\lambda,\,(\alpha,\,\beta)}(p)$ for $0 < \alpha \leq \beta \leq 1$,

(b) $w^W_{\nu}(p) \subseteq w^W_{\lambda}(p)$ for $0 < \alpha \leq \beta \leq 1$,

(c) $w^W_{\nu}(p) \subseteq w^W_{\lambda}(p)$.

Theorem 14 For $0 < \alpha \leq \beta \leq \gamma \leq 1$ and the sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$. If condition (3) holds, then

$$w^W_{\nu,\,(\gamma,\,\beta)}(p) \subseteq S^W_{\lambda,\,(\alpha,\,\beta)}.$$

Proof. Let $(A_k) \in w^W_{\nu,\,(\gamma,\,\beta)}(p)$ and for $\varepsilon > 0$. Thus we have

$$\lim_m \frac{1}{\nu_m^\gamma} \sum_{k \in C_m} \left[|\delta(z,\, A_k) - \delta(z,\, A)|^p \right]^\beta$$

$$= \lim_m \frac{1}{\nu_m^\gamma} \sum_{\substack{k \in C_m \\ \Psi \geq \varepsilon}} \left[|\delta(z,\, A_k) - \delta(z,\, A)|^p \right]^\beta$$

$$+ \lim_m \frac{1}{\nu_m^\gamma} \sum_{\substack{k \in C_m \\ \Psi < \varepsilon}} \left[|\delta(z,\, A_k) - \delta(z,\, A)|^p \right]^\beta, \text{ where } \Psi = |\delta(z,\, A_k) - \delta(z,\, A)|$$

$$\geq \lim_m \frac{1}{\nu_m^\gamma} \sum_{\substack{k \in C_m \\ \Psi \geq \varepsilon}} [|\delta(z,\, A_k) - \delta(z,\, A)|^p]^\beta$$

$$\geq \lim_m \frac{1}{\nu_m^\gamma} |\{k \in C_m :\ |\delta(z,\, A_k) - \delta(z,\, A)| \geq \varepsilon\}|^\beta . \varepsilon^p.$$

Thus, we have

$$\frac{1}{\nu_m^\gamma} \sum_{k \in C_m} \left[|\delta(z,\, A_k) - \delta(z,\, A)|^p \right]^\beta \geq \frac{\lambda_m^\alpha}{\nu_m^\gamma} \frac{1}{\lambda_m^\alpha} |\{k \in C_m :\ |\delta(z,\, A_k) - \delta(z,\, A)| \geq \varepsilon\}|^\beta . \varepsilon^p.$$

Since condition (3) holds and $(A_k) \in w^W_{\nu,\,\gamma,\,\beta}(p)$. Therefore from last inequality, we have $(A_k) \in S^W_{\lambda,\,(\alpha,\,\beta)}$. This completes the result. □

Corollary 8 For $0 < \alpha \leq \beta \leq 1$ and the sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$. If condition (3) holds, then

(a) $w^W_{\nu,\,(\alpha,\,\beta)}(p) \subseteq S^W_{\lambda,\,(\alpha,\,\beta)}$,

(b) $w^W_{\nu}(p) \subseteq S^W_{\lambda,\,(\alpha,\,\beta)}$,

(c) $w^W_{\nu}(p) \subseteq S^W_{\lambda}$.

Theorem 15 For $0 < \alpha \leq \beta \leq \gamma \leq 1$ and the sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$. If condition (3) holds, then

$$L_\infty \cap w^W_{\nu,\,(\alpha,\,\beta)}(p) \subseteq w^W_{\lambda,\,(\gamma,\,\beta)}(p).$$

Proof. Let $(A_k) \in L_\infty \cap w^W_{\nu,\,(\alpha,\,\beta)}(p)$ and suppose that condition (3) is satisfied.

Since $(A_k) \in L_\infty$, there exist $K > 0$ such that $|\delta(z, A_k) - \delta(z, A)| \leq K$ for all k. Since $\lambda_m \leq \nu_m$ and $C_m \subset J_m$ for all $m \in \mathbb{N}$, we have

$$\frac{1}{\nu_m^\gamma} \sum_{k \in J_m} \left[|\delta(z, A_k) - \delta(z, A)|^p \right]^\beta$$

$$= \frac{1}{\nu_m^\gamma} \sum_{k \in J_m - C_m} \left[|\delta(z, A_k) - \delta(z, A)|^p \right]^\beta + \frac{1}{\nu_m^\gamma} \sum_{k \in C_m} \left[|\delta(z, A_k) - \delta(z, A)|^p \right]^\beta$$

