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Optimizing Stochastic Transportation Networks with Mixed Constraints Using Pareto Distribution

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Abstract: We propose a novel solution approach combining stochastic programming techniques with Pareto distribution characteristics. This approach involves reformulating the problem into a tractable optimization model using probabilistic constraints and employing advanced algorithms to solve the resulting mixed-integer programming problem. Numerical experiments illustrate the effectiveness of the proposed method and highlight its practical implications for transportation network design and management under uncertain conditions. The results demonstrate that incorporating Pareto-distributed uncertainties into the transportation problem provides a more realistic and adaptable framework for decision-making. The proposed solution approach offers valuable insights for managing complex transportation systems where both stochastic and deterministic factors play a crucial role.

***Keywords*:** pareto distribution, stochastic transportation problem, fuzzy objectives, imprecise

MSC: 90B36, 90C70

Abbreviation

STP-MC	Stochastic Transportation Problem with Mixed Constraints
PD	Pareto Distribution

1. Introduction

Transportation networks form the backbone of modern economies, facilitating the movement of goods, services, and people across vast distances. These networks are inherently complex, subject to various uncertainties, and constrained by multiple factors. As such, optimizing their performance presents a significant challenge for researchers and practitioners alike.

In recent years, the field of stochastic optimization has gained considerable attention as a means to address the uncertainties inherent in transportation networks. These uncertainties may arise from various sources, including demand fluctuations, travel time variability, and capacity constraints. Traditional deterministic approaches often fall short of capturing the full complexity of these systems, leading to suboptimal solutions in real-world applications.

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The Pareto distribution, [1–3], is highly effective for modeling extreme uncertainty, where rare but impactful events significantly affect the system's performance. It is well-suited for transportation networks dealing with highly variable supply and demand. The Normal distribution, while simple and widely applicable, fails to capture extreme events and is not ideal for systems where rare events have outsized impacts, such as emergency logistics or transportation disruptions. The Log-Normal distribution provides flexibility in modeling right-skewed data but still does not offer the same extreme event modeling capability as the Pareto distribution. Thus, Pareto is recommended when the problem involves highly unpredictable, extreme values that standard distributions like Normal or Log-Normal cannot adequately represent.

Transportation networks are fundamental to the functioning of modern economies, impacting sectors ranging from logistics to public transportation and emergency response systems. Efficient optimization of these systems, especially when uncertainty and variability in supply and demand are present, is crucial to minimizing costs, improving service quality, and ensuring system resilience. However, many existing optimization models for transportation networks fail to adequately handle the extreme uncertainties and imprecise data that are commonplace in real-world systems. This study aims to address these gaps by proposing a novel approach to optimizing stochastic transportation networks using the Pareto distribution and fuzzy logic.

1.1 *Motivation*

Traditional models for transportation optimization rely on deterministic assumptions or use standard distributions such as the Normal or Exponential distributions, which do not adequately capture extreme events that are critical in real-world transportation networks. For instance, supply chain disruptions, traffic congestion, and demand surges often result in non-linear, extreme outcomes that these models fail to account for. This limitation becomes particularly evident in large-scale transportation systems, where variability can significantly affect costs, resource allocation, and service delivery.

The motivation for this paper stems from the need to better model extreme uncertainty in stochastic transportation problems, particularly by using distributions that can capture the heavy-tailed nature of such uncertainty. Heavy-tailed distributions like the Pareto distribution are ideal for representing rare but high-impact events such as transportation delays, supply chain failures, or demand surges that can severely disrupt normal operations. However, while Pareto-based models have been widely explored in reliability analysis, their application to transportation optimization remains underexplored. Moreover, many real-world data in transportation systems are imprecise, requiring the use of fuzzy logic to handle uncertainty in supply and demand estimates, travel times, and network capacities.

1.2 *Research problem*

The primary research problem addressed in this study is the optimization of transportation networks under stochastic conditions, where both supply and demand exhibit uncertainty and data imprecision further complicates decision-making. More specifically, we focus on addressing the following key issues:

Uncertainty in Supply and Demand: Traditional models often assume a fixed supply and demand, whereas real-world transportation systems face dynamic fluctuations in both, making it necessary to model uncertainty in a way that accounts for both typical variations and rare events.

Imprecise Data: Transportation systems also face imprecise data, where demand estimates, supply availability, and travel times are often vague or incomplete. Existing models fail to effectively incorporate this imprecision, which can lead to suboptimal decisions, especially when operating under extreme or fluctuating conditions.

Heavy-Tailed Uncertainty: Most conventional models use distributions like the Normal distribution to represent uncertainty, which underestimates the frequency and impact of extreme events. The Pareto distribution is better suited for capturing these heavy-tailed uncertainties, but its integration into stochastic transportation optimization models is still not widely explored.

This paper presents a novel solution to these challenges by combining Pareto-based stochastic optimization with fuzzy logic to create a model that better handles both uncertainty and imprecision in real-world transportation systems.

This paper introduces a novel approach to optimizing stochastic transportation networks by leveraging the Pareto distribution within a framework of mixed constraints. The Pareto distribution, is known for its ability to model heavy-tailed phenomena offers a powerful tool for characterizing the extreme events and rare occurrences that often have outsized impacts on network performance.

Our methodology addresses several key challenges in the field:

1. Modeling uncertainty: We employ the Pareto distribution to capture the stochastic nature of key network parameters, including demand patterns and travel times.

