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Research Article

# Adjusted Optimal Trimmed Theil-Sen Method for Multiple Regression Model, with Outliers Detection and Management

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**Abstract:** The objective of this research was to propose the adjusted optimal trimmed Theil-Sen (AOTS) method and compare the efficiency, with outliers, of four parametric and two nonparametric statistical point estimation methods for multiple regression analysis. The parametric methods consisted of the following: Ordinary Least Squares (OLS), Ordinary Least Trimmed Squares (OLTS), Parametric Bootstrap (PB), and Jackknife (JK) methods. The nonparametric methods consisted of Optimal Theil-Sen (OTS) and proposed AOTS methods. Data were simulated in a randomized manner in three instances of simulation: one, simulation of independent variables and errors without outliers; two, simulation of independent variables with outliers; and three, simulation of errors with outliers. Outliers were detected by an Interquartile Range (IQR) method. Both ends of the data were truncated to deal with outliers. $Y$-intercept and regression coefficient were estimated with six estimation methods. The measure for comparing the performances of these methods was a mean square error. For the parametric methods, when the independent variables had outliers with normal distribution, the PB method provided the least mean square error. It would be a good substitute for the OLS method. When the errors had outliers, for all normal, uniform, and gamma distributions, the performance of the OLTS method was better than the OLS method. For the nonparametric methods, when the independent variables had outliers with normal or uniform distributions, the proposed AOTS method performed better than the OTS method. In the same way, when the independent variables had outliers with gamma distribution, the proposed AOTS methods performed competitively to the OTS method. However, when the errors had outliers with normal, uniform, or gamma distribution, the OTS method edged over the proposed AOTS method.

***Keywords***: ordinary least squares, ordinary least trimmed squares, parametric bootstrap, jackknife, optimal Theil-Sen, adjusted optimal trimmed Theil-Sen

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## 1. Introduction

Multiple regression analysis is a statistical method for finding the relationship between independent variables and a dependent variable. The independent variables or variables $X$ can be just quantitative variables or both quantitative and


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qualitative variables. The dependent variable or variable $Y$ is quantitative. The analytical model is $Y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_n X_n + \varepsilon_i$. Parameter estimation can be separated into point estimation and interval estimation. Point estimation means using statistics obtained from a sample to estimate the population parameter. Statistics obtained from a sample can be equal to or deviate from the population parameter. Interval estimation is a range of estimated values of samples. This research focused on point estimation for multiple regression models [1–3]. Outliers are a group of data that deviates greatly from the majority [4, 5]. Statisticians need to test for outliers.

Outliers can be omitted from the data. However, if outliers are omitted, there may be significant changes in the research conclusion [6]. Variable values with Z-scores beyond this range may be further investigated to verify that they do not represent data entry errors or other issues. One should not automatically omit outliers from analysis because the sample size will be smaller. The Z-score method for identifying outliers states that a data value is an outlier if its Z-score is less than $-3$ or greater than 3. However, this method is limited: the mean and standard deviation used to calculate the Z-score are sensitive to outliers. This means that the presence or absence of an outlier can significantly affect the mean and standard deviation, making the Z-score method potentially unreliable for identifying outliers. Therefore, data analysts have developed more robust statistical methods for outlier detection, which are less sensitive to the presence of outliers [7].

One elementary robust method is to use the interquartile range (IQR). If the data were higher than $Q_3 + 1.5IQR$ or lower than $Q_1 - 1.5IQR$, the data were outliers [7].

There are several methods to deal with outliers. In the first method, we may remove outliers from the data to ensure that the number of outliers is not greater than 1-2%. The second method uses dividing continuous variables into groups, substituting outlier values into the first or second groups. The last method uses Winsorizing variables to find the Winsorized mean by replacing the values of the variables greater than the 99th percentile with the 99th percentile and replacing the values of the variables that are less than the 1st percentile with the 1st percentile [7].

Multiple regression analysis is one of the most important analytical techniques in statistical prediction studies. Multiple regression analysis aims to determine the relationship between the dependent variable and the independent variables. Nonparametric regression analysis methods are used in cases when the assumptions for the parametric regression method are not satisfied. The nonparametric method performs calculations with the median instead of the arithmetic mean. The parameter median does not reflect the effect of outliers on calculations. A weighted median incorporates the contribution of the outliers of the model when weighting is given to each observation [8]. In the presence of outliers, intercept estimation $\beta_0$, and regression coefficient $\beta_1$ (in simple linear regression analysis) can be determined in parametric and nonparametric statistical methods. The outlier detection method in our study was IQR, and the management method was data truncation or trimming. The determination of intercept estimation and regression coefficient required distinct analytical approaches (see Section 2.5). The ordinary least squares (OLS) method is a parametric statistical method. A study [5] compared OLS to three nonparametric statistical methods: Theil-Sen, Brown-Mood, and quantile. The measure was the mean square error (MSE) of $\beta_0$ and $\beta_1$. Three cases were considered: no outliers at all sample sizes; the independent variables and errors had outliers at the same position; and the independent variables and errors had outliers at different positions. OLS was the most effective in the estimation of $\beta_0$ and $\beta_1$ when there were no outliers. In the case that independent variables and errors had outliers at the same position, the Theil-Sen method was the best in the estimation of $\beta_0$ and $\beta_1$. Finally, in the case that independent variables and errors had outliers at different positions, the quantile method was the most effective in the estimation of $\beta_0$ and $\beta_1$ [5]. That study compared the following point parameter estimation methods for simple linear regression analysis in the presence of outliers: Bayesian method, OLS, and parametric bootstrap. Mean square error was the measure for evaluating the effectiveness of the estimation methods. Bayesian method was the most effective for all tested scenarios, followed by OLS and parametric bootstrap [9]. The Jackknife method was another estimation method commonly used in simple linear regression analysis (but, in our study, we used it for multiple linear regression analysis).

Sahinler and Topuz compared bootstrap sampling, jackknife, and OLS based on bias, standard error, and confidence intervals of regression coefficients. The Jackknife method had a higher bias, standard error, and a larger confidence interval of the regression coefficient than those of bootstrap and OLS methods. The jackknife percentile range was also wider than the bootstrap percentile range [10]. In another study, Shah et al. compared, in a simple linear regression analysis, parametric OLS and nonparametric Theil-Sen methods when the data included outliers. They considered the standard deviation, standard error, the square root of the mean square error, and the mean absolute error. When the data had no outliers, OLS performed better, but when the data contained outliers, the Theil-Sen method performed better [11].

For regression estimation in the presence of outliers, the Mean Square Errors (MSE) and bias were of interest. The performances of different estimation methods were investigated using a simulation study for outliers in the dependent variable. The estimates of the regression coefficients using nine methods were compared with OLS. The result showed that Tukey's M-estimator provided a lower total absolute bias and total mean square error than the others, for all sample sizes and for when the contamination is in the dependent variable. In addition, Ordinary Least Median Squares (OLMS) provided lower MSE and bias than Ordinary Least Trimmed Squares (OLTS) [12]. When there are outliers, two-point parameter estimation methods of multiple regression analysis-OLTS and OLMS-were compared. The research showed that OLTS was more efficient than OLMS [13].

For the outliers management of data using truncation on regression results, a study by Rousseeuw and Leroy concluded with a recommendation to use $h = n(1-\alpha)+1$ as $h$ is the remaining data amount after the data was truncated, where $\alpha$ is the percentage truncation [14]. In another study, Rousseeuw and Driessen proposed to replace OLMS with OLTS for a large sample size and reported that the best robustness condition for $h$ was $n/2$ [15]; in their cases, the truncated point was 50% [14]. Rousseeuw reported that a good truncation point for OLTS and OLMS was $h = (n+p+1)/2$ as $h$ is the remaining data amount after the data was truncated, $p$ is the number of independent variables and $n$ is the sample size [13]. Zaman et al. stated that OLTS is a good method for finding outliers. At times, this method can eliminate overestimated observations that may not provide a true regression relationship with the data [16]. Erilli concluded that the optimal Theil-Sen method did not depend on the assumption of symmetric distribution of $d_i = y_i - \hat{\beta}_1 x_i$, and it was suitable for dealing with outliers [8]. In addition, the use of a weighted median for simple linear regression analysis using the Theil-Sen method was investigated. In particular, OLS, Theil-Sen, Hodges-Lehman, optimal, weighted median, weighted median Hodges-Lehmann, and weighted median optimal methods were compared. OLS was the best performer when the data strictly followed the common assumptions of simple linear regression analysis. On the other hand, for the Theil-Sen method, the data did not have to follow as many assumptions as those of OLS, hence Theil-Sen could be applied even to very small data sets. For a simple linear regression analysis, Theil-Sen is a good alternative to OLS. In that study, weighted median instead of median was used in the Theil-Sen method. The median did not reflect the effect of outliers on the mean calculation. The weighted median added some effect to each observation because it gave a certain weight. Although the weighted median -was not as easy to calculate as the median, when there were a lot of outliers, the weighted median was quite successful. Moreover, a weighted median can also be used in nonparametric statistics [8].

Farooqi compared simple linear regression analysis with ordinary least quantile squares, Theil-Sen, and Theil-Sen Siegel methods when there were outliers. The measures were mean of bias, median of bias, standard deviation, standard error, root of mean square error, relative root mean square error, median absolute error, and relative median absolute error. The author concluded that under the assumption that there were no outliers, ordinary least quantile square method was optimal. However, when there were outliers for both $X$ and $Y$, the Theil-Sen Siegel method was optimal. Again, under the assumption that there were outliers, ordinary least quantile squares, Theil-Sen, and Theil-Sen Siegel methods provided similar performances [17]. Zaman and Alakus compared the estimation methods for parametric and nonparametric statistics for simple linear regression analysis when the data included outliers: OLS, Mood-Brown, Theil-Sen, optimal Theil-Sen, Theil-Sen-Hodges-Lehmann, Theil-Sen weighted mean-1, Theil-Sen weighted median-1, Theil-Sen weighted mean-2, and Theil-Sen weighted median-2 methods [18]. Their measure was mean absolute deviation. The optimal Theil-Sen method achieved the lowest mean absolute deviation, followed by Theil-Sen and Theil-Sen weighted median-1 methods. The optimal Theil-Sen was defined in the form of $d_i = y_i - \hat{\beta}_1 x_i$, where the calculation of $\hat{\beta}_1$ was the same as that of Theil-Sen. $\hat{\beta}_0$ was calculated as the median of all $d_i$, $\hat{\beta}_0 = median(d_i)$.

The research gap that the author would like to bridge was the lack of comparison between various estimation methods in multiple linear regression analysis, with outliers detection and management. The above-mentioned papers show that many researchers have investigated point parameter estimation for simple linear regression analysis; when the data has or does not have outliers. Only two studies investigated point parameter estimation of multiple regression analysis; when the data had outliers [12, 13]. It was a narrow comparison of ordinary least trimmed squares and ordinary least squares method, using only the median. The studies did not investigate many other methods, such as parametric bootstrap and jackknife method. Furthermore, the past studies did not compare between parametric and nonparametric statistics. Therefore, the authors of this study were motivated to compare the efficiency of the proposed adjusted nonparametric and parametric statistics point estimation methods in multiple regression models in the presence of outliers. The nonparametric estimation methods investigated were the following: optimal Theil-Sen and proposed adjusted optimal trimmed Theil-Sen. The parametric point estimation methods investigated were the following: OLS, trimmed OLS, bootstrap, and jackknife methods. OLS was reported in [5, 8, 10, 11, 17]. The ordinary least trimmed square method was reported in [12–16]. The parametric bootstrap method was reported in [9, 10]. The Jackknife method was originated in [19]. Optimal Theil-Sen method was based on [8, 18]. We proposed an adjusted optimal trimmed Theil-Sen method by weighting the optimal Theil-Sen statistic of the intercept $\beta_0$. We compared all of these methods based on mean square error and reported out findings. In this research, we just wanted to consider that when the data had outliers, which parametric and nonparametric statistics provided a lower mean square error for estimating the intercept of $\beta_0$ and regression coefficients of $\beta_1,\ \beta_2,\ \beta_3$. We focus on the outlier detection and management methods for the ordinary least trimmed squares method in parametric statistics and the adjusted optimal trimmed Theil-Sen method in nonparametric statistics. The study of multiple regression in this research covered a situation with two or more independent variables, a multiple regression.

This study investigated the intercept and regression coefficients under three continuous distributions: Normal, uniform, and gamma distributions (the special cases are exponential and chi-square distributions). Consideration of other situations for discrete or qualitative independent variables such as the binomial distribution, is not covered here. We cover only continuous distributions or quantitative independent variables for multiple regression analysis.

Our contribution to the field of statistics is a table suggesting proper point estimation methods for specific kinds of distributions, statistics, and outliers detection and management in doing multiple regression analysis.

## 2. Materials and methods

This research compared the efficiency of two nonparametric and four parametric statistics point estimation methods for multiple regression models, with outliers detection and management. The comparison follows the steps in the flowchart in Figure 1.

The flowchart covers three situations. In Situation 1, the independent variables and errors were randomly simulated without outliers. The independent variables were simulated to have a normal, uniform, or gamma distribution. The errors were simulated to have a standard normal, uniform, or gamma distribution. In Situation 2, the independent variables were randomly simulated with outliers. The independent variables were simulated to have a normal, uniform, or gamma distribution. Finally, In Situation 3, the errors were randomly simulated with outliers. The errors were simulated to have a standard normal, uniform, or gamma distribution.

The flowchart details our outlier detection and management method. In this study, we used an elementary yet robust statistical method for outlier detection, the Interquartile Range (IQR), which was less sensitive than the Z-score to the presence of outliers [7].

There were several methods to manage outliers. For convenience and expediency, we truncated the data to exclude outliers [7] in the case of the Ordinary Least Squares (OLS) method and the Optimal Thiel-Sen (OTS) method. A truncation percentage ($\alpha$) was suggested by [14]. In this study, the truncation percentage was set to 10% for the Ordinary Least Trimmed Squares (OLTS) method in parametric statistics and for the adjusted optimal trimmed Theil-Sen (AOTS) method in nonparametric statistics.

