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Research Article

# Parameter Identification and Prediction of the Rőssler System with Complete and Incomplete Information: Two Known and One Unknown State Variables

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**Abstract:** Parameters identification algorithms are formulated for the system of equations of Rőssler's type. The problem is solved for the cases of complete and incomplete information about functions of the system. If there is complete information about the state variables, it is possible to apply the algorithms discussed to any system of linear or nonlinear ordinary differential equations of arbitrary order in the Cauchy form that linearly depends on the unknown parameters (or groups of unknown parameters). The problems of parameter identification in the case of incomplete information about the state variables must be solved individually, depending on the possibility (or impossibility) of eliminating unknown steady states from the system of equations. Most real-world problems in fields such as chemical kinetics, mathematical ecology, predator-prey dynamics in game reserves, and the spread of infectious diseases belong to this class of problems, in which the algorithms discussed demonstrate their applicability. In the present paper the case of one unknown function is considered. Lemmas about possibilities on complete and incomplete parameter identification are formulated and proven. The algorithms of the parameter identification are formulated in the process of the constructive proofs of the lemmas. Numerical examples and graphs of solutions are considered which demonstrate efficiency and accuracy of the developed algorithms. In the proposed paper, the integration approach is used instead of the differential approach because it allows for the smoothing of discrete data, thereby reducing the estimation errors of the unknown parameters.

***Keywords*:** Rőssler's attractor, least squares, Chaotic systems, parameter identification, prediction, incomplete information

**MSC:** 65L05, 34K06, 34K28

## 1. Introduction

The Rőssler attractor [1] is one of the simplest dynamical systems which manifest chaotic behavior [2]. It is described by nonlinear system of three ordinary differential equations. Solutions of this system behave similarly to the solutions of the Lorenz attractor [3] but with one stable manifold. Originally the Rőssler system was considered as a prototype system to the Lorenz model of turbulence which contains only one nonlinearity of the second order in one variable. This system was proposed as model for hypothetical reaction in the field of chemical thermodynamics in which oscillations of concentration demonstrate chaotic behavior [1, 4]. In [5–8] it was demonstrated that both Lorenz and Rőssler attractors


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can simulate mechanical or electromechanical self-oscillating systems with inertial excitation. Hence, in these systems it is possible to expect both regular and chaotic effects analogous to the effects observing in the Rőssler and Lorenz systems depending on their parameters. Another field of application of the Rőssler model is electronic signals modulation for analysis of evolution of the optical absorptive effects exhibited by plasmonic nanoparticles [9]. Several authors refer to importance of the parameter identification of chaotic dynamical systems especially with application to high precision model synchronization. They developed special methods such as the observer/Kalman filter identification and bilinear transform discretization [10].

All applications of the Rőssler attractor including chemical kinetics deal with substantial simplifications of the original model. That is why it is interesting to consider models in which the Rőssler attractor exactly describe dynamics of the mechanical system with inertial or aperiodic excitation [5–8]. In this paper we considered situations with linear and nonlinear mechanical systems with linear and nonlinear feedbacks. These applications are important for the systems with limited power supply. Dynamics of actuators becomes important at present time due to a broad development of Microelectromechanical systems (MEMS) with low power actuators. In the present paper it is demonstrated that at specific feedback the problem of parameter identification of the MEMS-system can be converted into the problem of the parameter identification of the Rőssler system [11, 12].

Currently, many researchers pay much attention to solution of inverse problems of sparse identification of nonlinear dynamical systems using methods of artificial intelligence [13–17]. These works mainly consider systems with complete information about their state variables at discrete time instants. Some authors analyze dynamics of nonlinear and chaotic systems from incomplete observations of their state variables [18]. Despite the robustness and accuracy the abovementioned methods need substantial time for accurate parameter evaluation. In the present paper the authors follow the methods, which were demonstrated in [19, 20] with application to the Lorenz system, and develop fast and accurate methods of the parameter identification for the Rőssler system. The parameters are estimated from complete and incomplete information about the state variables. Incomplete information means that only two from three state variables are observed at particular time instants on a given time interval. Knowledge of the third state variable is limited mainly by its initial and terminal values. It is proven that in this case it is possible to fully identify the Rőssler system, i.e., estimate all unknown parameters and restore information about its unknown state variable between the initial and terminal time instants. Moreover, considering the terminal values of the state variables as new initial conditions it is possible to make prediction of the system behavior to a next finite time interval.

The abovementioned analogy between the Rőssler attractor and some electromechanical systems can help to identify these systems in terms of Rőssler's parameters which makes it possible the systematic topological characterization of the electromechanical systems [21, 22]. In this present paper, we have considered the system with two known and one unknown state variable of the Rőssler system in a deterministic case where random components are absent.