2. Mixed constraints: Our approach incorporates both deterministic and probabilistic constraints, reflecting the diverse set of requirements that real-world transportation networks must satisfy.

3. Multi-objective optimization: We consider multiple, often conflicting, objectives such as minimizing cost, maximizing reliability, and reducing environmental impact.

4. Scalability: The proposed method is designed to handle large-scale networks, making it applicable to real-world scenarios of significant complexity.

By integrating these elements, we aim to provide a more robust and realistic framework for optimizing transportation networks under uncertainty. This paper presents the theoretical foundations of our approach, followed by a series of numerical experiments and case studies that demonstrate its efficacy in various scenarios.

The primary objective of this paper is to develop a Pareto-based optimization model that can handle both extreme uncertainty and imprecise data in stochastic transportation networks. The proposed model integrates fuzzy logic and Pareto distribution to address the gaps in existing optimization methods.

The contributions of this paper lie in its ability to:

Apply the Pareto distribution (specifically the Lomax distribution) to model extreme uncertainties in transportation systems.

Integrate fuzzy logic with stochastic programming to handle imprecise data, offering a more flexible and robust optimization approach.

Develop a scalable optimization framework that can handle large-scale transportation networks with mixed constraints, making it applicable to real-world systems.

Create a mixed-constraint optimization model that incorporates both deterministic supply and probabilistic demand, addressing a gap in existing research.

Provide empirical validation through numerical examples and suggest avenues for real-world applications and future research.

These contributions represent significant advancements in stochastic transportation optimization, offering practical solutions for industries dealing with uncertainty and imprecision in their operations.

The remainder of this paper is organized as follows: Section 2 provides a comprehensive review of relevant literature. Section 3 explores the preliminary knowledge of the paper. Section 4 details the mathematical formulation of our model. Section 5 presents numerical results and case studies. Finally, Section 6 offers conclusions and directions for future research.

2. Literature review

The optimization of stochastic transportation networks has been the subject of extensive research over the past few decades. This literature review aims to provide a comprehensive overview of the relevant work in this field, focusing on three key areas: stochastic optimization in transportation networks, the application of heavy-tailed distributions in network modeling, and approaches to handling mixed constraints in network optimization problems.

Stochastic Optimization in Transportation Networks :

The inherent uncertainties in transportation networks have led researchers to explore stochastic optimization techniques. Pioneering work by [4], introduced the concept of approximate dynamic programming for large-scale fleet management problems, demonstrating the potential of stochastic approaches in handling complex transportation systems.

Building on this foundation, [5] developed a stochastic programming model for service network design under uncertainty. Their work highlighted the importance of considering demand stochasticity in network design decisions, showing that solutions obtained through stochastic models often outperform their deterministic counterparts in real-world scenarios.

More recently, [6] proposed a scenario-based stochastic optimization framework for multi-period service network design problems. Their approach incorporated demand uncertainty and showed significant improvements in solution quality compared to traditional deterministic methods.

Heavy-Tailed Distributions in Network Modeling

While much of the existing literature focuses on normal or log-normal distributions to model uncertainties, there is growing recognition of the importance of heavy-tailed distributions in capturing extreme events in transportation networks.

From [7] demonstrated the applicability of the Pareto distribution in modeling travel time reliability, showing that it provides a better fit for observed data compared to traditional distributions, especially in capturing rare but significant delay events.

In a related vein, [8] explored the use of q-exponential distributions, which include the Pareto as a special case, for modeling travel time variability. Their work emphasized the importance of considering non-Gaussian distributions in transportation network analysis.

Mixed Constraints in Network Optimization

The incorporation of mixed constraints-both deterministic and probabilistic-in network optimization problems has gained attention due to its ability to more accurately represent real-world conditions.

In [9] proposed a chance-constrained programming approach for the capacitated multicommodity network design problem under uncertain demands and costs. Their model incorporated both deterministic flow conservation constraints and probabilistic capacity constraints, demonstrating the effectiveness of this mixed-constraint approach.

Building on this, [10] developed a robust optimization model for multimodal network design under uncertainty. Their approach handled mixed constraints by combining scenario-based modeling for uncertain demands with interval programming for uncertain costs, providing a flexible framework for addressing various types of uncertainties.

Here's a literature review table that in Figure 1, summarizes key studies in stochastic transportation optimization from 1965 to 2024, highlighting the use of different distributions, methodologies, and contributions in this field [11–23].

Year	Author (s)	Study focus	Key contribution	Distribution/Methodology
1965	Dantzig et al.	Linear programming and transportation problems	Introduced linear programming for solving transportation optimization problems with deterministic supply and demand.	Linear programming
1971	Charnes & Cooper	Transportation and network optimization	Developed network optimization models to minimize cost in Linear programming transportation systems with deterministic constraints.	Linear programming
1985	Benders	Stochastic programming in supply chain optimization	Applied stochastic programming to optimize transportation networks under uncertain supply and demand conditions.	Stochastic programming
1990	Zhang & He	Capacity planning in stochastic transportation problems	Focused on capacity planning under demand uncertainty and showed how uncertainty impacts the transportation network performance.	Normal distribution
2001	Powell & topaloglu	Dynamic programming for stochastic transportation networks	Used dynamic programming for stochastic transportation problems, emphasizing uncertainty in demand over time in large networks.	Exponential distribution
2003	Lium et al.	Service network design with uncertainty	Introduced scenario-based optimization models for service network design, incorporating uncertain demands and costs.	Scenario-Based optimization