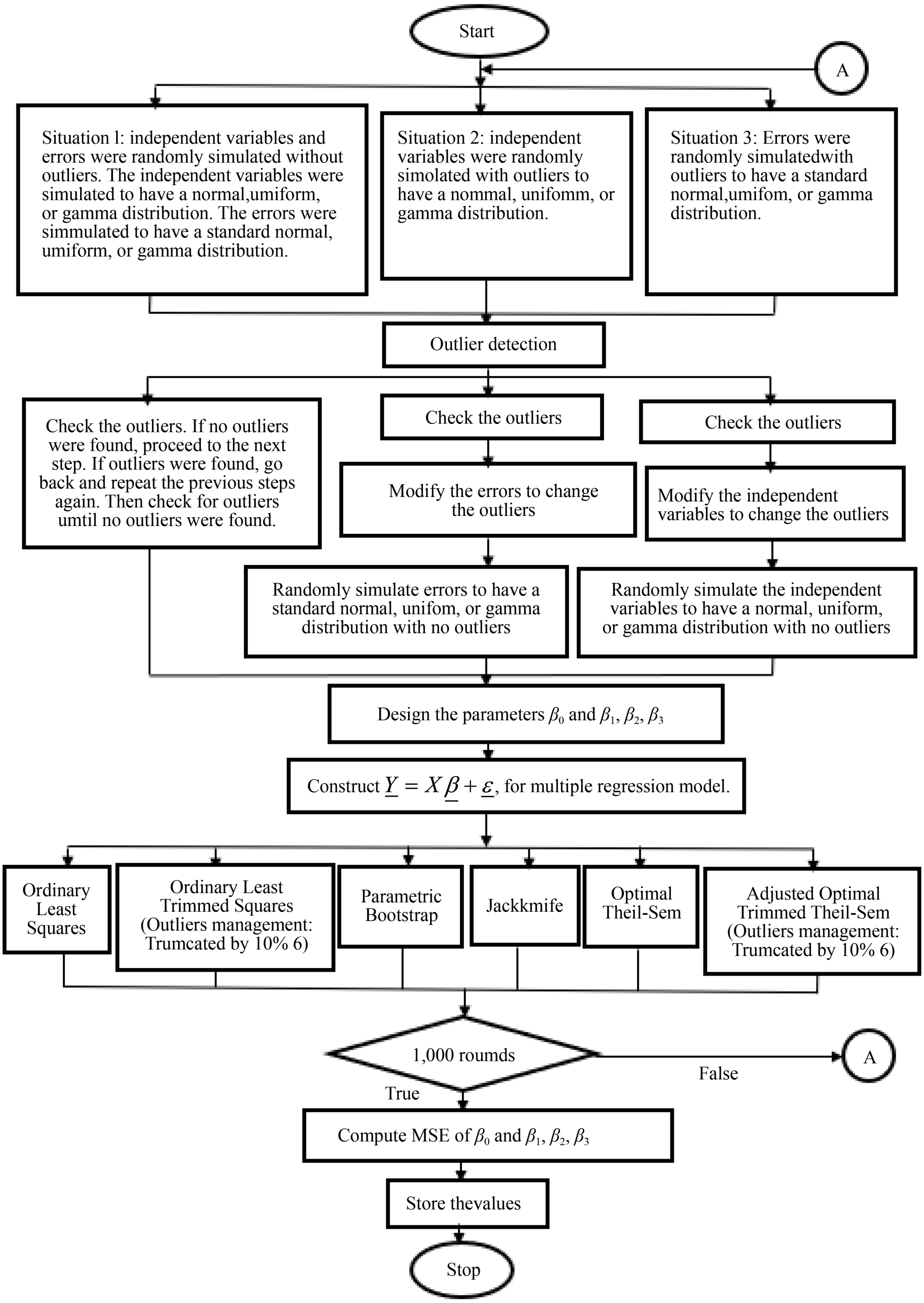


**Figure 1.** Flowchart of the comparison steps

## 2.1 *Data were randomized in simulations of independent variables and errors without outliers [20]*

(1) Each of the three independent variables was assigned $n$ random values. Using the R studio program, the values were assigned with three different distributions: a normal distribution of $N(1, 1)$ [21]; a uniform distribution of $Uniform(2, 5)$; and a gamma distribution of $Gamma(2, 5)$ [22].

(2) Errors were assigned $n$ random values. Using the R studio program, the values were assigned with three different distributions: a standard normal distribution of $N(0, 1)$; a uniform distribution of $Uniform(2, 5)$; and a gamma distribution of $Gamma(2, 5)$.

(3) Independent variables and errors were examined for outliers with a box plot diagram. If outliers were not found, the independent variables and errors were taken to determine the value of the dependent variable in 2.4. However, if outliers were found, (1) and (2) were repeated and re-examined until no outliers were found.

## 2.2 *Data were randomized in simulations of independent variables with outliers*

(1) The data values were randomly assigned with three different distributions, as described in (1) in 2.1.

(2) The random data of the independent variables from (1) were imported in ascending order. To include 10%, 20%, and 30% of outliers, the value of the largest or the second largest independent variable was increased to a higher value by an amount of $1IQR$, $2IQR$, and $3IQR$, and so on until the outliers of 10%, 20%, and 30% are found, as shown in Table 1, where $IQR = Q_3 - Q_1$, $IQR$ is interquartile range, $Q_1$ is the first quartile and $Q_3$ is the third quartile.

**Table 1.** Numbers of outliers and non-outliers in the data for the cases of 10%, 20%, and 30% of outliers, with small, medium, and large sample sizes

| Sample size ($n$) | Percentage of outliers | | | | | |
|---|---|---|---|---|---|---|
| | 10% | | 20% | | 30% | |
| | Number of outliers | Number of non-outliers | Number of outliers | Number of non-outliers | Number of outliers | Number of non-outliers |
| Small | | | | | | |
| 10 | 1 | 9 | 2 | 8 | 3 | 7 |
| 30 | 3 | 27 | 6 | 24 | 9 | 21 |
| Medium | | | | | | |
| 50 | 5 | 45 | 10 | 40 | 15 | 35 |
| 70 | 7 | 63 | 14 | 56 | 21 | 49 |
| Large | | | | | | |
| 90 | 9 | 81 | 18 | 72 | 27 | 63 |
| 110 | 11 | 99 | 22 | 88 | 33 | 77 |

(3) The data of independent variables in Microsoft Excel collected in (2) were checked for outliers using the R studio program with a box plot diagram. If the data were higher than $Q_3 + 1.5IQR$ or lower than $Q_1 - 1.5IQR$, the data were outliers. If there was no outlier, go back to step (2), but if there were a proper number of outliers for the specified 10%, 20%, and 30%, the independent variables data were processed to find the dependent variable in 2.4.

(4) Errors were randomly assigned with three different distributions, as described in (1) in 2.1. The errors were examined for outliers. If outliers were found, the errors must be resampled until outliers were not found.

## 2.3 *The data were randomized in simulations of errors with outliers*

(1) Errors were assigned $n$ random values, with three different distributions as described in (1) in 2.1.

(2) The random data of the errors from (1) were imported into Microsoft Excel in ascending order. To include 10%, 20%, and 30% of outliers, the value of the largest or the second largest independent variable was increased to a higher

value, and the value of the smallest or the second smallest independent variable was decreased to a lower value by an amount of $1IQR$, $2IQR$, and $3IQR$, and so on until the outliers of 10%, 20%, and 30% are found, as shown in Table 1.

(3) The data of errors from Microsoft Excel collected in (2) were checked for outliers using the R studio program with a box plot diagram. If there was no outlier, go back to step (2) but if a proper number of outliers were found corresponding to the specified outlier of 10%, 20%, and 30%, the error data were processed to find the dependent variable in 2.4.

(4) The independent variables were randomly assigned to have a normal distribution of $N(1, 1)$, a uniform distribution of $Uniform(2, 5)$, and a gamma distribution of $Gamma(2, 5)$. The independent variables were examined for outliers. If outliers were found, the independent variables had to be resampled until outliers were found.

## 2.4 *The dependent variables were constructed to have the same distributions as the independent variables*

The dependent variables were constructed to have a normal distribution of $N(1, 1)$, a uniform distribution of $Uniform(2, 5)$, and a gamma distribution of $Gamma(2, 5)$. The dependent variable data were calculated according to the relationship model of the multiple regression equation, $Y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon_i$, where $i = 1, 2, ..., n$, which is in matrix and vector form as follows: $\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$, where $\beta_0$ is the $Y$-intercept and $\beta_1, \beta_2, \beta_3$ are the slopes or the regression coefficients of the independent variables. In this research, $\beta_0$ was designated as 4 and $\beta_1, \beta_2, \beta_3$ were designated as 6, $-8$, and 10, respectively (see the Appendix: Tables A1-A12). The independent variables and errors were taken from 2.1, 2.2, and 2.3.

## 2.5 *Y-intercept ($\beta_0$) and regression coefficient ($\beta_1$, $\beta_2$, $\beta_3$) were estimated with six estimation methods*

Ordinary least squares, ordinary least trimmed squares, parametric bootstrap, Jackknife, optimal Theil-Sen, and proposed, adjusted optimal trimmed Theil-Sen methods as follows:

(1) Ordinary Least Squares Method (OLS)

OLS of estimating regression coefficients is based on the theory of linear estimation. The principle is to approximate the regression coefficients that minimize the sum of squares of the difference between the true and estimated values [23]. The regression model showing the relationship between independent and dependent variables is $\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$ or

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 1 & x_{11} & x_{12} & x_{13} \\ 1 & x_{21} & x_{22} & x_{23} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x_{n1} & x_{n2} & x_{n3} \end{bmatrix}_{n\times 4} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}_{4\times 1} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_n \end{bmatrix}_{n\times 1},$$

where, $\underline{Y}$ is an $n \times 1$ vector of the observations of the dependent variable, $X$ is an $n \times 4$ matrix of observations of independent variables, $\underline{\beta}$ is a $4 \times 1$ vector of the parameter $\beta_j$, where $j = 0, 1, 2, 3$, $\underline{\varepsilon}$ is an $n \times 1$ vector of the error, and $n$ is the sample size.

OLS of estimating regression coefficients ($\hat{\underline{\beta}}$) estimates the regression coefficients that provide the minimum sum of the squares of the difference between the true and the estimated values or sum of squares error ($SSE$). The sum of squares of the error is written in the matrix form as follows [24],

$$SSE = \left(\underline{Y} - \hat{\underline{\beta}}\right)' \left(\underline{Y} - X\hat{\underline{\beta}}\right) = \underline{Y}'\underline{Y} - 2\hat{\underline{\beta}}'X'\underline{Y} + \hat{\underline{\beta}}'X'X\hat{\underline{\beta}}.$$

$\underline{\hat{\beta}}$ is determined by taking the derivative of $SSE$ concerning $\underline{\hat{\beta}}$ and let it be equal to 0. Then,

$$\frac{\partial}{\partial \underline{\hat{\beta}}}\left(\underline{Y}'\underline{Y} - 2\underline{\hat{\beta}}'X'\underline{Y} + \underline{\hat{\beta}}'X'X\underline{\hat{\beta}}\right) = 0 \text{ or } -2X'\underline{Y} + 2X'X\underline{\hat{\beta}} = 0.$$

Therefore, the point estimate of the parameter $\underline{\beta}$ for OLS is $\underline{\hat{\beta}}_{OLS} = (X'X)^{-1}X'\underline{Y}$ [25, 26].

(2) Ordinary Least Trimmed Square Method (OLTS)

OLTS is derived from the OLS using both trimmed sides of the data to reduce the outliers. OLTS is as highly efficient as OLS [13]. OLTS is expressed as [12] min $\sum_{i=1}^{h} e_{i:n}^2$, where, $e_{1:n}^2 \leq e_{2:n}^2 \leq e_{3:n}^2 \leq \ldots \leq e_{h:n}^2 \leq \ldots \leq e_{n:n}^2$. $e_i^2$ is the squared residuals, arranged from the smallest to the largest. The use of $h = n(1-\alpha)+1$, where $h$ is the amount of the remaining data after the data was truncated, and $\alpha$ is truncation percentage, was suggested by [14]. OLTS replaces OLS for large sample sizes [15]. The most robust value for $h$ is $n/2$. In this case, the truncated point is 50% [14]. The truncated point of OLTS and OLS is $h = (n+p+1)/2$. OLTS has a convergence rate of $n^{-1/2}$ and converges at the same rate as the $M$ method [13]. OLTS is a good method for finding outliers. At times, it can eliminate overestimated observations [16]. In this research, $\alpha$ is a truncation percentage of 10%. Therefore, the point estimation of the parameter $\underline{\beta}$ for OLTS is $\underline{\hat{\beta}}_{OLTS} = (X'X)^{-1}X'\underline{Y}$.

For, $n = 30$, $\alpha = 30\% = 0.3$, $h = n(1-\alpha)+1 = 30(1-0.1)+1 = 28$, the number of outliers was 3; the number of removed outliers was 2; the number of remaining outliers was 3 − 2 = 1; and the number of non-outliers was 28 − 1 = 27.

From Table 2, for a small sample size ($n$) of 10, an outlier of 10% and percentage truncation of 10% means that 0 data point was the removed outlier; 1 data point was the remaining outlier; and 9 data points were non-outliers. An outlier of 20% means that 0 data points were the removed outlier; 2 data points were the remaining outliers; and 8 data points were non-outliers. An outlier of 30% means that 0 data points were the removed outlier; 3 data points were the remaining outliers; and 7 data points were non-outliers. For other sample sizes, the numbers are described similarly.