## 2. Parameters and conditions for their complete identification

In this paper we consider a system of equations of the Rőssler type:

$$\begin{cases} \dfrac{dX(t)}{dt} + K_1 Y(t) + K_2 Z(t) = 0, \\[2ex] \dfrac{dY(t)}{dt} - K_3 X(t) - K_4 Y(t) = 0, \\[2ex] \dfrac{dZ(t)}{dt} - K_5 - K_6 X(t) Z(t) + K_7 Z(t) = 0, \end{cases} \tag{1}$$

where $K_1, K_2, \ldots, K_7$ are constant unknown parameters and $X, Y, Z$ are state variables which are fully or partially known on $t \in [0, T]$ with $(N+1)$ points and at $t_i = \frac{T}{N}i$, $(i = 0, 1, \ldots, N)$. We present different cases in this section.

The main objective here is to identify as many parameters $K_1, K_2, \ldots, K_7$ as possible and find conditions of their complete identification. The second problem is to identify unknown state variables between $X, Y, Z$. The third problem is to predict behavior of system (1) with finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$. In this paper four cases are considered: in the first case we assume that complete information about state variables $X, Y, Z$ is given with $(N+1)$ points, i.e., that $3(N+1)$ values of $X_i = X\left(t_i = \frac{T}{N}i\right)$, $Y_i = Y(t_i)$, $Z_i = Y(t_i)$, $(i = 0, 1, \ldots, N)$, are known. In our second case we assume that information about $X(t)$ is unknown (or partially known at several time instances only), but information about $Y_i = Y(t_i)$, $Z_i = Y(t_i)$ is available. In the third case it is assumed that information about $Y(t)$ is absent (or partially known) and $X_i = X(t_i)$, $Z_i = Z(t_i)$ are known.

Finally, the fourth case deals with information about $Z(t)$ is assumed to be unknown (or partially known) and information about $X_i = X(t_i)$, $Y_i = Y(t_i)$ is available. The main results are formulated as lemmas and constructive ways of evaluation of parameters are formulated afterwards.

### 2.1 *Complete knowledge about state variables X, Y, Z*

Here it is assumed that state variables $X, Y, Z$ are known in $N+1 >> 1$ points, where $t_i = \frac{T}{N}i$, $(i = 0, 1, \ldots, N)$. In this case the following lemma holds true:

**Lemma 1** At complete knowledge of state variables $X, Y, Z$ in $N+1 >> 7$ points at $t_i = \frac{T}{N}i$, $(i = 0, 1, \ldots, N)$ all parameters $K_k$, $(k = 0, 1, \ldots, 7)$, can be identified and behavior of system (1) can be predicted for finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$.

**Proof.** We prove the lemma constructively demonstrating the method of identification of parameters $K_j$, $(j = 0, 1, \ldots, 7)$. Integrating equations of system (1) on $t \in (0, T]$. One can obtain:

$$
\begin{aligned}
&[\Delta X(t)] + K_1[J_2(t)] + K_2[J_3(t)] = 0, \\
&[\Delta Y(t)] - K_3[J_1(t)] - K_4[J_2(t)] = 0, \\
&[\Delta Z(t)] - K_5[t] - K_6[J_4(t)] + K_7[J_3(t)] = 0,
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
&\Delta X(t) = X(t) - X(0), \quad \Delta Y(t) = Y(t) - Y(0), \quad \Delta Z(t) = Z(t) - Z(0), \\
&J_1(t) = \int_0^t X(\tau)\,d\tau, \quad J_2(t) = \int_0^t Y(\tau)\,d\tau, \quad J_3(t) = \int_0^t Z(\tau)\,d\tau, \quad J_4(t) = \int_0^t X(\tau)Z(\tau)\,d\tau.
\end{aligned}
\tag{3}
$$

□

In the deterministic case, the integrals in Equation (3) and other integrals mentioned in this paper are calculated using the standard Simpson's rule. The errors of the numerical estimates are obtained using Simpson's method are well-known and are given in Tables 1 and 2 of Section 3.