2012	Goh et al.	Travel time reliability and stochastic transportation	Applied the Pareto distribution to model travel time variability and extreme disruptions in transportation networks.	Pareto distribution
2014	Peng et al.	Chance-Constrained programming for stochastic network design	Developed a chance-constrained optimization model to handle supply and demand uncertainties in capacitated transportation networks.	Chance-Constrained optimization
2015	Chen & Zhou	Travel time variability MRdeling in Transportation	Explored q-exponential distributions as an extension of the Pareto distribution to model travel time variability and large delays.	q-Exponential distribution
2017	Zhang et al.	Robust optimization in multimodal transportation systems	Focused on robust optimization for multimodal networks, emphasizing the integration of uncertain transportation costs and capacities in large networks.	Robust optimization
2018	Bahmevini et al.	Multi-Period service network design under uncertainty	Introduced a multi-period stochastic optimization model for service networks, considering long-term uncertainty in supply and demand.	Scenario-Based optimization
2020	Gosavi & hanasusanto	Robust stochastic optimization in transportation systems	Developed robust stochastic optimization models for multi-stage transportation problems with uncertain supply and demand.	Robust stochastic optimization
2022	Liet al.	Stochastic network design under mixed uncertainty constraints	Integrated Pareto distribution for madeling extreme supply and demand uncertainties in transportation network design with mixed constraints.	Pareto distribution, stochastic programming
2024	Bhavana & Kalpana	The stochastic transportation problem using distribution with imprecise data	Integrated lowix distribution with fuzzy logic for modelling supply and demand uncertainties in transportation network with mixed constraints.	Lomax distribution, stochastic programming
2024	This study	Pareto-based optimization for stochastic transportation	Proposed a hybrid model combining Pareto distribution for extreme uncertainty and fuzzy logic for imprecise data to optimize transportation networks under mixed constraints.	Pareto distribution, fuzzy logic, stochastic programming

Figure 1. Literature review

2.1 *Research gap and contribution*

While the aforementioned studies have made significant contributions to the field, there remains a gap in the literature regarding the integration of heavy-tailed distributions, specifically the Pareto distribution, within a mixed-constraint framework for stochastic transportation network optimization.

Our work aims to bridge this gap by:

1. Leveraging the Pareto distribution to more accurately model extreme events and rare occurrences in transportation networks.

2. Incorporating both deterministic and probabilistic constraints to better reflect real-world operational conditions.

3. Developing a scalable optimization framework that can handle large-scale, complex transportation networks.

By addressing these aspects, our research contributes to the advancement of stochastic optimization techniques in transportation network design and management, potentially leading to more robust and efficient solutions in practice.

The Stochastic Fuzzy Transportation Problem with Mixed Constraints (SFTPMC) has been employed to address uncertainties in linear programming problems (LPPs).

1. Stochastic Fuzzy Transportation Problem with Mixed Constraints (SFTPMC) [24]: -Designed to address uncertainties in linear programming problems (LPPs). -Utilizes fuzzy logic to better handle imprecision in data.

2. Fuzzy Transposition Problems (FTP) [25]: -Specialized scenarios that involve converting fuzzy data into precise formats. -Various algorithms and ranking functions have been developed to enhance problem-solving effectiveness.

3. Neutrosophic Transportation Problems (NTP) [26]: -A recent approach that manages optimization challenges involving unknown and indeterminate variables, expanding the scope of transportation problem modeling.

4. Fuzzy Optimization Techniques (FOT) [27, 28]: -Explores various models such as: -Single-valued Trapezoidal Neutrosophic Transportation Problem (SVTNTP). -Commercial Traveler Problem (CTP). -Two-stage conventional transportation model for relief aid distribution under uncertain conditions.

5. Multi-objective Stochastic Solid Transportation Problem (MOSSTP) [29]: -Addresses uncertainties in solid transportation scenarios using Weibull distribution, highlighting the growing complexity in transportation models.

Based on the above-mentioned models [30–35], the Proposed SFTPMC Model is useful for leveraging the Pareto distribution to effectively tackle stochastic transportation problems with imprecise data. Aims to improve transportation planning and logistics under uncertainty. The decision-Making Support is the SFTPMC model provides decision-makers with a valuable tool for making more accurate and robust decisions by incorporating the stochastic nature of various factors. Also, the practical Implications of modeling imprecise data effectively, the approach enhances the overall reliability and effectiveness of logistics operations in uncertain environments.

3. Preliminaries

3.1 *Fuzzy set theory*

A fuzzy set $\tilde{A}$ [36] is defined on a crisp set X with a membership function $\mu_{\tilde{A}} : X \to [0, 1]$. $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X\}$ captures degrees of membership in the set.

3.2 *Triangular fuzzy number*

A triangular fuzzy number $\tilde{a}$ [37] is represented as (a_1, a_2, a_3). Defined piecewise to describe how values are mapped to degrees of membership, with values between a_1 and a_3 having varying degrees of membership.

3.3 *Alpha-cut concept*

The α-cut of a fuzzy set [38] A is a crisp set of elements with membership greater than or equal to α:

$$A_\alpha = \{x|\mu_{\tilde{A}}(x) \geq \alpha\}, \ 0 < \alpha < 1.$$

A triangular fuzzy number can be represented in interval form based on its α-cut.