**Table 2.** Numbers of removed outliers, remaining outliers, and non-outliers in the data for the cases of 10%, 20%, and 30% of outliers, percentage truncation of 10%, with small, medium, and large sample sizes

| Sample size ($n$) | Percentage of outliers | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 10% | | | 20% | | | 30% | | |
| | Number of removed outliers | Number of remaining outliers | Number of non-outliers | Number of removed outliers | Number of remaining outliers | Number of non-outliers | Number of removed outliers | Number of remaining outliers | Number of non-outliers |
| Small | | | | | | | | | |
| 10 | 0 | 1 | 9 | 0 | 2 | 8 | 0 | 3 | 7 |
| 30 | 2 | 1 | 27 | 2 | 4 | 24 | 2 | 7 | 21 |
| Medium | | | | | | | | | |
| 50 | 4 | 1 | 45 | 4 | 6 | 40 | 4 | 11 | 35 |
| 70 | 6 | 1 | 63 | 6 | 8 | 56 | 6 | 15 | 49 |
| Large | | | | | | | | | |
| 90 | 8 | 1 | 81 | 8 | 10 | 72 | 8 | 19 | 63 |
| 110 | 10 | 1 | 99 | 10 | 12 | 88 | 10 | 23 | 77 |

Note: For, $n = 10$, $\alpha = 10\% = 0.1$, $h = n(1-\alpha)+1 = 10(1-0.1)+1 = 10$, the number of outliers was 1; the number of removed outliers was 0; the number of remaining outliers was 1 − 0 = 1; and the number of non-outliers were 10 − 1 = 9.

OLS and OLTS were followed by the assumption that the population does have a normal distribution.

(3) Parametric Bootstrap Method (PB)

The parametric bootstrap method used in this research is called the empirical bootstrap method. This method was originally proposed by [19] and was later developed to use an n-size replacement sampling method from a single random sample to generate a possible n-size sample by sampling from the empirical distribution function (F) of the sampled data [27].

Let $X_1, X_2, \ldots, X_n$ be independent random samples from a normally distributed population with parameter $\underline{\beta}$. PB collects samples one at a time for $n$ times. A new set of random samples is obtained $X_1^*, X_2^*, \ldots, X_n^*$. PB estimation is as follows.

In the first round, one sample at a time is randomly sampled with a replacement for $n$ times from $X_1, X_2, \ldots, X_n$ to get $X_1^*, X_2^*, \ldots, X_n^*$, and then by OLS, the estimation of $\underline{\beta}_1$ is $\hat{\underline{\beta}}_1^*$. In the later rounds until the B$^{\text{th}}$ round, one sample at a time is randomly sampled with replacement for $n$ times as before to get the final $X_1^*, X_2^*, \ldots, X_n^*$, and then by OLS, the estimation of $\underline{\beta}_3, \underline{\beta}_4, \ldots, \underline{\beta}_B$ are $\hat{\underline{\beta}}_3^*, \hat{\underline{\beta}}_4^*, \ldots, \hat{\underline{\beta}}_B^*$. Each estimate has the same probability $1/B$. Therefore, the point estimation of the parameter $\underline{\beta}$ for PB is $\hat{\underline{\beta}}_{PB} = \sum_{i=1}^{B} \frac{\hat{\underline{\beta}}_i^*}{B}$. B is the number of repeated rounds until 1,000.

(4) Jackknife Method (JK)

Let be random samples with a normal distribution. A new set of samples is generated by omitting the $i^{\text{th}}$ value, and a new set of samples of size $n-1$ is obtained. The omitted value is returned to the set of samples before the next sample is generated. Do this n times with the following steps [28, 29]:

Step 1: Random $\underline{\varepsilon}$ from the distribution of errors is obtained, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Step 2: In the first round, $X_1$ is omitted from the sample to obtain $X_2, X_3, \ldots, X_n$ and $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$. In the second round, $X_2$ is omitted from the sample to obtain and $\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_n$. This is done for $n$ times. Step 3: By OLS, an estimation of $\hat{\underline{\beta}}_{LS}$ is obtained to calculate a parameter estimate of $\underline{\beta}$. Step 4: Substitute the values of $X$, $\underline{\varepsilon}$ and $\hat{\underline{\beta}}_{LS}$ in the equation $\underline{Y} = X\hat{\underline{\beta}}_{LS} + \underline{\varepsilon}$ to obtain $\underline{Y} = X\hat{\underline{\beta}}_{LS} + \underline{\varepsilon}$. Step 5: Take the obtained $X_2, X_3, \ldots, X_n, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$ and $Y_2, Y_3, \ldots, Y_n$ to estimate parameter $\hat{\underline{\beta}}_{JK}$. Step 6: Repeat steps 2, 3, and 4 for $n$ times. Therefore, the point estimate of parameters $\beta_0$ and $\beta_1$ for JK are $\hat{\beta}_{0_{JK}} = \frac{1}{n}\sum_{\text{j}=1}^{\text{n}} \hat{\beta}_{0j}$ and $\hat{\beta}_{1_{JK}} = \frac{1}{n}\sum_{\text{j}=1}^{\text{n}} \hat{\beta}_{1j}$, respectively. In the same way, $\hat{\beta}_{3_{JK}}$ and $\hat{\beta}_{2_{JK}}$ can be obtained.

(5) Optimal Theil-Sen Method (OTS)

Theil's method was first proposed by [30]. Later, Sen [31] proposed a method for Kendall's tau and named it the Theil-Sen method (TS). It is also called the Theil-Kendal method [18]. OTS was developed from TS.

TS is a method for estimating the parameter $Y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_n X_n + \varepsilon_i$, where, $i = 1, 2, \ldots$, n, $\beta_1$ is for $X_1$, $\beta_2$ is for $X_2$, etc [32, 33]. The steps are as follows. Step 1: Calculate the slope $S_{ij} = \frac{y_j - y_i}{x_j - x_i}$, $1 \le i \le j \le n$ to obtain $N = \frac{n(n-1)}{2}$ slope values. Step 2: Estimate $\beta_1$; the point estimate of the parameter $\beta_1$ for TS is $\hat{\beta}_{1_{TS}} = median(S_{ij},\ 1 \le i \le j \le n)$. $\hat{\beta}_{2_{TS}}$ and $\hat{\beta}_{3_{TS}}$ can also be found in the same way. Then, the point estimate of the parameter $\beta_0$ for TS is $\hat{\beta}_{0_{TS}} = \text{median}(y_i) - \hat{\beta}_1 \text{median}(x_{1i}) - \hat{\beta}_2 \text{median}(x_{2i}) - \hat{\beta}_3 \text{median}(x_{3i})$, where $\text{median}(y_i)$ and $median(x_i)$ are median of $y_i$ and $x_i$, respectively. Step 3: let the variable $d_i = y_i - \hat{\beta}_1 x_{1i} - \hat{\beta}_2 x_{2i} - \hat{\beta}_3 x_{3i}$. There is no need for an assumption of symmetric distribution of $d_i$. $\hat{\beta}_0$ is the arithmetic mean of $d_i$ [18, 34]. Therefore, the point estimate of the parameter $\beta_1$ for OTS is $\hat{\beta}_{1_{OTS}} = median(S_{ij},\ 1 \le i \le j \le n)$. $\hat{\beta}_{2_{OTS}}$ and $\hat{\beta}_{3_{OTS}}$ can be found in the same way. The point estimate of the parameter $\beta_0$ is $\hat{\beta}_{0_{OTS}} = \text{median}(d_i)$.

(6) Adjusted Optimal Trimmed Theil-Sen Method (AOTS)

In this method, both ends of the data are truncated by 10%, as described in Table 2. Let the variable $d_i = y_i - \hat{\beta}_1 x_{1i} - \hat{\beta}_2 x_{2i} - \hat{\beta}_3 x_{3i}$, with no need to assume a symmetric distribution of $d_i$. $\hat{\beta}_0$ is the arithmetic mean of $d_i$, and $w_{ij} = x_j - x_i$ is the weight. The resulting estimate is the weighted median of $d_i$. Therefore, the point estimate of the regression coefficient of $\beta_1$ for AOTS is $\hat{\beta}_{1_{AOTS}} = \text{median}(S_{ij},\ 1 \le i \le j \le n)$. $\hat{\beta}_{2_{AOTS}}$ and $\hat{\beta}_{3_{AOTS}}$ can be found in the same way. The point estimate of the Y-intercept of $\beta_0$ for AOTS is $\hat{\beta}_{0_{AOTS}} = \text{median}(|w_{ij} d_i|)$.

### 2.6 *The mean square error of MSE(β₀), MSE(β₁), MSE(β₂), and MSE(β₃) for the 6 methods was calculated*

The method with the lowest MSE was the most efficient, where $MSE_{b_0} = \sum_{i=1}^{1,000} \left(\beta_0 - \hat{\beta}_{0i}\right)^2 \Big/ 1,000$, $MSE_{b_1} = \sum_{i=1}^{1,000} \left(\beta_1 - \hat{\beta}_{1i}\right)^2 \Big/ 1,000$, $MSE_{b_2} = \sum_{i=1}^{1,000} \left(\beta_2 - \hat{\beta}_{2i}\right)^2 \Big/ 1,000$ and $MSE_{b_3} = \sum_{i=1}^{1,000} \left(\beta_3 - \hat{\beta}_{3i}\right)^2 \Big/ 1,000$. $\hat{\beta}_{0i}$ is the parameter estimator of $\beta_0$ and $\hat{\beta}_{1i}$, $\hat{\beta}_{2i}$, $\hat{\beta}_{3i}$ are the parameter estimators of $\beta_1$, $\beta_2$, $\beta_3$, respectively from the approximation of the $i^{th}$ cycle, $i$ = 1, 2, …, 1,000.

## 3. Results of a simulation study

Table 3 shows the case that both independent variables and errors were normally distributed with no outliers. For parametric statistics, OLS provided the minimum mean square error, followed by PB. For nonparametric statistics, OTS provided a lower mean square error than AOTS. In addition, in estimating $\beta_0$ and $\beta_2$ for all sample sizes, OTS provided a lower mean square error than AOTS. In estimating $\beta_1$ and $\beta_3$ for most of the sample sizes, OTS provided a lower mean square error than AOTS.

Table 4 shows the case that only independent variables were normally distributed with outliers, but errors were without outliers. For parametric statistics, PB provided the minimum mean square error, followed by OLS. For nonparametric statistics, AOTS provided a lower mean square error than OTS. In addition, in estimating $\beta_0$ for all sample sizes, AOTS provided a lower mean square error than OTS. In estimating $\beta_1$ and $\beta_2$ for most of the sample sizes, OTS provided a lower mean square error than AOTS. Moreover, in estimating $\beta_3$, OTS provided a slightly lower mean square error than AOTS.

Table 5 shows the case that only errors were normally distributed with outliers, but independent variables were without outliers. For parametric statistics, OLTS provided the minimum mean square error, followed by OLS. For nonparametric statistics, OTS provided a lower mean square error than AOTS. In addition, in estimating $\beta_0$ for all sample sizes, AOTS provided a lower mean square error than OTS. In estimating $\beta_1$, $\beta_2$ and $\beta_3$ for most of the sample sizes, OTS provided a lower mean square error than AOTS.

Table 6 shows the case that both independent variables and errors were uniformly distributed with no outliers. For parametric statistics, OLS provided the minimum mean square error, followed by OLTS. For nonparametric statistics, OTS provided a lower mean square error than AOTS. In addition, in estimating $\beta_0$ and $\beta_2$ for all sample sizes, OTS provided a lower mean square error than AOTS. In estimating $\beta_1$ and $\beta_3$ for most of the sample sizes, OTS provided a lower mean square error than AOTS.

Table 7 shows the case that only independent variables were uniformly distributed with outliers, but errors were without outliers. For parametric statistics, OLS provided the minimum mean square error, followed by PB. For nonparametric statistics, AOTS provided a lower mean square error than OTS. In addition, in estimating of $\beta_0$, all sample sizes, AOTS provided a lower mean square error than OTS, except for $n$ = 110 and outliers = 30. In estimating $\beta_1$ and $\beta_3$ for most of the sample sizes, OTS provided a lower mean square error than AOTS. Moreover, in estimating $\beta_2$, OTS provided a slightly lower mean square error than AOTS.

Table 8 shows the case that only errors were uniformly distributed with outliers, but independent variables were without outliers. For parametric statistics, OLTS provided the minimum mean square error, followed by OLS. For nonparametric statistics, OTS provided a lower mean square error than AOTS. In addition, in estimating $\beta_0$ for all sample sizes, AOTS provided a lower mean square error than OTS. In estimating $\beta_1$, $\beta_2$ and $\beta_3$ for most of the sample sizes, OTS provided a lower mean square error than AOTS.

Table 9 shows the case that both independent variables and errors were gamma-distributed with no outliers. For parametric statistics, PB provided the minimum mean square error, followed by OLTS. For nonparametric statistics, OTS provides a lower mean square error than AOTS. In addition, in estimating $\beta_0$ and $\beta_2$ for all sample sizes, OTS provided a lower mean square error than AOTS. In estimating $\beta_1$ and $\beta_3$ for most of the sample sizes, OTS provided a lower mean square error than AOTS.

Table 10 shows the case that only independent variables were gamma-distributed with outliers, but errors were without outliers. For parametric statistics, OLS provided the minimum mean square error, followed by PB. For nonparametric statistics, OTS provided a slightly lower mean square error than AOTS. In addition, in estimating $\beta_0$ for most of the sample sizes, AOTS provided a lower mean square error than OTS. In estimating $\beta_1$ and $\beta_2$ for most of the sample sizes, OTS provided a lower mean square error than AOTS. In estimating $\beta_3$, AOTS provided a slightly lower mean square error than OTS.

Table 11 shows the case that only errors were gamma-distributed with outliers, but independent variables were without outliers. For parametric statistics, OLTS provided the minimum mean square error, followed by OLS. For nonparametric statistics, OTS provided a lower mean square error than AOTS. In estimating $\beta_0$, $\beta_1$, $\beta_2$ and $\beta_3$ for most of the sample sizes, OTS provided a lower mean square error than AOTS.