$$\leq \left(\frac{\nu_m - \lambda_m}{\nu_m^\gamma} \right) K^p + \frac{1}{\nu_m^\gamma} \sum_{k \in C_m} \left[|\delta(z, A_k) - \delta(z, A)|^p \right]^\beta$$

$$\leq \left(\frac{\nu_m - \lambda_m^\gamma}{\nu_m^\gamma} \right) K^p + \frac{1}{\nu_m^\gamma} \sum_{k \in C_m} \left[|\delta(z, A_k) - \delta(z, A)|^p \right]^\beta$$

$$\leq \left(\frac{\nu_m - \lambda_m^\gamma}{\lambda_m^\gamma} \right) K^p + \frac{\lambda_m^\alpha}{\nu_m^\gamma} \frac{1}{\lambda_m^\alpha} \sum_{k \in C_m} \left[|\delta(z, A_k) - \delta(z, A)|^p \right]^\beta$$

$$\leq \left(\frac{\nu_m}{\lambda_m^\gamma} - 1 \right) K^p + \frac{\lambda_m^\alpha}{\nu_m^\gamma} \frac{1}{\lambda_m^\alpha} \sum_{k \in C_m} \left[|\delta(z, A_k) - \delta(z, A)|^p \right]^\beta.$$

This implies that $(A_k) \in w^W_{\lambda,\,(\gamma,\,\beta)}(p)$. Hence $L_\infty \cap w^W_{\nu,\,(\alpha,\,\beta)}(p) \subseteq w^W_{\lambda,\,(\gamma,\,\beta)}(p)$. □

Corollary 9 Suppose that the sequences $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ in Λ such that $\lambda_m \leq \nu_m$ for all $m \in \mathbb{N}$. If condition (3) holds, then

(a) $L_\infty \cap w^W_{\nu,\,(\alpha,\,\beta)}(p) \subseteq w^W_{\lambda,\,(\gamma,\,\beta)}(p)$ for $0 < \alpha \leq \beta \leq \gamma \leq 1$,

(b) $L_\infty \cap w^W_{\lambda,\,(\alpha,\,\beta)}(p) \subseteq w^W_{\nu}(p)$ for $0 < \alpha \leq \beta \leq 1$,

(c) $L_\infty \cap w^W_{\lambda}(p) \subseteq w^W_{\nu}(p)$.

Theorem 16 Consider $\mathscr{M}$ be a Musielak-Orlicz function and if $\inf_k p_k > 0$, then limit of the sequence (A_k) in $w^W_{\lambda,\,\alpha,\,\beta}[\mathscr{M},\, p]$ is unique.

Proof. Suppose $\lim_k p_k = s > 0$ and $A_k \to A_1 \left(w^W_{\lambda,\,(\alpha,\,\beta)}[\mathscr{M},\, p] \right)$ and $A_k \to A_2 \left(w^W_{\lambda,\,(\alpha,\,\beta)}[\mathscr{M},\, p] \right)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{m\to\infty}\frac{1}{\lambda_m^{\alpha}}\sum_{k\in C_m}\left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_k)-\delta(z,\,A_1)}{\rho}\right)\right]^{p_k}\right]^{\beta}=0$$

and

$$\lim_{m\to\infty}\frac{1}{\lambda_m^{\alpha}}\sum_{k\in C_m}\left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_k)-\delta(z,\,A_2)}{\rho}\right)\right]^{p_k}\right]^{\beta}=0$$

Let $\rho=\max\{2\rho_1,\ 2\rho_2\}$. Since $\mathscr{M}$ is convex and non-decreasing, we get

$$\frac{1}{\lambda_m^{\alpha}}\sum_{k\in C_m}\left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_1)-\delta(z,\,A_2)}{\rho}\right)\right]^{p_k}\right]^{\beta}$$

$$\leq\frac{D}{\lambda_m^{\alpha}}\sum_{k\in C_m}\left(\frac{1}{2^{p_k}}\left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_k)-\delta(z,\,A_1)|}{\rho}\right)\right]^{p_k}\right]^{\beta}\right.$$

$$\left.+\left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_k)-\delta(z,\,A_2)|}{\rho}\right)\right]^{p_k}\right]^{\beta}\right)$$

$$\frac{D}{\lambda_m^{\alpha}}\sum_{k\in C_m}\left(\left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_k)-\delta(z,\,A_1)|}{\rho}\right)\right]^{p_k}\right]^{\beta}\right.$$

$$\left.+\frac{D}{\lambda_m^{\alpha}}\sum_{k\in C_m}\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_k)-\delta(z,\,A_2)|}{\rho}\right)\right]^{p_k}\right)^{\beta}\to 0 \text{ as } m\to\infty,$$

where $\sup_k p_k=H$ and $D=\max(1,\ 2^{H-1})$. Thus, we have

$$\lim_{m\to\infty}\frac{1}{\lambda_m^{\alpha}}\sum_{k\in C_m}\left[\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_1)-\delta(z,\,A_2)}{\rho}\right)\right]^{p_k}\right]^{\beta}=0.$$