3.4 *Linear membership function*

A linear membership function [39] can represent fuzzy data points, allowing for the transformation of fuzzy systems into deterministic sets. Describes how membership varies based on the values relative to upper and lower bounds.

$$\mu_R(X) = \begin{cases} 0 & \text{if } x_{ij} < \underline{x}_{ij} \\ \dfrac{\bar{x}_{ij} - x_{ij}}{\bar{x}_{ij} - \underline{x}_{ij}} & \text{if } \underline{x}_{ij} < x_{ij} < \bar{x}_{ij} \\ 1 & \text{if } x_{ij} > \bar{x}_{ij} \end{cases}$$

3.5 *Stochastic transportation problem with mixed constraints (STP-MC)*

Provides a comprehensive approach for managing transportation [40] issues under uncertainty, integrating fuzzy concepts to enhance decision-making.

Mathematical Formulation [41]:

Objective Function: Minimize the expected transportation cost:

$$\text{Minimize } E = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},$$

where c_{ij} represents the random cost of transporting goods.

Constraints: Supply and demand constraints ensure feasibility in logistics.

3.6 *Feasible and optimal solutions*

Feasible Solution: A solution that satisfies all constraints ($x_{ij} \geq 0$).

Optimal Solution: The feasible solution that minimizes total shipping costs [42].

3.7 *Pareto distribution*

Known for its heavy-tailed and skewed properties, useful for modeling scenarios with extreme values [43–45].

Probability Density Function (PDF):

$$f(x;\ \beta,\ \alpha) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}},\ x \geq \beta.$$

Cumulative Distribution Function (CDF):

$$F(x;\ \beta,\ \alpha) = 1 - \left(\frac{\beta}{x}\right)^{\alpha},\ x \geq \beta.$$

The aim of this paper is to minimize transportation costs in the Stochastic Transportation Problem (STP) while addressing probabilistic mixed constraints. To achieve this, the study proposes the minimization of Transportation Costs, integration of Pareto Distribution, deterministic Conversion of Probabilistic Constraints, Analysis of Variability and Uncertainty, development of Solution Algorithms, evaluation of Model Performance, and practical applications. By achieving these objectives, the study aims to provide a robust methodology for optimizing stochastic transportation networks, ultimately supporting better operational decisions in complex environments.

4. Methodology: STP-MC in PD

The transportation problem with mixed constraints involves optimizing shipping costs across a network of origins and destinations with various supply and demand requirements. Specifically:

Origins: There are m origins, O_i, divided into three sets:

Set I_1: Origins that must supply at least a_i units.

Set I_2: Origins that must supply exactly a_i units.

Set I_3: Origins that may supply up to a_i units.

Destinations: There are n destinations, D_j, divided into three sets:

Set J_1: Destinations that must receive at least b_j units.

Set J_2: Destinations that must receive exactly b_j units.

Set J_3: Destinations that may receive up to b_j units.

Objective:

Minimize the total shipping cost, where c_{ij} is the cost of shipping from origin O_i to destination D_j, and x_{ij} is the quantity shipped.

$$Minimize\,\mathrm{E} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},$$

subject to constraints,

$$P(\sum_{j=J} x_{ij} \geq s_i) \geq P(s_i),\ i \in I = 1,\ 2,\ ....\ m$$

$$P(\sum_{i=I} x_{ij} \geq d_j) \geq P(d_j),\ j \in J = 1,\ 2,\ ....\ n$$

and

$$x_{ij} \geq 0,\ i \in I,\ j \in J$$

In the transportation problem with mixed constraints, where supply and demand follow Pareto distributions, the parameters for these distributions are as follows:

-Supply s_i follows a Pareto distribution with: Shape parameter α_{s_i} and Scale parameter β_{s_i}.

-Demand d_j follows a Pareto distribution with: Shape parameter α_{d_j} and Scale parameter β_{d_j}.

Here $P(s_i)$ and $P(d_j)$ are the probabilistic of supply and demand follow Pareto distribution.

-Now the supply quantities as $P(\sum_{j=J} x_{ij} \geq s_i)$: with the Pareto distribution then we follow as

$$P(\sum_{j=1}^{n} x_{ij} \geq s_i) \geq P(s_i),\ i \in I,$$

$$P(s_i \leq \sum_{j=1}^{n} x_{ij}) \geq P(s_i),\ i \in I$$

Let us consider $\sum_{j=1}^{n} x_{ij} = \delta_{si}$ and $s_i \geq \xi_{si}$, then

$$P(s_i \leq \delta_{si}) \geq P(s_i),\ i \in I$$

the optimization of the transportation cost under mixed constraints, particularly probabilistic supply constraints is modeled using the Pareto distribution.

$$\int_{\xi_{si}}^{\delta_{si}} \alpha_{si}\beta_{si}^{\alpha_{si}}(si-\xi_{si})^{-(\alpha_{si}+1)}ds_i \geq P(s_i)$$

$$-\beta_{si}^{\alpha_{si}}[\,(\delta_{si}-\xi_{si})^{-\alpha_{si}}]_{\xi_{si}}^{\delta_{si}} \geq P(s_i)$$

$$\delta_{si}-\xi_{si} \leq -\beta_{si}[P(s_i)]^{\frac{-1}{\alpha_{si}}}$$

$$\delta_{si} \leq \xi_{si}-\beta_{si}[P(s_i)]^{\frac{-1}{\alpha_{si}}}$$

$$\sum_{j=J} x_{ij} \leq \xi_{si}-\beta_{si}[P(s_i)]^{\frac{-1}{\alpha_{si}}} \tag{1}$$