**Table 3.** Mean square errors in the case that independent variables and errors are distributed normally with no outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110

| $n$ | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\beta_o$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_o$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| 10 | 0.66 | 1.39 | 0.73 | 2.64 | 0.19 | 2.72 | 0.20 | 0.77 | 0.20 | 2.69 | 0.22 | 0.78 | 0.17 | 3.48 | 0.21 | 0.69 | 61.33 | 152.09 | 30.04 | 16.84 | 25.58 | 347.53 | 18.52 | 12.98 |
| 30 | 0.15 | 0.27 | 0.16 | 0.61 | 0.04 | 1.24 | 0.03 | 0.15 | 0.03 | 1.23 | 0.04 | 0.16 | 0.04 | 1.30 | 0.05 | 0.16 | 12.52 | 67.34 | 7.24 | 7.22 | 5.36 | 275.09 | 4.09 | 6.59 |
| 50 | 0.08 | 0.15 | 0.09 | 0.35 | 0.02 | 0.99 | 0.03 | 0.09 | 0.02 | 0.96 | 0.03 | 0.08 | 0.02 | 1.10 | 0.03 | 0.09 | 6.62 | 61.11 | 3.94 | 6.44 | 3.31 | 275.63 | 2.46 | 5.13 |
| 70 | 0.07 | 0.10 | 0.06 | 0.25 | 0.02 | 0.98 | 0.01 | 0.06 | 0.02 | 0.95 | 0.01 | 0.07 | 0.02 | 0.92 | 0.01 | 0.06 | 4.90 | 60.29 | 2.85 | 5.75 | 2.35 | 269.69 | 1.70 | 4.94 |
| 90 | 0.04 | 0.07 | 0.05 | 0.18 | 0.01 | 0.84 | 0.02 | 0.05 | 0.01 | 1.23 | 0.02 | 0.04 | 0.02 | 0.79 | 0.01 | 0.05 | 3.90 | 49.89 | 2.34 | 5.18 | 1.88 | 263.90 | 1.40 | 4.74 |
| 110 | 0.04 | 0.06 | 0.05 | 0.15 | 0.01 | 0.73 | 0.02 | 0.04 | 0.01 | 0.75 | 0.02 | 0.04 | 0.01 | 0.75 | 0.02 | 0.04 | 2.83 | 52.59 | 1.67 | 4.95 | 1.37 | 261.63 | 0.99 | 4.74 |

**Table 4.** Mean square errors in the case that only independent variables were normally distributed with outliers. The errors were without outliers; $\beta_0 = 4$, $\beta_1 = 6$, $\beta_2 = -8$, $\beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110; and the numbers of outliers were 10%, 20%, and 30% of the sample size

| | | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | Outliers | $\beta_0$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_0$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| 10 | 10 | 2.37 | 2.37 | 2.43 | 9.50 | 3.35 | 3.35 | 3.24 | 13.41 | 1.17 | 1.17 | 1.13 | 4.71 | 3.80 | 3.80 | 3.87 | 15.20 | 260.41 | 37.49 | 7.49 | 7.49 | 230.41 | 230.41 | 8.48 | 8.48 |
| | 20 | 0.77 | 0.77 | 0.71 | 3.07 | 2.18 | 2.18 | 1.94 | 8.72 | 0.83 | 0.83 | 0.72 | 3.33 | 0.12 | 0.12 | 0.11 | 0.47 | 125.63 | 40.61 | 3.61 | 3.61 | 177.92 | 177.92 | 16.97 | 16.97 |
| | 30 | 1.12 | 1.12 | 1.08 | 4.48 | 1.50 | 1.50 | 1.33 | 6.00 | 1.61 | 1.61 | 1.40 | 6.42 | 1.70 | 1.70 | 1.61 | 6.78 | 184.15 | 118.98 | 5.23 | 5.23 | 193.16 | 193.16 | 14.70 | 14.70 |
| 30 | 10 | 0.04 | 0.11 | 0.05 | 0.19 | 0.42 | 1.48 | 0.47 | 1.68 | 0.45 | 1.61 | 0.46 | 1.80 | 0.16 | 0.20 | 0.16 | 0.66 | 88.49 | 13.61 | 12.48 | 11.57 | 269.56 | 268.97 | 5.99 | 6.05 |
| | 20 | 0.10 | 0.16 | 0.12 | 0.40 | 1.51 | 2.48 | 1.69 | 6.02 | 0.74 | 1.30 | 0.76 | 2.94 | 0.46 | 0.53 | 0.49 | 1.83 | 97.58 | 35.79 | 17.44 | 18.99 | 276.84 | 280.02 | 6.24 | 5.92 |
| | 30 | 0.11 | 0.26 | 0.12 | 0.46 | 1.06 | 2.27 | 1.02 | 4.24 | 0.71 | 1.38 | 0.68 | 2.83 | 0.11 | 0.11 | 0.10 | 0.44 | 86.03 | 60.49 | 18.14 | 17.74 | 258.88 | 247.42 | 8.60 | 9.70 |
| 50 | 10 | 0.09 | 0.14 | 0.08 | 0.36 | 0.22 | 0.41 | 0.21 | 0.87 | 0.21 | 0.58 | 0.20 | 0.85 | 0.39 | 0.89 | 0.38 | 1.58 | 583.76 | 86.77 | 8.32 | 8.48 | 355.92 | 339.92 | 0.49 | 1.01 |
| | 20 | 0.06 | 0.09 | 0.07 | 0.26 | 0.26 | 0.88 | 0.27 | 1.06 | 0.15 | 0.18 | 0.17 | 0.61 | 0.65 | 1.41 | 0.68 | 2.60 | 598.30 | 178.55 | 6.73 | 7.35 | 365.63 | 368.21 | 0.11 | 0.17 |
| | 30 | 0.04 | 0.06 | 0.05 | 0.17 | 0.15 | 0.65 | 0.14 | 0.60 | 0.03 | 0.08 | 0.04 | 0.12 | 0.24 | 1.13 | 0.23 | 0.95 | 663.17 | 401.12 | 8.16 | 7.47 | 389.19 | 389.58 | 0.01 | 0.03 |
| 70 | 10 | 0.06 | 0.08 | 0.05 | 0.22 | 0.86 | 2.25 | 0.94 | 3.45 | 0.15 | 0.27 | 0.16 | 0.60 | 1.08 | 2.43 | 1.15 | 4.30 | 89.39 | 1.27 | 1.84 | 1.93 | 238.12 | 232.40 | 3.41 | 4.01 |
| | 20 | 0.08 | 0.16 | 0.09 | 0.34 | 0.91 | 2.34 | 0.93 | 3.63 | 0.23 | 0.51 | 0.24 | 0.91 | 0.88 | 2.13 | 0.96 | 3.53 | 46.48 | 2.86 | 0.36 | 0.69 | 201.31 | 206.70 | 7.64 | 7.05 |
| | 30 | 0.04 | 0.10 | 0.03 | 0.17 | 0.63 | 1.14 | 0.66 | 2.51 | 0.06 | 0.16 | 0.07 | 0.25 | 0.71 | 1.43 | 0.74 | 2.84 | 41.66 | 14.13 | 0.24 | 0.14 | 195.93 | 191.48 | 8.26 | 8.78 |
| 90 | 10 | 0.05 | 0.06 | 0.04 | 0.18 | 0.51 | 0.81 | 0.47 | 2.04 | 0.49 | 1.32 | 0.48 | 1.98 | 0.80 | 0.99 | 0.74 | 3.19 | 118.53 | 5.65 | 4.53 | 5.65 | 240.70 | 259.04 | 6.97 | 4.96 |
| | 20 | 0.06 | 0.07 | 0.05 | 0.22 | 0.85 | 1.06 | 0.84 | 3.39 | 0.49 | 0.90 | 0.44 | 1.95 | 0.77 | 0.99 | 0.73 | 3.07 | 97.68 | 7.34 | 4.23 | 5.22 | 219.98 | 232.17 | 9.61 | 8.13 |
| | 30 | 0.10 | 0.11 | 0.11 | 0.43 | 1.31 | 1.39 | 1.36 | 5.23 | 0.62 | 0.62 | 0.61 | 2.47 | 0.83 | 0.89 | 0.82 | 3.32 | 78.82 | 13.72 | 4.67 | 4.30 | 198.40 | 201.45 | 13.61 | 13.03 |
| 110 | 10 | 0.09 | 0.10 | 0.08 | 0.34 | 0.99 | 2.47 | 0.90 | 3.98 | 0.85 | 0.96 | 0.76 | 3.39 | 1.62 | 2.36 | 1.44 | 6.49 | 138.87 | 12.07 | 1.76 | 1.98 | 219.00 | 220.47 | 7.16 | 7.34 |
| | 20 | 0.09 | 0.11 | 0.08 | 0.36 | 0.80 | 2.30 | 0.71 | 3.19 | 1.15 | 1.56 | 1.09 | 4.58 | 1.43 | 2.57 | 1.32 | 5.73 | 148.38 | 35.66 | 2.27 | 2.74 | 223.93 | 227.36 | 6.81 | 6.76 |
| | 30 | 0.05 | 0.06 | 0.07 | 0.20 | 0.82 | 2.41 | 0.02 | 3.26 | 0.73 | 0.98 | 0.02 | 2.94 | 0.88 | 2.09 | 0.02 | 3.50 | 154.27 | 140.03 | 2.56 | 3.14 | 227.10 | 231.36 | 6.57 | 6.45 |

**Table 5.** Mean square errors in the case that only errors were normally distributed with outliers. The independent variables were without outliers; $\beta_0 = 4$, $\beta_1 = 6$, $\beta_2 = -8$, $\beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110; the numbers of outliers were 10%, 20%, and 30% of the sample size

| $n$ | Outliers | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\beta_0$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_0$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| 10 | 10 | 0.71 | 0.71 | 1.80 | 2.84 | 0.20 | 0.20 | 0.54 | 0.79 | 0.19 | 0.19 | 0.50 | 0.77 | 0.20 | 0.19 | 0.45 | 0.77 | 63.11 | 45.74 | 31.61 | 31.61 | 25.63 | 25.63 | 18.43 | 18.43 |
| | 20 | 1.65 | 1.65 | 4.50 | 6.60 | 0.42 | 0.42 | 1.06 | 1.66 | 0.42 | 0.42 | 1.09 | 1.70 | 0.41 | 0.42 | 1.05 | 1.63 | 62.31 | 46.83 | 31.10 | 31.10 | 25.20 | 25.20 | 18.74 | 18.74 |
| | 30 | 2.85 | 2.85 | 7.31 | 11.44 | 0.81 | 0.81 | 2.14 | 3.24 | 0.83 | 0.83 | 2.13 | 3.30 | 0.80 | 0.83 | 2.05 | 3.21 | 63.10 | 45.05 | 31.53 | 31.53 | 25.86 | 25.86 | 19.12 | 19.12 |
| 30 | 10 | 0.32 | 0.23 | 0.89 | 1.30 | 0.08 | 0.04 | 0.15 | 0.31 | 0.07 | 0.05 | 0.16 | 0.30 | 0.08 | 0.04 | 0.16 | 0.32 | 12.46 | 7.17 | 6.71 | 7.53 | 5.93 | 6.10 | 4.51 | 4.29 |
| | 20 | 0.39 | 0.30 | 0.99 | 1.55 | 0.11 | 0.07 | 0.21 | 0.42 | 0.10 | 0.09 | 0.22 | 0.41 | 0.11 | 0.07 | 0.23 | 0.44 | 12.47 | 7.35 | 6.73 | 7.54 | 5.98 | 6.14 | 4.54 | 4.38 |
| | 30 | 0.64 | 0.57 | 1.69 | 2.55 | 0.17 | 0.14 | 0.35 | 0.69 | 0.17 | 0.16 | 0.37 | 0.67 | 0.18 | 0.13 | 0.36 | 0.72 | 12.61 | 7.13 | 6.85 | 7.59 | 5.98 | 6.30 | 4.60 | 4.45 |
| 50 | 10 | 0.17 | 0.09 | 0.34 | 0.66 | 0.04 | 0.02 | 0.10 | 0.17 | 0.05 | 0.02 | 0.10 | 0.20 | 0.05 | 0.03 | 0.10 | 0.20 | 7.34 | 4.57 | 4.15 | 4.62 | 3.46 | 3.76 | 2.57 | 2.78 |
| | 20 | 0.24 | 0.16 | 0.48 | 0.97 | 0.07 | 0.05 | 0.14 | 0.26 | 0.08 | 0.05 | 0.15 | 0.31 | 0.08 | 0.06 | 0.15 | 0.31 | 7.38 | 4.49 | 4.22 | 4.67 | 3.49 | 3.78 | 2.61 | 2.80 |
| | 30 | 0.34 | 0.26 | 0.68 | 1.35 | 0.10 | 0.09 | 0.22 | 0.40 | 0.11 | 0.08 | 0.22 | 0.45 | 0.11 | 0.09 | 0.22 | 0.46 | 7.53 | 4.57 | 4.25 | 4.69 | 3.57 | 3.81 | 2.63 | 2.86 |
| 70 | 10 | 0.07 | 0.04 | 0.15 | 0.28 | 0.02 | 0.01 | 0.04 | 0.09 | 0.02 | 0.01 | 0.04 | 0.08 | 0.02 | 0.01 | 0.05 | 0.09 | 5.06 | 3.51 | 2.94 | 2.96 | 2.38 | 2.59 | 1.67 | 1.90 |
| | 20 | 0.11 | 0.08 | 0.25 | 0.46 | 0.04 | 0.02 | 0.07 | 0.14 | 0.03 | 0.02 | 0.07 | 0.14 | 0.04 | 0.02 | 0.07 | 0.14 | 5.06 | 3.42 | 2.94 | 2.98 | 2.38 | 2.59 | 1.68 | 1.91 |
| | 30 | 0.17 | 0.13 | 0.38 | 0.68 | 0.05 | 0.04 | 0.10 | 0.20 | 0.05 | 0.04 | 0.10 | 0.20 | 0.05 | 0.04 | 0.10 | 0.20 | 5.08 | 3.40 | 2.96 | 3.00 | 2.41 | 2.61 | 1.70 | 1.92 |
| 90 | 10 | 0.06 | 0.03 | 0.12 | 0.22 | 0.02 | 0.01 | 0.04 | 0.08 | 0.02 | 0.01 | 0.04 | 0.08 | 0.02 | 0.01 | 0.04 | 0.07 | 3.92 | 3.07 | 2.21 | 2.35 | 1.76 | 1.93 | 1.36 | 1.30 |
| | 20 | 0.08 | 0.06 | 0.18 | 0.33 | 0.03 | 0.02 | 0.06 | 0.12 | 0.03 | 0.02 | 0.06 | 0.12 | 0.03 | 0.02 | 0.06 | 0.11 | 3.94 | 3.04 | 2.23 | 2.36 | 1.77 | 1.93 | 1.37 | 1.31 |
| | 30 | 0.13 | 0.11 | 0.28 | 0.50 | 0.04 | 0.03 | 0.09 | 0.18 | 0.05 | 0.03 | 0.09 | 0.18 | 0.04 | 0.03 | 0.09 | 0.17 | 3.99 | 2.89 | 2.25 | 2.38 | 1.79 | 1.93 | 1.38 | 1.34 |
| 110 | 10 | 0.07 | 0.02 | 0.14 | 0.29 | 0.02 | 0.01 | 0.05 | 0.10 | 0.02 | 0.01 | 0.05 | 0.09 | 0.02 | 0.01 | 0.05 | 0.10 | 2.77 | 2.55 | 1.58 | 1.89 | 1.46 | 1.65 | 1.05 | 1.17 |
| | 20 | 0.11 | 0.07 | 0.22 | 0.44 | 0.04 | 0.02 | 0.08 | 0.15 | 0.04 | 0.02 | 0.07 | 0.14 | 0.04 | 0.03 | 0.07 | 0.15 | 2.82 | 2.45 | 1.59 | 1.90 | 1.47 | 1.67 | 1.06 | 1.18 |
| | 30 | 0.14 | 0.11 | 0.27 | 0.56 | 0.05 | 0.04 | 0.10 | 0.19 | 0.05 | 0.04 | 0.10 | 0.18 | 0.05 | 0.04 | 0.09 | 0.19 | 2.86 | 2.32 | 1.60 | 1.91 | 1.48 | 1.68 | 1.08 | 1.20 |