Since $\lim_k p_k=s$, we have

$$\lim_{m\to\infty}\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_1)-\delta(z,\,A_2)}{\rho}\right)\right]^{p_k}=\left[\mathfrak{F}_k\left(\frac{|\delta(z,\,A_1)-\delta(z,\,A_2)}{\rho}\right)\right]^{s}.$$

So $A_1=A_2$. Hence limit is unique. □

5. Conclusion

In this paper, we defined and developed the concept of Wijsman λ-statistical convergence of order (α, β), providing a solid theoretical framework and proving essential properties of this convergence type. By extending the analysis to include the Musielak-Orlicz function, the study has demonstrated the adaptability of this convergence framework to more generalized function spaces. The investigation of topological properties has further enriched the understanding of the behavior of Wijsman λ-statistical convergence within various mathematical structures. In future research can explore extending Wijsman λ-statistical convergence of order (α, β) to broader spaces, such as Banach or Fréchet spaces, and examine its relationship with other convergence types like strong or weak convergence. Applications in approximation theory, optimization, and dynamical systems could be further investigated. The impact of different Musielak-Orlicz functions on convergence and the study of more complex topological properties, such as compactness, are potential areas for exploration.

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Conflict of interest

The authors declare no competing financial interest.

References

[1] Zygmund A. *Trigonometrical Series*. Monografias de Matematicas: Warszawa-Lwow; 1935.

[2] Schoenberg IJ. The integrability of certain functions and related summability methods. *The American Mathematical Monthly*. 1959; 66(5): 361-375.

[3] Steinhaus H. Sur La Convergence Ordinaire Et La Convergence Asymptotique [Regarding Ordinary Convergence and Asymptotic Convergence]. *Colloquium Mathematicum*. 1951; 2: 73-74.

[4] Fast H. Sur La Convergence Statistique [Regarding Statistical Convergence]. *Colloquium Mathematicae*. 1951; 2(3-4): 241-244.

[5] Buck RC. Generalized asymptotic density. *American Journal of Mathematics.* 1953; 75(2): 335-346.

[6] Esi A. The A-statistical and strongly (Ap)-Cesaro convergence of sequences. *Pure and Applied Mathematika Sciences*. 1996; 43(1-2): 89-94.

[7] Fridy JA. On statistical convergence. *Analysis*. 1985; 5(4): 301-313.

[8] Šalát T. On statistically convergent sequences of real numbers. *Mathematica Slovaca*. 1980; 30(2): 139-150.

[9] Mursaleen M. λ-statistical convergence. *Mathematica Slovaca*. 2000; 50(1): 111-115.

[10] Cai QB, Sharma SK, Mursaleen MA. A note on lacunary sequence spaces of fractional difference operator of order (α, β). *Journal of Function Spaces*. 2022; 2022(1): 2779479.

[11] Colak R. On λ-statistical convergence. In: *Conference on Summability and Applications.* Turkey: Istanbul; 2011. p.12-13.

[12] Mohiuddine SA, Sharma SK, Abuzaid DA. Some seminormed difference sequence spaces over *n*-normed spaces defined by a musielak-orlicz function of order (α, β). *Journal of Function Spaces*. 2018; 2018(1): 4312817.

[13] Sharma SK, Mohiuddine SA, Sharma AK, Sharma TK. Sequence spaces over *n*-normed spaces defined by Musielak-Orlicz function of order(α, β). *Facta Universitatis, Series: Mathematics and Informatics*. 2019; 33(5): 721-738.

[14] Hazarika B, Esi A. On generalized statistical convergence of sequences of sets of order α. *Miskolc Mathematical Notes*. 2016; 17(2): 893-210.

[15] Gülle E, Ulusu U. Wijsman deferred invariant statistical and strong p-deferred invariant equivalence of order α. *Fundamental Journal of Mathematics and Applications.* 2023; 6(4): 211-217.

[16] Kişi Ö, Gülle E. $\mathscr{I}$-Cesàro summability of a sequence of order α of random variables in probability. *Fundamental Journal of Mathematics and Applications.* 2018; 1(2): 157-161.

[17] Kişi Ö. On $\mathscr{I}_\theta$-convergence in neutrosophic normed spaces. *Fundamental Journal of Mathematics and Applications.* 2021; 4(2): 67-76.