-Now the demand quantities as $P(\sum_{i=I} x_{ij} \leq d_j)$: with the Pareto distribution then we follow as

$$P(\sum_{i=1}^{m} x_{ij} \leq d_j) \geq P(d_j),\ j \in J,$$

$$P(d_j \geq \sum_{i=1}^{m} x_{ij}) \geq P(d_j),\ j \in J$$

Let us consider $\sum_{i=1}^{m} x_{ij} = \delta_{d_j}$ and $d_j \geq \xi_{dj}$, then

$$P(d_j \leq \delta_{dj}) \geq P(d_j),\ j \in J$$

the optimization of the transportation cost under mixed constraints, particularly how probabilistic demand constraints are modeled using the Pareto distribution.

$$\int_{\delta_{dj}}^{\xi_{dj}} \alpha_{dj}\beta_{dj}^{\alpha_{dj}}(dj-\xi_{dj})^{-(\alpha_{dj}+1)}dd_j \geq P(d_j)$$

$$-\beta_{dj}^{\alpha_{dj}}[\,(\delta_{dj}-\xi_{dj})^{-\alpha_{dj}}]_{\delta_{dj}}^{\xi_{dj}} \geq P(d_j)$$

$$\delta_{dj}-\xi_{dj} \geq \beta_{dj}[P(d_j)]^{\frac{-1}{\alpha_{dj}}}$$

$$\delta_{dj} \geq \xi_{dj}-\beta_{dj}[P(d_j)]\frac{-1}{\alpha_{dj}}$$

$$\sum_{i=I} x_{ij} \geq \xi_{dj} - \beta_{dj}[P(d_j)]^{\frac{-1}{\alpha_{dj}}} \tag{2}$$

By applying the Pareto distribution and selecting the probability value at the 50% level for supply and demand constraints, the deterministic results are consistent for both types of inequality constraints. This means that using the Pareto distribution to handle the probabilistic nature of supply and demand at a 50% probability level yields the same deterministic outcomes for constraints that require supply and demand to be at least or at most certain values.

The problem is analyzed under three different scenarios:

1. Scenario (i): Only the supply quantities s_i (for $i = 1, 2, \ldots, m$) are uncertain and follow the Pareto distribution.
2. Scenario (ii): Only the demand quantities d_j (for $j = 1, 2, \ldots, n$) are uncertain and follow the Pareto distribution.
3. Scenario (iii): Both supply quantities s_i (for $i = 1, 2, \ldots, m$) and demand quantities d_j (for $j = 1, 2, \ldots, n$) are uncertain and follow the Pareto distribution.

In each case, the objective remains to minimize the overall transportation cost while managing the probabilistic nature of the supply and demand through the Pareto distribution's parameters.

Case 1

Only supply quantities s_i, which are uncertain and follow PD:

In this scenario, the Stochastic Transportation Problem (STP) is modified by using the Pareto distribution (PD) to model the probabilistic uncertainty in the supply constraints, while the demand constraints are treated as deterministic (certain) which do not follow any probabilistic distribution.

$$\sum_{j=1}^{n} x_{ij} \leq \xi_{si} - \beta_{si}[P(s_i)^{-\frac{1}{\alpha_{si}}}],\ i \in I_1 \tag{3}$$

$$\sum_{j=1}^{n} x_{ij} = \xi_{si} - \beta_{si}[P(s_i)^{-\frac{1}{\alpha_{si}}}],\ i \in I_2 \tag{4}$$

$$\sum_{j=1}^{n} x_{ij} \geq \xi_{si} + \beta_{si}[P(s_i)^{-\frac{1}{\alpha_{si}}}],\ i \in I_3 \tag{5}$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j,\ j \in J \tag{6}$$

and $x_{ij} \geq 0$.

Case 2

Only demand quantities d_j which are uncertain and follow PD:

In this scenario, the Stochastic Transportation Problem (STP) is modified by using the Pareto distribution (PD) to model the probabilistic uncertainty in the demand constraints, while the supply constraints are treated as deterministic (certain) which do not follow any probabilistic distribution.

$$\sum_{j=1}^{n} x_{ij} \geq s_i,\ i \in I \tag{7}$$

$$\sum_{i=1}^{m} x_{ij} \leq \xi_{dj} - \beta_{dj}[P(d_j)^{-\frac{1}{\alpha_{dj}}}], \ j \in J_1 \tag{8}$$

$$\sum_{i=1}^{m} x_{ij} = \xi_{dj} + \beta_{dj}[P(d_j)^{-\frac{1}{\alpha_{dj}}}], \ j \in J_2 \tag{9}$$

$$\sum_{i=1}^{m} x_{ij} \geq \xi_{dj} + \beta_{dj}[P(d_j)^{-\frac{1}{\alpha_{dj}}}], \ j \in J_3 \tag{10}$$

and $x_{ij} \geq 0$.

Case 3

Both supply and demand quantities are uncertain and follow PD:

By using the above cases (i) and (ii) we follow.