**Table 6.** Mean square errors in the case that independent variables and errors were uniformly distributed with no outliers; $\beta_0 = 4$, $\beta_1 = 6$, $\beta_2 = -8$, $\beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110

| $n$ | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\beta_0$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_0$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| 10 | 17.70 | 17.83 | 54.86 | 71.12 | 0.18 | 2.77 | 0.18 | 0.71 | 0.18 | 3.28 | 0.19 | 0.72 | 0.18 | 2.75 | 0.20 | 0.73 | 638.99 | 1,998.07 | 28.39 | 9.40 | 20.73 | 303.73 | 17.20 | 7.94 |
| 30 | 13.66 | 12.98 | 51.01 | 54.65 | 0.04 | 1.53 | 0.04 | 0.16 | 0.04 | 1.52 | 0.03 | 0.16 | 0.03 | 1.47 | 0.04 | 0.15 | 160.03 | 1,444.44 | 6.90 | 5.69 | 5.20 | 269.85 | 4.27 | 5.02 |
| 50 | 13.17 | 12.76 | 49.05 | 51.31 | 0.02 | 1.53 | 0.03 | 0.09 | 0.02 | 1.59 | 0.03 | 0.08 | 0.03 | 1.60 | 0.02 | 0.09 | 99.21 | 1,335.05 | 4.14 | 4.95 | 3.28 | 261.75 | 2.51 | 4.67 |
| 70 | 12.75 | 12.64 | 49.50 | 51.47 | 0.02 | 1.40 | 0.02 | 0.06 | 0.02 | 1.39 | 0.02 | 0.06 | 0.01 | 1.40 | 0.02 | 0.06 | 69.59 | 1,291.60 | 2.71 | 4.58 | 2.35 | 259.85 | 1.65 | 4.48 |
| 90 | 12.60 | 12.32 | 49.49 | 50.40 | 0.01 | 1.44 | 0.02 | 0.05 | 0.02 | 1.43 | 0.01 | 0.05 | 0.01 | 1.30 | 0.02 | 0.05 | 53.09 | 1,303.52 | 2.04 | 4.44 | 1.80 | 259.37 | 1.35 | 4.30 |
| 110 | 12.72 | 12.58 | 49.79 | 51.03 | 0.01 | 1.31 | 0.01 | 0.04 | 0.01 | 1.27 | 0.01 | 0.04 | 0.01 | 1.30 | 0.01 | 0.04 | 43.83 | 1,305.10 | 1.67 | 4.45 | 1.50 | 260.08 | 1.03 | 4.18 |

**Table 7.** Mean square error in the case that only independent variables were uniformly distributed with outliers. The errors were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110; and the numbers of outliers were 10%, 20%, and 30% of the sample size

| | | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | Outliers | $\beta_o$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_o$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| 10 | 10 | 33.28 | 33.28 | 66.69 | 133.24 | 4.82 | 4.81 | 4.54 | 19.27 | 14.45 | 14.45 | 14.30 | 57.83 | 3.44 | 3.44 | 3.65 | 13.78 | 7,242.97 | 2,628.16 | 12.10 | 12.10 | 405.70 | 405.70 | 0.15 | 0.15 |
| | 20 | 33.31 | 33.31 | 66.84 | 133.26 | 4.84 | 4.84 | 4.57 | 19.40 | 14.05 | 14.05 | 13.62 | 56.14 | 2.77 | 2.77 | 2.86 | 11.09 | 7,347.22 | 4,303.84 | 12.42 | 12.42 | 409.98 | 409.98 | 0.10 | 0.10 |
| | 30 | 32.94 | 32.94 | 67.45 | 131.68 | 4.90 | 4.90 | 4.71 | 19.61 | 14.47 | 14.47 | 14.38 | 57.87 | 2.71 | 2.72 | 2.81 | 10.90 | 7,404.62 | 5,540.32 | 12.53 | 12.53 | 412.76 | 412.76 | 0.07 | 0.07 |
| 30 | 10 | 12.71 | 13.22 | 49.08 | 50.81 | 0.39 | 0.72 | 0.41 | 1.58 | 0.73 | 1.64 | 0.77 | 2.92 | 0.04 | 0.09 | 0.05 | 0.16 | 7,690.89 | 3,674.87 | 27.09 | 27.61 | 464.67 | 471.32 | 0.80 | 0.85 |
| | 20 | 12.49 | 12.45 | 48.88 | 49.97 | 0.22 | 0.28 | 0.23 | 0.89 | 0.46 | 0.65 | 0.48 | 1.85 | 0.03 | 0.03 | 0.02 | 0.10 | 10,891.97 | 10,389.85 | 45.35 | 47.59 | 584.77 | 604.10 | 0.18 | 0.17 |
| | 30 | 12.43 | 12.40 | 48.79 | 49.71 | 0.17 | 0.17 | 0.16 | 0.69 | 0.24 | 0.23 | 0.24 | 0.94 | 0.05 | 0.05 | 0.04 | 0.19 | 7,822.08 | 7,646.10 | 24.20 | 24.85 | 475.91 | 484.86 | 0.27 | 0.21 |
| 50 | 10 | 12.32 | 12.35 | 49.21 | 49.30 | 0.95 | 1.02 | 0.89 | 3.78 | 0.02 | 0.04 | 0.03 | 0.10 | 0.97 | 1.00 | 0.91 | 3.87 | 4,274.22 | 1,825.33 | 7.80 | 6.36 | 315.47 | 295.53 | 1.69 | 2.69 |
| | 20 | 12.32 | 12.34 | 49.20 | 49.28 | 0.26 | 1.48 | 0.28 | 1.06 | 0.02 | 0.03 | 0.02 | 0.08 | 0.27 | 1.51 | 0.29 | 1.07 | 5,092.64 | 3,256.29 | 7.26 | 6.98 | 352.88 | 325.95 | 0.34 | 0.89 |
| | 30 | 12.40 | 12.31 | 49.13 | 49.60 | 1.39 | 1.67 | 1.29 | 5.57 | 1.37 | 1.36 | 1.27 | 5.49 | 0.27 | 1.18 | 0.29 | 1.07 | 3,156.98 | 2,409.11 | 3.88 | 3.88 | 267.15 | 267.48 | 2.61 | 2.62 |
| 70 | 10 | 12.38 | 12.25 | 49.27 | 49.51 | 1.06 | 1.15 | 1.02 | 4.24 | 0.25 | 0.39 | 0.26 | 1.03 | 1.05 | 1.04 | 1.05 | 4.19 | 3,453.20 | 1,168.07 | 5.47 | 5.65 | 277.88 | 270.23 | 2.65 | 3.16 |
| | 20 | 12.26 | 12.19 | 49.14 | 49.03 | 0.36 | 1.16 | 0.33 | 1.45 | 0.25 | 0.31 | 0.27 | 1.00 | 0.46 | 1.02 | 0.48 | 1.84 | 3,187.70 | 1,808.53 | 5.75 | 5.07 | 264.65 | 266.33 | 3.73 | 3.33 |
| | 30 | 12.37 | 12.22 | 49.16 | 49.47 | 1.17 | 1.08 | 1.05 | 4.67 | 0.31 | 0.35 | 0.32 | 1.26 | 1.12 | 1.07 | 0.99 | 4.47 | 3,688.43 | 1,682.90 | 5.78 | 5.92 | 283.45 | 279.95 | 2.51 | 2.68 |
| 90 | 10 | 12.18 | 12.69 | 48.87 | 48.74 | 0.06 | 0.07 | 0.05 | 0.22 | 0.08 | 0.13 | 0.07 | 0.29 | 0.10 | 0.12 | 0.09 | 0.39 | 5,939.77 | 3,165.19 | 11.10 | 10.55 | 373.69 | 375.38 | 0.19 | 0.18 |
| | 20 | 12.20 | 12.34 | 48.99 | 48.78 | 0.06 | 0.07 | 0.05 | 0.23 | 0.06 | 0.11 | 0.07 | 0.26 | 0.10 | 0.12 | 0.09 | 0.41 | 6,052.07 | 5,442.12 | 10.72 | 11.19 | 381.89 | 394.60 | 0.15 | 0.05 |
| | 30 | 12.29 | 12.37 | 49.03 | 49.17 | 2.18 | 2.57 | 2.02 | 8.74 | 0.08 | 0.08 | 0.07 | 0.31 | 2.30 | 2.69 | 2.10 | 9.20 | 6,199.93 | 5,910.83 | 10.48 | 10.92 | 383.15 | 389.59 | 0.14 | 0.08 |
| 110 | 10 | 12.38 | 12.22 | 49.42 | 49.52 | 0.15 | 0.64 | 0.16 | 0.60 | 0.02 | 0.06 | 0.03 | 0.09 | 0.20 | 0.86 | 0.19 | 0.80 | 1,421.27 | 174.98 | 0.50 | 25.76 | 194.35 | 84.59 | 9.80 | 81.51 |
| | 20 | 12.37 | 12.24 | 49.40 | 49.48 | 0.23 | 0.26 | 0.24 | 0.91 | 0.03 | 0.03 | 0.02 | 0.09 | 0.30 | 0.32 | 0.33 | 1.21 | 1,292.96 | 322.29 | 0.13 | 27.55 | 189.22 | 80.60 | 9.67 | 84.80 |
| | 30 | 12.25 | 12.23 | 49.28 | 49.01 | 0.03 | 0.04 | 0.04 | 0.13 | 0.04 | 0.05 | 0.05 | 0.19 | 0.13 | 0.14 | 0.15 | 0.52 | 1,228.09 | 1,348.56 | 0.04 | 28.32 | 183.85 | 78.86 | 10.91 | 86.14 |

**Table 8.** Mean square errors in the case that only errors were uniformly distributed with outliers. The independent variables were without outliers; $\beta_0 = 4$, $\beta_1 = 6$, $\beta_2 = -8$, $\beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110; and the numbers of outliers were 10%, 20%, and 30% of the sample size