[18] Hazarika B, Esi A. On asymptotically Wijsman lacunary statistical convergence of set sequences in ideal context. *Filomat*. 2017; 31(9): 2691-2703.

[19] Esi A, Subramanian N. Wijsman rough statistical convergence on triple sequences. *International Journal of Sciences: Basic and Applied Research.* 2017; 32(3): 13-27.

[20] Esi A, Subramanian N, Aiyub M. Wijsman rough λ-statistical convergence of order α of triple sequence of functions. *Asia Pasific Journal of Mathematics*. 2017; 4(1): 49-61.

[21] Lindenstrauss J, Tzafriri L. On Orlicz sequence spaces. *Israel Journal of Mathematics*. 1971; 10: 379-390.

[22] Zygmund A. *Trigonometric Series*. Cambridge, UK: Cambridge Univ. Press; 1979.

[23] Leindler L. Über die verallgemeinerte de la Vallée-Pousinsche Summierbarkeit allgemeiner Orthogonalreihen [On the generalised de la Vallée-Pousinsche summation of general orthogonal series]. *Acta Mathematica Academiae Scientiarum Hungarica*. 1965; 16: 375-387.

[24] Wijsman RA. Convergence of sequences of convex sets, cones and functions. *Bulletin of the American Mathematical Society*. 1964; 70: 186-188.

[25] Wijsman RA. Convergence of sequences of convex sets, cones and functions II. *Transactions of the American Mathematical Society*. 1966; 123(1): 32-45.

[26] Kumar R, Sharma SK, Sharma AK. Asymptotically Wijsman lacunary sequences of order (α, β). *The Journal of Analysis*. 2024; 32(4): 2037-2055.

[27] Kumar R, Mohiuddine SA, Sharma SK, Sharma AK. Asymptotically Wijsman lacunary sequences of order (α, β) by using ideal. *Honam Mathematical Journal*. 2024; 46(3): 485-499.

[28] Nuray F, Rhoades BE. Statistical convergence of sequences of sets. *Fasciculi Mathematici*. 2012; 49(2): 87-99.

[29] Hazarika B. Wijsman Orlicz asymptotically ideal ϕ-statistical equivalent sequences. *Journal of Function Spaces*. 2013; 2013(1): 257181.

[30] Hazarika B, Esi A. On λ-asymptotically Wijsman generalized statistical convergence of sequences of sets. *Tatra Mountains Mathematical Publications*. 2013; 56(1): 67-77.

[31] Hazarika B, Esi A. Statistically almost λ-convergence of sequence of sets. *European Journal of Pure and Applied Mathematics*. 2013; 6(2): 137-146.

[32] Hazarika B, Esi A, Braha NL. On asymptotically Wijsman lacunary σ-statistical equivalent set sequences. *Journal of Mathematical Analysis*. 2013; 4(3): 33-46.

[33] Kumar R, Sharma SK. Asymptotically lacunary statistical equivalence set sequences of order (α, β). *Advanced Studies: Euro-Tbilisi Mathematical Journal*. 2023; 16(3): 67-75.

[34] Gadjiev AD, Orhan C. Some approximation theorems via statistical convergence. *The Rocky Mountain Journal of Mathematics*. 2012; 32(1): 129-138.

[35] Çolak R. *Modern Methods in Analysis and Its Applications*. New Delhi, India: Anamaya Pub.; 2010.

[36] Çolak R, Bektaş CA. λ-statistical convergence of order α. *Acta Mathematica Scientia*. 2011; 31(3): 953-959.

[37] Et M, Çinar M, Karakaş M. On λ-statistical convergence of order α of sequences of function. *Journal of Inequalities and Applications*. 2013; 204: 8.

[38] Et M, Mohiuddine SA, Alotaibi A. On λ-statistical convergence and strongly λ-summable functions of order α. *Journal of Inequalities and Applications*. 2013; 469: 8.

[39] Subramanian N, Esi A. Wijsman rough convergence of triple sequences. *Matematychni Studii*. 2017; 48(2): 171-179.

[40] Subramanian N, Esi A. Wijsman rough λ-weak statistical convergence of order α of triple sequence of functions. *Asia Pasific Journal of Mathematics*. 2017; 4(1): 49-61.

[41] Subramanian N, Esi A. Wijsman rough lacunary statistical convergence on I Cesaro triple sequences. *International Journal of Analysis and Applications*. 2018; 16(5): 643-665.

[42] Ruckle WH. FK spaces in which the sequence of coordinate vectors is bounded. *Canadian Journal of Mathematics*. 1973; 25(5): 973-978.

[43] Musielak J. *Orlicz Spaces and Modular Spaces*. Springer; 2006.