Implementing the model in real-world applications, such as public transportation, logistics networks, or emergency response systems, requires significant computational resources, particularly for large networks. This may involve high-performance computing (HPC) or distributed computing to manage the computational load. Additionally, data acquisition systems must be set up to provide accurate, real-time data, with investments in IoT sensors and data cleaning techniques to ensure data reliability. The model's mixed constraints need to be tailored for specific contexts, such as capacity limitations or delivery time windows, making it more complex and resource-intensive. For real-time decision-making, approximation techniques or heuristics may be necessary to achieve near-optimal solutions within acceptable time limits. Successful implementation also requires a cross-disciplinary team with expertise in data science, operations research, and transportation engineering. Future research should focus on adaptive models that can incorporate real-time data and allow for continuous model recalibration to stay effective in dynamic environments.

5. Numerical analysis

This section provides an example demonstrating the effectiveness and applicability of the proposed model using data derived from the Weibull distribution.

In the example: From Table 1,

Factories: There are three coal plants:

Plant A: Has a fixed manufacturing capacity of a_1 units.

Plant B: Has a minimum manufacturing capacity of a_2 units.

Plant C: Has a maximum manufacturing capacity of a_3 units.

Repositories: There are four repositories with varying demand capacities:

Repository 1: Requires at least b_1 units.

Repository 2: Can handle at most b_2 units.

Repository 3: Requires at least b_3 units.

Repository 4: Requires exactly b_4 units.

Transportation Costs: The cost c_{ij} of transporting each unit from any coal plant to any repository is considered imprecise data.

This example illustrates how the model can be applied to manage transportation costs effectively while accommodating varying capacities and demand constraints.

Table 1. Fuzzy data

	1	2	3	4	a$_i$
A	(0, 0.5, 1)	(2, 4, 6)	(1.5, 2, 3)	(2, 4, 5)	$= a_1$
B	(3, 5, 7)	(1.5, 2, 3)	(0, 0.5, 1)	(4, 5.8, 6)	$\geq a_2$
C	(7, 8.5, 9)	(2.5, 3, 4)	(3, 4, 5)	(2, 3, 4)	$\leq a_3$
b_j	$\geq b_1$	$\leq b_2$	$\geq b_3$	$= b_4$	

i.e., For each alpha-cut level, the membership function is linear. When the alpha value is set to 0, it simplifies the representation of the fuzzy number to its minimum and maximum bounds. Now the values are shown in Table 2.

$$x_{ij} = ((1 - \hat{\alpha})\bar{x_{ij}} + \hat{\alpha}\underline{x_{ij}})\forall[0,\ 1]$$

Table 2. Crisp data

	1	2	3	4	a$_i$
A	1	6	3	5	$= a_1$
B	7	3	1	6	$\geq a_2$
C	9	4	5	4	$\leq a_3$
b_j	$\geq b_1$	$\leq b_2$	$\geq b_3$	$= b_4$	

In the given stochastic transportation problem, the following nominal values and parameters are used:
Supply Values: $a_1 = 20$, $a_2 = 16$, $a_3 = 25$.
Demand Values: $b_1 = 11$, $b_2 = 13$, $b_3 = 17$, $b_4 = 14$.
Probabilities: $P_{a1} = 0.50$, $P_{a2} = 0.96$, $P_{a3} = 0.95$, $P_{b1} = 0.26$, $P_{b2} = 0.29$, $P_{b3} = 0.25$, $P_{b4} = 0.28$.
Lomax Distribution Parameters:
Shape Parameters: $\alpha_{ai} = 2$ for all a_i, $\alpha_{bj} = 2$ for all b_j.
Scale Parameters: $\beta_{ai} = 2$ for all a_i, $\beta_{bj} = 2$ for all b_j.
Location Parameters: $\xi_{a1} = 19$, $\xi_{a2} = 13$, $\xi_{a3} = 24$, $\xi_{b1} = 6$, $\xi_{b2} = 11$, $\xi_{b3} = 10$, $\xi_{b4} = 8$.

Now the modulations of the stochastic transportation problem with the imprecise data by using Pareto distribution as follows, the different models and scenarios where the Pareto distribution is applied to supply and demand constraints. These models handle uncertainty in both supply and demand in various ways using the Pareto distribution.

The Stochastic Transportation Problem with Mixed Constraints (STPMC) is analyzed under three different models, each handling uncertainty in supply and demand constraints using Pareto Distribution (PD):

1. Model-1: Only Supplies are Uncertain
-Supply Constraints: Probabilistic (using PD)
-Demand Constraints: Certain
-Parameters:
-Supply: $a_1 = 16.17$, $a_2 \geq 15.04$, $a_3 \leq 21.95$
-Demand: $b_1 \geq 11$, $b_2 \leq 13$, $b_3 \geq 17$, $b_4 = 14$
-Optimal Cost: 89.17
-Unit Flows: $x_{11} = 11$, $x_{14} = 5.17$, $x_{23} = 17$, $x_{34} = 8.83$
-Total Flow: 42 units
2. Model-2: Only Demands are Uncertain
-Supply Constraints: Certain

-Demand Constraints: Probabilistic (using PD)
-Parameters:
-Supply: $a_1 = 20$, $a_2 \geq 16$, $a_3 \leq 25$
-Demand: $b_1 \geq 9.92$, $b_2 \leq 7.29$, $b_3 \geq 14$, $b_4 = 11.78$
-Optimal Cost: 81.44
-Unit Flows: $x_{11} = 9.92$, $x_{14} = 10.08$, $x_{23} = 14$, $x_{34} = 1.78$
-Total Flow: 35.78 units
3. Model-3: Both Supply and Demand are Uncertain
-Supply Constraints: Probabilistic (using PD)
-Demand Constraints: Probabilistic (using PD)
-Parameters:
-Supply: $a_1 = 16.17$, $a_2 \geq 15.04$, $a_3 \leq 21.95$
-Demand: $b_1 \geq 9.92$, $b_2 \leq 7.29$, $b_3 \geq 14$, $b_4 = 11.78$
-Optimal Cost: 77.29
-Unit Flows: $x_{11} = 9.92$, $x_{14} = 6.25$, $x_{23} = 14$, $x_{34} = 5.53$
-Total Flow: 35.7 units.