| $n$ | Outliers | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\beta_o$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_o$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| | 10 | 20.71 | 20.71 | 69.18 | 82.87 | 0.30 | 0.30 | 0.78 | 1.20 | 0.30 | 0.30 | 0.70 | 1.22 | 0.30 | 0.30 | 0.80 | 1.20 | 605.33 | 433.43 | 27.90 | 27.90 | 22.34 | 22.34 | 16.63 | 16.63 |
| 10 | 20 | 26.88 | 26.88 | 87.95 | 107.57 | 0.45 | 0.45 | 1.17 | 1.80 | 0.47 | 0.47 | 1.06 | 1.87 | 0.45 | 0.45 | 1.24 | 1.78 | 615.63 | 440.69 | 28.12 | 28.12 | 22.73 | 22.73 | 16.82 | 16.82 |
| | 30 | 40.30 | 40.30 | 124.03 | 161.43 | 0.77 | 0.77 | 1.90 | 3.07 | 0.79 | 0.79 | 1.79 | 3.18 | 0.74 | 0.74 | 2.02 | 2.96 | 638.13 | 458.29 | 28.65 | 28.65 | 22.64 | 22.64 | 17.28 | 17.28 |
| | 10 | 16.23 | 0.91 | 61.46 | 64.92 | 0.05 | 0.30 | 0.11 | 0.20 | 0.05 | 0.32 | 0.11 | 0.21 | 0.05 | 0.32 | 0.11 | 0.22 | 165.15 | 113.19 | 7.61 | 7.01 | 6.25 | 6.50 | 4.31 | 4.55 |
| 30 | 20 | 17.14 | 1.43 | 63.13 | 68.57 | 0.09 | 1.71 | 0.20 | 0.36 | 0.09 | 1.81 | 0.19 | 0.37 | 0.09 | 1.87 | 0.18 | 0.38 | 166.44 | 112.51 | 7.66 | 7.07 | 6.23 | 6.51 | 4.37 | 4.56 |
| | 30 | 18.92 | 4.95 | 68.63 | 75.70 | 0.12 | 2.49 | 0.27 | 0.48 | 0.13 | 2.52 | 0.25 | 0.52 | 0.13 | 2.69 | 0.25 | 0.53 | 165.91 | 114.52 | 7.64 | 7.10 | 6.24 | 6.50 | 4.42 | 4.62 |
| | 10 | 15.81 | 12.13 | 60.64 | 63.25 | 0.06 | 0.02 | 0.11 | 0.25 | 0.06 | 0.02 | 0.12 | 0.23 | 0.05 | 0.02 | 0.11 | 0.22 | 92.27 | 71.04 | 3.68 | 4.28 | 3.68 | 3.65 | 2.49 | 2.72 |
| 50 | 20 | 20.46 | 16.21 | 74.29 | 81.83 | 0.11 | 0.07 | 0.20 | 0.43 | 0.10 | 0.07 | 0.22 | 0.42 | 0.10 | 0.08 | 0.21 | 0.42 | 95.87 | 75.08 | 3.71 | 4.35 | 3.37 | 3.72 | 2.52 | 2.79 |
| | 30 | 20.97 | 16.99 | 74.87 | 83.89 | 0.12 | 0.09 | 0.23 | 0.45 | 0.12 | 0.09 | 0.25 | 0.48 | 0.12 | 0.09 | 0.23 | 0.47 | 98.23 | 75.11 | 3.75 | 4.34 | 3.38 | 3.73 | 2.56 | 2.83 |
| | 10 | 14.34 | 11.58 | 53.44 | 57.35 | 0.03 | 0.02 | 0.07 | 0.12 | 0.03 | 0.02 | 0.07 | 0.13 | 0.03 | 0.02 | 0.07 | 0.13 | 73.41 | 51.87 | 2.90 | 2.95 | 2.20 | 2.77 | 1.76 | 1.79 |
| 70 | 20 | 17.17 | 14.09 | 62.90 | 68.66 | 0.05 | 0.04 | 0.11 | 0.22 | 0.06 | 0.04 | 0.11 | 0.23 | 0.05 | 0.04 | 0.12 | 0.21 | 75.23 | 54.45 | 2.95 | 2.98 | 2.23 | 2.78 | 1.79 | 1.84 |
| | 30 | 23.07 | 19.86 | 84.30 | 92.26 | 0.09 | 0.07 | 0.18 | 0.34 | 0.09 | 0.08 | 0.18 | 0.35 | 0.08 | 0.07 | 0.18 | 0.33 | 81.53 | 60.31 | 2.99 | 3.03 | 2.26 | 2.81 | 1.82 | 1.87 |
| | 10 | 15.36 | 14.04 | 60.90 | 61.46 | 0.05 | 0.03 | 0.09 | 0.19 | 0.05 | 0.03 | 0.10 | 0.19 | 0.05 | 0.03 | 0.09 | 0.19 | 54.75 | 48.47 | 2.17 | 2.48 | 1.84 | 1.96 | 1.39 | 1.45 |
| 90 | 20 | 19.19 | 18.49 | 75.48 | 76.75 | 0.07 | 0.06 | 0.14 | 0.29 | 0.07 | 0.06 | 0.16 | 0.29 | 0.08 | 0.07 | 0.15 | 0.30 | 56.85 | 51.99 | 2.17 | 2.54 | 1.87 | 2.00 | 1.42 | 1.47 |
| | 30 | 21.40 | 21.31 | 82.89 | 85.58 | 0.09 | 0.07 | 0.16 | 0.34 | 0.09 | 0.07 | 0.18 | 0.34 | 0.09 | 0.08 | 0.18 | 0.35 | 59.34 | 54.91 | 2.19 | 2.55 | 1.88 | 2.01 | 1.44 | 1.50 |
| | 10 | 13.44 | 12.25 | 53.22 | 53.75 | 0.02 | 0.01 | 0.04 | 0.08 | 0.02 | 0.01 | 0.04 | 0.08 | 0.02 | 0.01 | 0.04 | 0.08 | 49.39 | 38.80 | 1.91 | 2.02 | 1.58 | 1.70 | 1.14 | 1.21 |
| 110 | 20 | 15.92 | 15.05 | 62.44 | 63.68 | 0.04 | 0.03 | 0.07 | 0.15 | 0.04 | 0.03 | 0.07 | 0.15 | 0.04 | 0.03 | 0.07 | 0.15 | 51.13 | 39.75 | 1.92 | 2.03 | 1.60 | 1.71 | 1.15 | 1.22 |
| | 30 | 16.16 | 15.53 | 62.78 | 64.62 | 0.04 | 0.03 | 0.08 | 0.16 | 0.04 | 0.03 | 0.08 | 0.17 | 0.04 | 0.03 | 0.08 | 0.16 | 51.86 | 41.70 | 1.94 | 2.05 | 1.60 | 1.73 | 1.16 | 1.23 |

**Table 9.** Mean square errors in the case that independent variables and errors were gamma-distributed with no outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110

| $n$ | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\beta_o$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_o$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| 10 | 0.26 | 0.29 | 0.77 | 1.03 | 0.23 | 3.88 | 0.24 | 0.92 | 0.26 | 4.21 | 0.24 | 1.02 | 0.25 | 3.70 | 0.22 | 0.98 | 8.68 | 10.47 | 35.88 | 23.10 | 28.97 | 393.41 | 20.76 | 16.98 |
| 30 | 0.18 | 0.17 | 0.66 | 0.71 | 0.05 | 1.19 | 0.04 | 0.18 | 0.05 | 1.38 | 0.04 | 0.18 | 0.04 | 1.49 | 0.05 | 0.16 | 1.79 | 11.83 | 6.88 | 9.17 | 5.36 | 292.28 | 4.66 | 7.87 |
| 50 | 0.17 | 0.16 | 0.64 | 0.68 | 0.02 | 1.03 | 0.03 | 0.09 | 0.03 | 1.01 | 0.02 | 0.09 | 0.03 | 1.02 | 0.02 | 0.09 | 0.98 | 12.23 | 3.86 | 6.98 | 3.07 | 276.46 | 2.32 | 5.82 |
| 70 | 0.17 | 0.16 | 0.65 | 0.69 | 0.02 | 0.88 | 0.01 | 0.07 | 0.02 | 0.85 | 0.01 | 0.07 | 0.02 | 0.79 | 0.01 | 0.07 | 0.83 | 12.19 | 3.02 | 6.47 | 2.16 | 271.22 | 1.64 | 5.44 |
| 90 | 0.17 | 0.16 | 0.65 | 0.67 | 0.01 | 0.83 | 0.02 | 0.05 | 0.01 | 0.81 | 0.02 | 0.05 | 0.02 | 0.85 | 0.01 | 0.05 | 0.62 | 12.38 | 2.07 | 6.03 | 1.50 | 270.48 | 1.20 | 5.01 |
| 110 | 0.17 | 0.16 | 0.65 | 0.65 | 0.01 | 0.73 | 0.02 | 0.04 | 0.01 | 0.66 | 0.02 | 0.04 | 0.01 | 0.71 | 0.01 | 0.04 | 0.48 | 12.33 | 1.61 | 5.45 | 1.27 | 264.16 | 1.02 | 5.02 |

**Table 10.** Mean square error in the case that only independent variables were gamma distributed with outliers. The errors were without outliers; $\beta_0 = 4$, $\beta_1 = 6$, $\beta_2 = -8$, $\beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110; and the numbers of outliers were 10%, 20%, and 30% of the sample size

| $n$ | Outliers | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\beta_o$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_o$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| | 10 | 0.19 | 0.19 | 0.66 | 0.75 | 3.20 | 3.20 | 3.38 | 12.95 | 6.17 | 6.17 | 6.80 | 24.73 | 1.30 | 1.30 | 1.41 | 5.20 | 43.66 | 10.42 | 39.27 | 39.27 | 346.89 | 346.89 | 4.06 | 4.06 |
| 10 | 20 | 0.20 | 0.20 | 0.68 | 0.80 | 12.32 | 12.32 | 12.57 | 49.18 | 3.57 | 3.57 | 4.03 | 14.36 | 2.76 | 2.76 | 2.97 | 10.96 | 57.00 | 9.17 | 65.25 | 65.25 | 398.00 | 398.00 | 3.76 | 3.76 |
| | 30 | 0.24 | 0.24 | 0.74 | 1.02 | 18.45 | 18.45 | 17.82 | 74.76 | 10.60 | 10.60 | 10.10 | 42.18 | 1.89 | 1.89 | 1.85 | 8.14 | 42.35 | 11.60 | 18.12 | 18.12 | 268.77 | 268.77 | 6.95 | 6.95 |
| | 10 | 0.16 | 0.17 | 0.64 | 0.66 | 0.81 | 1.49 | 0.79 | 3.24 | 0.22 | 3.48 | 0.21 | 0.90 | 0.62 | 0.77 | 0.65 | 2.48 | 34.33 | 13.03 | 19.00 | 24.64 | 597.71 | 618.09 | 1.01 | 1.06 |
| 30 | 20 | 0.16 | 0.17 | 0.64 | 0.66 | 1.01 | 1.22 | 1.02 | 4.07 | 0.21 | 8.89 | 0.19 | 0.86 | 0.77 | 3.54 | 0.82 | 3.09 | 38.51 | 12.53 | 25.34 | 28.49 | 647.58 | 663.78 | 0.65 | 0.70 |
| | 30 | 0.17 | 0.20 | 0.65 | 0.70 | 0.94 | 1.25 | 0.82 | 3.74 | 0.22 | 14.01 | 0.19 | 0.88 | 0.81 | 3.77 | 0.73 | 3.25 | 147.28 | 7.15 | 22.76 | 23.40 | 681.27 | 711.13 | 1.44 | 2.35 |
| | 10 | 0.16 | 0.17 | 0.64 | 0.65 | 0.70 | 3.07 | 0.72 | 2.79 | 3.63 | 5.25 | 3.69 | 14.51 | 1.89 | 2.778 | 1.90 | 7.58 | 16.02 | 13.56 | 4.73 | 5.91 | 213.54 | 211.00 | 10.15 | 11.62 |
| 50 | 20 | 0.16 | 0.18 | 0.64 | 0.65 | 0.17 | 3.52 | 0.16 | 0.68 | 1.78 | 3.16 | 1.85 | 7.11 | 1.98 | 2.27 | 1.92 | 7.90 | 12.99 | 13.08 | 3.53 | 4.23 | 198.81 | 191.52 | 13.55 | 15.11 |
| | 30 | 0.16 | 0.17 | 0.64 | 0.67 | 0.16 | 6.49 | 0.17 | 0.65 | 1.17 | 3.68 | 1.30 | 4.67 | 1.17 | 1.55 | 1.24 | 4.68 | 40.21 | 13.16 | 4.91 | 5.57 | 203.04 | 191.44 | 13.79 | 15.53 |
| | 10 | 0.16 | 0.17 | 0.64 | 0.66 | 1.20 | 1.73 | 1.24 | 4.79 | 0.29 | 1.41 | 0.30 | 1.18 | 2.05 | 2.38 | 2.06 | 8.20 | 32.31 | 13.49 | 4.24 | 4.46 | 278.66 | 292.36 | 2.17 | 1.30 |
| 70 | 20 | 0.16 | 0.17 | 0.64 | 0.66 | 1.52 | 3.53 | 1.63 | 6.07 | 0.41 | 1.72 | 0.40 | 1.63 | 2.78 | 4.62 | 2.89 | 11.13 | 23.60 | 13.59 | 2.13 | 2.09 | 240.58 | 244.71 | 4.52 | 3.86 |
| | 30 | 0.17 | 0.16 | 0.66 | 0.68 | 2.60 | 7.79 | 2.78 | 10.40 | 0.48 | 2.27 | 0.49 | 1.90 | 4.27 | 6.44 | 4.53 | 17.07 | 48.78 | 11.63 | 2.30 | 2.84 | 235.84 | 247.15 | 5.47 | 4.63 |
| | 10 | 0.16 | 0.17 | 0.64 | 0.65 | 1.95 | 3.71 | 1.94 | 7.77 | 1.01 | 1.35 | 0.99 | 4.02 | 3.69 | 6.77 | 3.41 | 14.77 | 24.63 | 13.59 | 1.21 | 1.00 | 265.78 | 263.01 | 1.22 | 1.16 |
| 90 | 20 | 0.16 | 0.17 | 0.64 | 0.65 | 2.27 | 2.29 | 2.18 | 9.08 | 1.84 | 2.57 | 1.85 | 7.38 | 3.96 | 5.19 | 3.68 | 15.85 | 27.17 | 12.88 | 1.57 | 1.91 | 278.11 | 287.17 | 0.76 | 0.57 |
| | 30 | 0.17 | 0.18 | 0.66 | 0.71 | 2.28 | 2.39 | 2.06 | 9.12 | 3.13 | 6.46 | 3.26 | 12.53 | 5.83 | 11.24 | 6.51 | 23.33 | 62.11 | 10.82 | 0.97 | 1.23 | 256.72 | 264.36 | 1.47 | 1.19 |
| | 10 | 0.16 | 0.17 | 0.64 | 0.66 | 2.29 | 3.57 | 2.56 | 9.17 | 0.89 | 1.45 | 0.96 | 3.56 | 0.74 | 1.93 | 0.75 | 2.95 | 7.85 | 13.41 | 0.07 | 0.39 | 165.81 | 177.84 | 16.92 | 14.20 |
| 110 | 20 | 0.16 | 0.17 | 0.64 | 0.65 | 2.64 | 3.06 | 2.91 | 10.54 | 0.93 | 1.07 | 2.91 | 3.72 | 1.19 | 1.52 | 1.16 | 4.75 | 6.04 | 12.77 | 0.06 | 0.09 | 156.51 | 169.89 | 18.62 | 15.91 |
| | 30 | 0.16 | 0.17 | 0.64 | 0.66 | 2.80 | 2.82 | 2.67 | 11.22 | 0.52 | 0.59 | 0.54 | 2.09 | 2.55 | 2.55 | 2.49 | 10.21 | 41.40 | 12.70 | 0.03 | 0.08 | 161.80 | 157.95 | 15.53 | 15.83 |