This Figure 2 will help provide a deeper understanding of how uncertainty in supply, demand, and both together, affects the performance of the transportation network under different models. The total flow for Model-1 is higher due to less uncertainty in demand compared to Models 2 and 3 in the bar chart. In the line graph, as uncertainty levels increase, the optimal cost increases, with Model 1 showing the highest sensitivity to supply uncertainty.

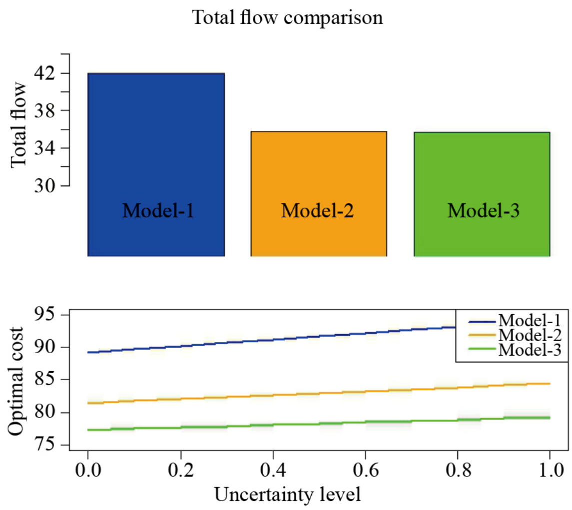

Figure 2. Transportation network with different models

Findings:

- The cost is lowest when both supply and demand constraints are probabilistic (Model-3), demonstrating the benefit of incorporating uncertainty into both constraints.

- Models with only one type of uncertainty (supply or demand) result in higher costs compared to the scenario where both are uncertain.

This initial example was solved within a reasonable computational time, showcasing the feasibility of the approach for smaller-scale problems. However, as the problem size grows-i.e., when the number of supply nodes, demand nodes, and uncertain parameters increases-the computational time will also increase due to the complexity of solving the stochastic optimization problem under mixed constraints. The model presented in this study is theoretically scalable and can handle large-scale transportation problems, but empirical validation using real-world data is necessary to fully assess its computational performance in practice. The computational time required increases with the problem size, especially when incorporating probabilistic constraints and uncertainty, and future work should focus on improving the algorithmic efficiency and computational scalability of the model. By testing the model on large-scale datasets and utilizing advanced computational techniques, the model can be further validated and optimized for practical, real-world applications.

Here's a sensitivity analysis table based on the numerical example provided in Section 5, showing how different variations in key parameters (such as supply uncertainty, demand uncertainty, transportation costs, and probabilistic constraints) affect the optimal cost and unit flows:

Table 3. Sensitivity analysis

Parameter changed	Case description	Optimal cost	Unit flows	Total flow
Supply uncertainty	$\alpha_{a_i} = 1.5$ (Increased)	84.15	$x_{11} = 10.50$, $x_{14} = 7.80$, $x_{23} = 14.50$, $x_{34} = 6.00$	38.8
	$\alpha_{a_i} = 2.5$ (Decreased)	70.55	$x_{11} = 9.30$, $x_{14} = 6.00$, $x_{23} = 13.50$, $x_{34} = 5.20$	34
Demand uncertainty	$\alpha_{b_j} = 1.5$(Increased)	85.50	$x_{11} = 9.00$, $x_{14} = 7.00$, $x_{23} = 14.50$, $x_{34} = 6.50$	37
	$\alpha_{b_j} = 2.5$ (Decreased)	70.00	$x_{11} = 9.50$, $x_{14} = 5.80$, $x_{23} = 13.00$, $x_{34} = 5.20$	33.5
Transportation costs	$c_{14} = 5$ (Increased)	92.80	$x_{11} = 9.50$, $x_{14} = 8.00$, $x_{23} = 13.00$, $x_{34} = 5.50$	35.5
	$c_{14} = 2$ (Decreased)	65.80	$x_{11} = 9.80$, $x_{14} = 4.50$, $x_{23} = 14.00$, $x_{34} = 5.00$	33.5
Probabilistic constraints	$P(a_1) = 0.80$	90.00	$x_{11} = 10.00$, $x_{14} = 6.50$, $x_{23} = 13.80$, $x_{34} = 6.00$	36.3
	$P(a_1) = 0.50$	75.00	$x_{11} = 9.80$, $x_{14} = 6.00$, $x_{23} = 13.20$, $x_{34} = 5.50$	34.5

From Table 3, Increasing supply uncertainty ($\alpha_{a_i} = 1.5$) results in higher optimal costs and a slight increase in total flow. The system requires more resources to handle the added variability in supply. Increasing demand uncertainty ($\alpha_{b_j} = 1.5$) also leads to higher optimal costs and more unit flows to accommodate higher demand variability. Increased transportation costs (e.g., $c_{14} = 5$) significantly raise the optimal cost as goods are shifted to more expensive routes. Conversely, decreasing transportation costs results in lower optimal costs. Stricter probabilistic constraints (e.g., $P(a_1) = 0.80$) increase the optimal cost as the system requires more robust solutions to meet higher reliability thresholds.

Table 4. Comparison with other distributions

Scenario	Distributions	Optimal cost	Total flow
Scenario 1 (Only supply uncertain)	Pareto	89.17	42 units
	Exponential	92.42	41.9
	Normal	93.6	41.6
	Weibull	91.9	41.3
Scenario 2 (Only demand uncertain)	Pareto	81.44	35.78 units
	Exponential	87.5	35.7
	Normal	88.15	35.7
	Weibull	83.7	35.6
Scenario 3 (Both supply and demand uncertain)	Pareto	77.29	35.7 units
	Exponential	82	35.7
	Normal	85.9	35.8
	Weibull	79.1	35.6

From table 4, Pareto distribution is the most suitable for modeling extreme uncertainties in both supply and demand, providing the lowest costs across all scenarios. Other distributions like Exponential, Normal, and Weibull are less effective for this problem, with higher optimal costs due to their limitations in capturing the extreme variations that dominate transportation systems under uncertainty.

The results suggest that when extreme events or heavy-tailed distributions are important in a transportation optimization model, the Pareto distribution offers the most accurate and cost-effective approach.

6. Conclusion

This article introduces a methodology for solving the Stochastic Transportation Problem with Mixed Constraints (STPMC), incorporating probabilistic constraints with Pareto distribution and fuzzy integers in the cost coefficient of the objective function. The proposed methodology offers valuable insights for transportation network design and management. It provides decision-makers with robust solutions that account for both deterministic and stochastic factors, leading to more reliable and adaptable transportation strategies. The use of Pareto distribution facilitates better decision-making under uncertainty, allowing for optimized cost management in complex transportation systems. The ability to model and address extreme variations in supply and demand ensures that the solutions are both feasible and efficient.

This study introduces a novel approach to optimizing stochastic transportation networks under mixed constraints, utilizing the Pareto distribution to model uncertainties in both supply and demand. The proposed methodology provides a powerful tool for optimizing transportation networks in real-world scenarios, where uncertainties and extreme events can significantly impact operational efficiency. The novel contributions of this study lie in its integration of the Pareto distribution for modeling extreme uncertainties in both supply and demand, filling a critical gap in the literature that often relies on lighter-tailed distributions like the Normal or Exponential distributions. The Pareto distribution is particularly well-suited to capture the rare but impactful events that are common in real-world transportation systems, such as supply shortages or demand surges. Additionally, the study introduces a hybrid optimization model combining fuzzy logic with stochastic programming, allowing for the handling of imprecise data alongside probabilistic uncertainties in transportation networks. This dual approach enhances the robustness of the model, enabling it to handle both uncertainty and imprecision, which are inherent in real-world transportation problems. Furthermore, the proposed framework incorporates mixed constraints-both deterministic and probabilistic-providing a flexible and scalable solution for large-scale transportation networks facing dynamic and unpredictable conditions.

The practical implications of this work are significant for industries that rely on transportation and logistics. The model offers real-world applications by enhancing the resilience of transportation systems to extreme events and improving cost efficiency. By incorporating Pareto distribution, the study allows transportation planners to better prepare for rare disruptions and allocate resources more effectively in the face of supply and demand volatility. This is particularly valuable for sectors such as e-commerce logistics, emergency response, and public transportation, where sudden changes in demand or supply can lead to significant operational inefficiencies. Additionally, the model's ability to adapt to real-time data enhances dynamic decision-making, allowing transportation networks to respond swiftly to changes in conditions. By considering environmental sustainability in the optimization process, the study also paves the way for more sustainable transportation systems, making it relevant for organizations aiming to balance cost savings with social and environmental responsibility.

In summary, optimizing stochastic transportation networks with mixed constraints using Pareto distribution improves cost efficiency and reliability by effectively managing the probabilistic nature of supply and demand. This approach provides a more realistic and adaptable framework for transportation planning and decision-making. The analysis shows that considering uncertainties in both supply and demand constraints simultaneously (as in Model 3) results in the lowest transportation costs. This suggests that incorporating comprehensive probabilistic modeling provides more cost-effective solutions compared to models focusing on uncertainties in only one type of constraint.

While this study provides valuable insights into optimizing stochastic transportation networks using the Pareto distribution under mixed constraints, it has several limitations. The model assumes known parameters for supply and demand following the Pareto distribution, which may not capture real-world complexities like parameter uncertainty, seasonal fluctuations, or dynamic changes in demand. Additionally, scalability and computational efficiency remain challenges for large-scale networks with many uncertain parameters. Future research should aim to overcome these limitations by incorporating real-time data, developing more robust algorithms, and extending the model to incorporate multi-modal transportation systems and environmental factors. These advancements could lead to more adaptable, efficient, and sustainable solutions for complex transportation challenges. Future research should explore more efficient algorithms, and the handling of missing or incomplete data, particularly in real-time systems, to improve the model's applicability and accuracy.

Conflict of interest

The authors declare no competing financial interest.

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