**Table 11.** Mean square errors in the case that only errors were gamma distributed with outliers. The independent variables were without outliers; $\beta_0 = 4$, $\beta_1 = 6$, $\beta_2 = -8$, $\beta_3 = 10$; sample sizes were 10, 30, 50, 70, 90, and 110; and numbers of outliers were 10%, 20%, and 30% of the sample size

| $n$ | Outliers | MSE of parametric statistics | | | | | | | | | | | | | | | | MSE of nonparametric statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\beta_o$ | | | | $\beta_1$ | | | | $\beta_2$ | | | | $\beta_3$ | | | | $\beta_o$ | | $\beta_1$ | | $\beta_2$ | | $\beta_3$ | |
| | | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OLS | OLTS | PB | JK | OTS | AOTS | OTS | AOTS | OTS | AOTS | OTS | AOTS |
| 10 | 10 | 0.40 | 0.40 | 1.19 | 1.59 | 0.43 | 0.43 | 1.41 | 1.72 | 0.43 | 0.43 | 1.05 | 1.74 | 0.46 | 0.46 | 1.22 | 1.85 | 7.71 | 7.83 | 31.82 | 31.82 | 27.72 | 27.72 | 20.95 | 20.95 |
| | 20 | 0.62 | 0.62 | 1.79 | 2.47 | 0.79 | 0.79 | 2.47 | 3.14 | 0.78 | 0.78 | 1.93 | 3.14 | 0.83 | 0.83 | 2.18 | 3.31 | 7.99 | 7.77 | 32.33 | 32.33 | 27.77 | 27.77 | 21.50 | 21.50 |
| | 30 | 0.77 | 0.77 | 2.63 | 3.07 | 0.52 | 0.52 | 1.73 | 2.07 | 0.52 | 0.52 | 1.22 | 2.12 | 0.52 | 0.52 | 1.53 | 2.08 | 8.16 | 7.48 | 32.37 | 32.37 | 27.35 | 27.35 | 21.42 | 21.42 |
| 30 | 10 | 0.25 | 0.01 | 0.93 | 1.00 | 0.06 | 0.36 | 0.13 | 0.26 | 0.06 | 0.38 | 0.14 | 0.24 | 0.06 | 0.33 | 0.14 | 0.25 | 1.80 | 7.90 | 7.53 | 7.62 | 5.27 | 6.45 | 3.78 | 4.53 |
| | 20 | 0.34 | 0.01 | 1.26 | 1.36 | 0.11 | 1.10 | 0.22 | 0.45 | 0.11 | 1.05 | 0.24 | 0.41 | 0.11 | 1.11 | 0.25 | 0.45 | 1.87 | 7.81 | 7.52 | 7.63 | 5.35 | 6.63 | 3.83 | 4.57 |
| | 30 | 0.52 | 0.04 | 2.00 | 2.08 | 0.08 | 0.83 | 0.17 | 0.34 | 0.08 | 0.84 | 0.18 | 0.31 | 0.08 | 0.86 | 0.19 | 0.33 | 2.04 | 7.55 | 7.51 | 7.55 | 5.35 | 6.51 | 3.82 | 4.52 |
| 50 | 10 | 0.26 | 0.01 | 0.95 | 1.03 | 0.05 | 0.55 | 0.09 | 0.18 | 0.04 | 0.52 | 0.09 | 0.17 | 0.04 | 0.54 | 0.10 | 0.17 | 1.07 | 8.02 | 3.75 | 4.45 | 3.12 | 3.28 | 2.48 | 2.53 |
| | 20 | 0.35 | 0.01 | 1.27 | 1.40 | 0.07 | 1.87 | 0.16 | 0.29 | 0.07 | 1.73 | 0.14 | 0.29 | 0.07 | 1.76 | 0.16 | 0.28 | 1.16 | 7.93 | 3.81 | 4.51 | 3.19 | 3.36 | 2.51 | 2.58 |
| | 30 | 0.64 | 0.07 | 2.47 | 2.56 | 0.05 | 0.89 | 0.11 | 0.21 | 0.05 | 0.91 | 0.10 | 0.21 | 0.05 | 0.87 | 0.11 | 0.20 | 1.48 | 7.52 | 3.77 | 4.47 | 3.13 | 3.32 | 2.49 | 2.55 |
| 70 | 10 | 0.16 | 0.13 | 0.60 | 0.64 | 0.02 | 0.01 | 0.04 | 0.07 | 0.02 | 0.01 | 0.04 | 0.07 | 0.02 | 0.01 | 0.04 | 0.07 | 0.75 | 8.02 | 2.73 | 2.91 | 1.94 | 2.32 | 1.57 | 1.67 |
| | 20 | 0.23 | 0.19 | 0.84 | 0.91 | 0.04 | 0.03 | 0.07 | 0.13 | 0.04 | 0.03 | 0.07 | 0.12 | 0.04 | 0.03 | 0.07 | 0.13 | 0.81 | 7.93 | 2.74 | 2.92 | 1.97 | 2.37 | 1.59 | 1.68 |
| | 30 | 0.42 | 0.40 | 1.61 | 1.67 | 0.02 | 0.01 | 0.05 | 0.09 | 0.02 | 0.01 | 0.05 | 0.08 | 0.02 | 0.01 | 0.05 | 0.09 | 1.03 | 7.61 | 2.74 | 2.91 | 1.95 | 2.34 | 1.58 | 1.67 |
| 90 | 10 | 0.20 | 0.12 | 0.78 | 0.79 | 0.02 | 0.01 | 0.04 | 0.08 | 0.02 | 0.01 | 0.04 | 0.08 | 0.02 | 0.01 | 0.04 | 0.08 | 0.60 | 7.98 | 1.94 | 2.31 | 1.64 | 1.81 | 1.18 | 1.22 |
| | 20 | 0.28 | 0.18 | 1.09 | 1.13 | 0.05 | 0.02 | 0.07 | 0.14 | 0.04 | 0.02 | 0.08 | 0.14 | 0.04 | 0.02 | 0.08 | 0.15 | 0.65 | 7.91 | 1.97 | 2.33 | 1.68 | 1.83 | 1.20 | 1.24 |
| | 30 | 0.48 | 0.36 | 1.89 | 1.93 | 0.03 | 0.02 | 0.05 | 0.10 | 0.03 | 0.02 | 0.06 | 0.11 | 0.03 | 0.02 | 0.06 | 0.11 | 0.87 | 7.61 | 1.95 | 2.33 | 1.66 | 1.81 | 1.18 | 1.23 |
| 110 | 10 | 0.16 | 0.11 | 0.62 | 0.62 | 0.02 | 0.01 | 0.02 | 0.03 | 0.02 | 0.01 | 0.02 | 0.03 | 0.02 | 0.01 | 0.02 | 0.03 | 0.47 | 7.97 | 1.56 | 1.83 | 1.29 | 1.53 | 0.97 | 1.10 |
| | 20 | 0.20 | 0.14 | 0.80 | 0.82 | 0.02 | 0.01 | 0.03 | 0.05 | 0.02 | 0.01 | 0.03 | 0.06 | 0.02 | 0.01 | 0.03 | 0.05 | 0.50 | 7.91 | 1.57 | 1.83 | 1.30 | 1.54 | 0.98 | 1.12 |
| | 30 | 0.26 | 0.20 | 1.03 | 1.04 | 0.02 | 0.01 | 0.03 | 0.04 | 0.02 | 0.01 | 0.02 | 0.05 | 0.02 | 0.01 | 0.02 | 0.05 | 0.57 | 7.80 | 1.57 | 1.83 | 1.29 | 1.53 | 0.97 | 1.11 |

## 4. Discussions

This research aimed to compare the efficiencies of two nonparametric and four parametric statistics point estimation methods in multiple regression models in the presence of outliers. The parametric point estimation methods to compare were Ordinary Least Squares (OLS), Ordinary Least Trimmed Squares (OLTS), Parametric Bootstrap (PB), and Jackknife (JK). The nonparametric methods were Optimal Theil-Sen (OTS) and Adjusted Optimal Trimmed Theil-Sen (AOTS) (proposed by the authors).

The data were randomized from simulations of three cases: (1) independent variables and errors were distributed without outliers, (2) only independent variables were distributed with outliers, and (3) only errors were distributed with outliers. $Y$-intercept and regression coefficients were estimated with the six estimation methods.

In the case that independent variables and errors were normally distributed with no outliers, parametric OLS and nonparametric OTS provided the minimum mean squared error. This parametric OLS result agrees with the result reported in [11] that compared OLS and Theil-Sen (TS) when there were no outliers in the data. The authors concluded that OLS performance was better than TS.

In the case that only the independent variables were normally distributed with outliers, the errors were normally distributed without outliers. For parametric methods, PB provided the minimum mean squared error. For nonparametric statistics, AOTS provided a lower mean square error than OTS. These results are consistent with the results reported by [18] that compared several parametric and nonparametric estimation methods in a simple linear regression analysis. They considered a mean absolute deviation. The results showed that OTS provided the lowest mean absolute deviation, followed by TS, then Theil-Sen weighted median-1 method.

Finally, for the case that only the errors were normally distributed with outliers, while the independent variables were normally distributed without outliers, parametric OLTS and nonparametric OTS provided the minimum mean squared error. These results agree well with the results reported by [12] that OLTS was more efficient than the Ordinary Least Median Squares (OLMS) method. The authors of [15] proposed that OLTS should replace OLMS for a large sample size. The authors of [18] compare many parametric and nonparametric estimation methods in simple linear regression analysis. They considered a mean absolute deviation. The results showed that OTS provided the lowest mean absolute deviation, followed by TS, then Theil-Sen weighted median-1 method. Finally, the experimentation of [8] concluded that OTS does not necessarily satisfy the assumption of the symmetric distribution of $d_i$. This method is suitable for statistics with outliers.

Previous research has not investigated the uniform and gamma distributions in their experiments on data with outliers for multiple regression. Therefore, the authors cannot compare their results with ours. Nevertheless, in the conclusion, the authors summarize the results of these two distributions in this study in all 3 situations.

## 5. Conclusions

This study compared the efficiency of various methods for estimating parameters in a multiple regression model with outliers. We evaluated traditional methods like Ordinary Least Squares (OLS) alongside more robust techniques like Ordinary Least Trimmed Squares (OLTS), Parametric Bootstrap (PB), Jackknife (JK), Optimal Theil-Sen (OTS), and our proposed Adjusted Optimal Trimmed Theil-Sen (AOTS).

Our findings, summarized in Table 12, show that the best-performing methods varied depending on the distribution of the data and the location of the outliers (in the independent variables or the errors).

**Table 12.** The highest number and the second highest number of minimum mean square error for each situation in three cases

| Distributions | Statistics | Independent variables and errors have no outliers | Independent variables have outliers | Errors have outliers |
|---|---|---|---|---|
| Normal | Parametric | OLS, PB | PB, OLS | OLTS, OLS |
| | Nonparametric | OTS, AOTS | AOTS, OTS | OTS, AOTS |
| Uniform | Parametric | OLS, OLTS | OLS, PB | OLTS, OLS |
| | Nonparametric | OTS, AOTS | AOTS, OTS | OTS, AOTS |
| Gamma | Parametric | PB, OLS | OLS, PB | OLTS, OLS |
| | Nonparametric | OTS, AOTS | OTS, AOTS | OTS, AOTS |

Generally, for parametric methods:

PB performed well with outliers in the independent variables for normal and gamma distributions.

OLTS was most effective with outliers in the errors across all distributions.

For nonparametric methods:

AOTS often outperformed OTS with outliers in the independent variables, particularly for normal and uniform distributions.

OTS was generally better with outliers in the errors.

Key takeaways:

AOTS is a strong alternative to OTS, especially with outliers in the independent variables.

AOTS offers the advantages of nonparametric methods (fewer assumptions about the data) while providing comparable or better accuracy than OTS in the presence of outliers.

This study highlights the importance of choosing an appropriate estimation method based on the specific characteristics of the data. AOTS provides a robust and efficient option for handling outliers in multiple regression models.

Recommended future works may include other non-parametric methods, a bigger sample size (200, 500), and a higher number of outliers (40%, 50%) in the investigation.

## Data availability

The data used to support this study were simulated from normal, uniform, and gamma distributions, using the R studio program.

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## Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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# Appendix

Parametric point estimation

**Table A1.** The values of independent variables and dependent variable for the ordinary least squares and the ordinary least trimmed squares methods in the case that the independent variables and errors are distributed normally with no outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10

| No. | $X_1$ | $X_2$ | $X_3$ | $Y$ |
|---|---|---|---|---|
| 1 | 0.2621 | 0.3546 | 0.4657 | 7.4190 |
| 2 | 1.2543 | 1.4344 | −0.8643 | −8.8928 |
| 3 | −0.1121 | 0.5406 | 2.4363 | 23.6227 |
| 4 | 1.8164 | 0.4995 | 1.0093 | 22.0630 |
| 5 | 3.1115 | 0.2322 | 0.5855 | 26.3104 |
| 6 | 1.5811 | 2.0497 | 1.4449 | 11.5539 |
| 7 | 1.7927 | 1.3649 | 0.9961 | 14.4089 |
| 8 | 0.7818 | 0.1332 | −0.4092 | 3.2929 |
| 9 | 0.6134 | 2.1914 | −0.1718 | −12.7707 |
| 10 | 2.1249 | 2.9609 | 1.8039 | 11.6332 |

**Table A2.** The values of independent variables and dependent variable for the parametric bootstrap method in the case that the independent variables and errors are distributed normally with no outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10

| No. | $X_1$ | $X_2$ | $X_3$ | $Y$ | $Y$ bootstrap |
|---|---|---|---|---|---|
| 1 | 0.4131 | −0.4129 | 0.6219 | 16.4967 | 25.7273 |
| 2 | 1.0666 | −0.7929 | 0.4126 | 22.3790 | 24.4782 |
| 3 | 0.6060 | 0.7973 | 1.2366 | 12.9270 | 17.6251 |
| 4 | 0.4386 | 0.1766 | −0.5243 | −1.2908 | −25.6442 |
| 5 | 0.7785 | 1.6148 | 0.7906 | 2.6680 | 14.1114 |
| 6 | 1.6449 | −2.2881 | 1.8379 | 51.4135 | −0.5373 |
| 7 | −0.4673 | 2.6642 | 2.4310 | 5.2794 | 16.2310 |
| 8 | 0.5146 | 3.7416 | 0.8815 | −12.2905 | 41.9157 |
| 9 | −0.6393 | −0.0103 | 0.4275 | 3.3460 | 6.5415 |
| 10 | 0.9733 | 0.2944 | 2.2333 | 30.9207 | −4.6048 |

**Table A3.** The values of independent variables and dependent variable for the jackknife method in the case that the independent variables and errors are distributed normally with no outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10

| No. | $X_1$ | $X_2$ | $X_3$ | $Y$ ols | $Y$ jackknife |
|---|---|---|---|---|---|
| 1 | 0.2621 | 0.3546 | 0.4657 | 7.4190 | 7.3193 |
| 2 | 1.2543 | 1.4344 | −0.8643 | −8.8928 | −9.5020 |
| 3 | −0.1121 | 0.5406 | 2.4363 | 23.6227 | 24.1761 |
| 4 | 1.8164 | 0.4995 | 1.0093 | 22.0630 | 22.3919 |
| 5 | 3.1115 | 0.2322 | 0.5855 | 26.3104 | 26.7264 |
| 6 | 1.5811 | 2.0497 | 1.4449 | 11.5539 | 11.7699 |
| 7 | 1.7927 | 1.3649 | 0.9961 | 14.4089 | 14.5954 |
| 8 | 0.7818 | 0.1332 | −0.4092 | 3.2929 | 2.9808 |
| 9 | 0.6134 | 2.1914 | −0.1718 | −12.7707 | −13.3380 |
| 10 | 2.1249 | 2.9609 | 1.8039 | 11.6332 | |

**Table A4.** The values of independent variables and dependent variable for the ordinary least squares and the ordinary least trimmed squares methods in the case that only the independent variables were normally distributed with outliers. The errors were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10; and the number of outliers was 10% of the sample size

| No. | $X_1$ | $X_2$ | $X_3$ | $Y$ |
|---|---|---|---|---|
| 1 | −0.1121 | 0.1332 | −0.8643 | −6.1265 |
| 2 | 0.2621 | 0.2322 | −0.4092 | −1.0032 |
| 3 | 0.6134 | 0.3546 | −0.1718 | 2.8959 |
| 4 | 0.7818 | 0.4995 | 0.4657 | 8.3426 |
| 5 | 1.2543 | 0.5406 | 0.5855 | 11.3220 |
| 6 | 1.5811 | 1.3649 | 0.9961 | 10.8462 |
| 7 | 1.7927 | 1.4344 | 1.0093 | 11.5592 |
| 8 | 1.8164 | 2.0497 | 1.4449 | 12.4080 |
| 9 | 2.1249 | 2.1914 | 1.8039 | 15.6874 |
| 10 | 4.2665 | 5.9709 | 5.1331 | 35.1317 |

Note: The numbers in bold were outliers

**Table A5.** The values of independent variables and dependent variable for the parametric bootstrap method in the case that only the independent variables were normally distributed, but with outliers. The errors were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10; and the number of outliers was 10% of the sample size

| No. | $X_1$ | $X_2$ | $X_3$ | $Y$ | $Y$ bootstrap |
|---|---|---|---|---|---|
| 1 | −0.1121 | 0.1332 | −0.8643 | −7.2419 | 6.0679 |
| 2 | 0.2621 | 0.2322 | −0.4092 | 0.0242 | 9.1149 |
| 3 | 0.6134 | 0.3546 | −0.1718 | 3.3031 | 29.5331 |
| 4 | 0.7818 | 0.4995 | 0.4657 | 8.3634 | −8.2364 |
| 5 | 1.2543 | 0.5406 | 0.5855 | 12.6148 | 41.3100 |
| 6 | 1.5811 | 1.3649 | 0.9961 | 12.8498 | 4.9242 |
| 7 | 1.7927 | 1.4344 | 1.0093 | 13.3428 | 11.1506 |
| 8 | 1.8164 | 2.0497 | 1.4449 | 13.7036 | 34.6552 |
| 9 | 2.1249 | 2.1914 | 1.8039 | 17.9927 | −0.8997 |
| 10 | 4.2665 | 5.9709 | 5.1331 | 34.2866 | −25.4630 |

**Table A6.** The values of independent variables and dependent variable for the jackknife method in the case that only the independent variables were normally distributed, but with outliers. The errors were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10; and the number of outliers was 10% of the sample size

| No. | $X_1$ | $X_2$ | $X_3$ | $Y$ ols | $Y$ jackknife |
|---|---|---|---|---|---|
| 1 | −0.1121 | 0.1332 | −0.8643 | −6.1265 | −5.8221 |
| 2 | 0.2621 | 0.2322 | −0.4092 | −1.0032 | −1.3550 |
| 3 | 0.6134 | 0.3546 | −0.1718 | 2.8959 | 1.8649 |
| 4 | 0.7818 | 0.4995 | 0.4657 | 8.3426 | 7.4703 |
| 5 | 1.2543 | 0.5406 | 0.5855 | 11.3220 | 9.1838 |
| 6 | 1.5811 | 1.3649 | 0.9961 | 10.8462 | 9.5094 |
| 7 | 1.7927 | 1.4344 | 1.0093 | 11.5592 | 9.7251 |
| 8 | 1.8164 | 2.0497 | 1.4449 | 12.4080 | 11.9121 |
| 9 | 2.1249 | 2.1914 | 1.8039 | 15.6874 | 14.7517 |
| 10 | 4.2665 | 5.9709 | 5.1331 | 35.1317 | |

**Table A7.** The values of independent variables and dependent variable for the ordinary least squares and the ordinary least trimmed squares methods in the case that only the errors were normally distributed, but with outliers. The independent variables were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10; and the number of outliers was 10% of the sample size

| No. | $X_1$ | $X_2$ | $X_3$ | Error | $Y$ |
|---|---|---|---|---|---|
| 1 | 0.2621 | 0.3546 | 0.4657 | −1.2014 | 24.1617 |
| 2 | 1.2543 | 1.4344 | −0.8643 | −0.3568 | 29.3960 |
| 3 | −0.1121 | 0.5406 | 2.4363 | −0.3008 | 30.4386 |
| 4 | 1.8164 | 0.4995 | 1.0093 | −0.2406 | −11.5751 |
| 5 | 3.1115 | 0.2322 | 0.5855 | 0.016 | −1.4051 |
| 6 | 1.5811 | 2.0497 | 1.4449 | 0.026 | −5.3325 |
| 7 | 1.7927 | 1.3649 | 0.9961 | 0.2572 | −0.8618 |
| 8 | 0.7818 | 0.1332 | −0.4092 | 0.5315 | −4.5203 |
| 9 | 0.6134 | 2.1914 | −0.1718 | 0.6104 | −1.1425 |
| 10 | 2.1249 | 2.9609 | 1.8039 | 2.5650 | 13.9375 |

**Table A8.** The values of independent variables and dependent variable for the parametric bootstrap method in the case that only the errors were normally distributed, but with outliers. The independent variables were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10; and the number of outliers was 10% of the sample size

| No. | $X_1$ | $X_2$ | $X_3$ | Error | $Y$ | $Y$ bootstrap |
|---|---|---|---|---|---|---|
| 1 | 2.7531 | −1.7425 | −0.4538 | −1.2014 | 28.7187 | 15.0207 |
| 2 | −0.1274 | 0.8577 | 1.7592 | −0.3568 | 13.6089 | 32.1856 |
| 3 | 2.9081 | 1.2277 | 1.8807 | −0.3008 | 30.1334 | 5.8990 |
| 4 | 0.7676 | 1.5763 | 1.1410 | −0.2406 | 7.1645 | 20.0256 |
| 5 | 0.9198 | 1.1584 | 1.3816 | 0.016 | 14.0836 | 7.8059 |
| 6 | 0.9526 | 1.4824 | 0.6723 | 0.026 | 4.6053 | 8.0166 |
| 7 | 0.9988 | 2.4626 | 0.4123 | 0.2572 | −5.3277 | 59.1874 |
| 8 | −0.2016 | 0.4997 | 1.1556 | 0.5315 | 10.8803 | 15.9080 |
| 9 | 2.2386 | 1.4295 | 2.7742 | 0.6104 | 34.3484 | 1.5718 |
| 10 | −1.0898 | 1.3244 | 1.5326 | 2.5650 | 4.7571 | −3.3202 |

**Table A9.** The values of independent variables and dependent variable for the jackknife method in the case that only the errors were normally distributed, but with outliers. The independent variables were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10; and the number of outliers was 10% of the sample size

| No. | $X_1$ | $X_2$ | $X_3$ | Error | $Y$ ols | $Y$ jackknife |
|---|---|---|---|---|---|---|
| 1 | 2.6935 | 0.7817 | 1.1456 | −1.2014 | 24.1617 | 23.5546 |
| 2 | 1.2302 | −1.0724 | 0.9792 | −0.3568 | 29.3960 | 29.4471 |
| 3 | 1.7861 | −0.7018 | 1.0409 | −0.3008 | 30.4386 | 30.2409 |
| 4 | 1.0240 | 3.8116 | 0.9014 | −0.2406 | −11.5751 | −11.4412 |
| 5 | 1.3893 | 2.0074 | 0.2302 | 0.016 | −1.4051 | −1.3537 |
| 6 | 1.4581 | 1.1998 | −0.8508 | 0.026 | −5.3325 | −5.1949 |
| 7 | −1.2086 | 0.5878 | 0.6835 | 0.2572 | −0.8618 | 0.2737 |
| 8 | 0.8571 | 2.4274 | 0.5225 | 0.5315 | −4.5203 | −4.2700 |
| 9 | −0.0652 | 1.2708 | 0.4804 | 0.6104 | −1.1425 | −0.4839 |
| 10 | 0.6938 | 0.6522 | 0.8428 | 2.5650 | 13.9375 | |

Nonparametric point estimation

**Table A10.** The values of independent variables and dependent variable for the optimal Theil-Sen and the adjusted optimal trimmed Theil-Sen methods in the case that both the independent variables and errors are distributed normally with no outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10

| No. | $X_1$ | $X_2$ | $X_3$ | Error | $Y$ |
|---|---|---|---|---|---|
| 1 | 0.2621 | 0.3546 | 0.4657 | 0.0260 | 7.4190 |
| 2 | 1.2543 | 1.4344 | −0.8643 | −0.3008 | −8.8928 |
| 3 | −0.1121 | 0.5406 | 2.4363 | 0.2572 | 23.6227 |
| 4 | 1.8164 | 0.4995 | 1.0093 | 1.0678 | 22.0630 |
| 5 | 3.1115 | 0.2322 | 0.5855 | −0.3568 | 26.3104 |
| 6 | 1.5811 | 2.0497 | 1.4449 | 0.0160 | 11.5539 |
| 7 | 1.7927 | 1.3649 | 0.9961 | 0.6104 | 14.4089 |
| 8 | 0.7818 | 0.1332 | −0.4092 | −0.2406 | 3.2929 |
| 9 | 0.6134 | 2.1914 | −0.1718 | −1.2014 | −12.7707 |
| 10 | 2.1249 | 2.9609 | 1.8039 | 0.5315 | 11.6332 |

**Table A11.** The values of independent variables and dependent variable for the optimal Theil-Sen and the adjusted optimal trimmed Theil-Sen methods in the case that only the independent variables were normally distributed, but with outliers. The errors were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10; and the number of outliers was 10% of the sample size

| No. | $X_1$ | $X_2$ | $X_3$ | Error | $Y$ |
|---|---|---|---|---|---|
| 1 | −0.1121 | 0.1332 | −0.8643 | 0.2547 | −6.1265 |
| 2 | 0.2621 | 0.2322 | −0.4092 | −0.6262 | −1.0032 |
| 3 | 0.6134 | 0.3546 | −0.1718 | −0.2297 | 2.8959 |
| 4 | 0.7818 | 0.4995 | 0.4657 | −1.0092 | 8.3426 |
| 5 | 1.2543 | 0.5406 | 0.5855 | −1.7340 | 11.3220 |
| 6 | 1.5811 | 1.3649 | 0.9961 | −1.6822 | 10.8462 |
| 7 | 1.7927 | 1.4344 | 1.0093 | −1.8148 | 11.5592 |
| 8 | 1.8164 | 2.0497 | 1.4449 | −0.5418 | 12.4080 |
| 9 | 2.1249 | 2.1914 | 1.8039 | −1.5698 | 15.6874 |
| 10 | 4.2665 | 5.9709 | 5.1331 | 1.9689 | 35.1317 |

**Table A12.** The values of independent variables and dependent variable for the optimal Theil-Sen and the adjusted optimal trimmed Theil-Sen methods in the case that only the errors were normally distributed, but with outliers. The independent variables were without outliers; $\beta_0 = 4,\ \beta_1 = 6,\ \beta_2 = -8,\ \beta_3 = 10$; the sample size was 10; and the number of outliers was 10% of the sample size

| No. | $X_1$ | $X_2$ | $X_3$ | Error | $Y$ |
|---|---|---|---|---|---|
| 1 | 1.6935 | −0.2183 | 0.1456 | −1.2014 | 16.1617 |
| 2 | 0.2302 | −2.0724 | −0.0208 | −0.3568 | 21.3960 |
| 3 | 0.7861 | −1.7018 | 0.0409 | −0.3008 | 22.4386 |
| 4 | 0.0240 | 2.8116 | −0.0986 | −0.2406 | −19.5751 |
| 5 | 0.3893 | 1.0074 | −0.7698 | 0.016 | −9.4051 |
| 6 | 0.4581 | 0.1998 | −1.8508 | 0.026 | −13.3325 |
| 7 | −2.2086 | −0.4122 | −0.3165 | 0.2572 | −8.8618 |
| 8 | −0.1429 | 1.4274 | −0.4775 | 0.5315 | −12.5203 |
| 9 | −1.0652 | 0.2708 | −0.5196 | 0.6104 | −9.1425 |
| 10 | −0.3062 | −0.3478 | −0.1572 | 2.5650 | 5.9375 |