Composing three goal functions:

$$G_1 = G_1(K_1, K_2) = \frac{1}{2}\sum_{j=1}^{N}\left\{K_1\left[J_{2j}\right] + K_2\left[J_{3j}\right] + \Delta X_j\right\}^2,$$

$$G_2 = G_2(K_3, K_4) = \frac{1}{2}\sum_{j=1}^{N}\left\{K_3\left[J_{1j}\right] + K_4\left[J_{2j}\right] - \Delta Y_j\right\}^2, \tag{4}$$

$$G_3 = G_3(K_5, K_6, K_7) = \frac{1}{2}\sum_{j=1}^{N}\left\{K_5\left[t_j\right] + K_6\left[J_{4j}\right] + K_7\left[-J_{3j}\right] - \Delta Z_j\right\}^2,$$

where $J_{1j} = J_1(t_j)$, $J_{2j} = J_2(t_j)$, $J_{3j} = J_3(t_j)$, $J_{4j} = J_4(t_j)$, $\Delta X_j = \Delta X(t_j)$, $\Delta Y_j = \Delta Y(t_j)$, $\Delta Z_j = \Delta Z(t_j)$, $(j = 1, 2, \ldots, N)$. Minimizing goal functions (4) so that

$$\frac{\partial G_1}{\partial K_1} = \frac{\partial G_1}{\partial K_2} = \frac{\partial G_2}{\partial K_3} = \frac{\partial G_2}{\partial K_4} = \frac{\partial G_3}{\partial K_5} = \frac{\partial G_3}{\partial K_6} = \frac{\partial G_3}{\partial K_7} = 0 \tag{5}$$

we obtain three systems of linear algebraic equations;

$$\begin{cases} K_1\sum_{j=1}^{N}\left[J_{2j}^2\right] + K_2\sum_{j=1}^{N}\left[J_{2j}J_{3j}\right] = -\sum_{j=1}^{N}\left[J_{2j}\Delta X_j\right], \\ K_1\sum_{j=1}^{N}\left[J_{2j}J_{3j}\right] + K_2\sum_{j=1}^{N}\left[J_{3j}^2\right] = -\sum_{j=1}^{N}\left[J_{3j}\Delta X_j\right], \end{cases} \tag{6}$$

$$\begin{cases} K_3\sum_{j=1}^{N}\left[J_{1j}^2\right] + K_4\sum_{j=1}^{N}\left[J_{1j}J_{2j}\right] = \sum_{j=1}^{N}\left[J_{1j}\Delta Y_j\right], \\ K_3\sum_{j=1}^{N}\left[J_{1j}J_{2j}\right] + K_4\sum_{j=1}^{N}\left[J_{2j}^2\right] = \sum_{j=1}^{N}\left[J_{2j}\Delta Y_j\right], \end{cases} \tag{7}$$

$$\begin{cases} K_5\sum_{j=1}^{N}\left[t_j^2\right] + K_6\sum_{j=1}^{N}\left[t_jJ_{4j}\right] + K_7\sum_{j=1}^{N}\left[-t_jJ_{3j}\right] = \sum_{j=1}^{N}\left[t_j\Delta Z_j\right], \\ K_5\sum_{j=1}^{N}\left[t_jJ_{4j}\right] + K_6\sum_{j=1}^{N}\left[J_{4j}^2\right] + K_7\sum_{j=1}^{N}\left[-J_{4j}J_{3j}\right] = \sum_{j=1}^{N}\left[J_{4j}\Delta Z_j\right], \\ K_5\sum_{j=1}^{N}\left[-t_jJ_{3j}\right] + K_6\sum_{j=1}^{N}\left[-J_{4j}J_{3j}\right] + K_7\sum_{j=1}^{N}\left[J_{3j}^2\right] = -\sum_{j=1}^{N}\left[J_{3j}\Delta Z_j\right], \end{cases} \tag{8}$$

The solution for the above equations may be written as follows:

$$[\bar{K}_1, \bar{K}_2]^T = \left(L_1^T L_1\right)^{-1}\left(L_1^T R_1\right),$$

$$[\bar{K}_3, \bar{K}_4]^T = \left(L_2^T L_2\right)^{-1}\left(L_2^T R_2\right), \tag{9}$$

$$[\bar{K}_5, \bar{K}_6, \bar{K}_7]^T = \left(L_3^T L_3\right)^{-1}\left(L_3^T R_3\right),$$

where $\bar{K}_k$ are estimations of parameters $K_k$, $(k = 1, 2, \ldots, 7)$, sign $(\ldots)^T$ denotes matrix transposition, sign $(\ldots)^{-1}$ means inversion of matrix, coma is used for separation of rows and/or columns, and

$$\underset{(N\times 2)}{L_1} = \left[J_{2j}, J_{3j}\right], \qquad \underset{(N\times 1)}{R_1} = \left[-\Delta X_j\right],$$

$$\underset{(N\times 2)}{L_2} = \left[J_{1j}, J_{2j}\right], \qquad \underset{(N\times 1)}{R_2} = \left[\Delta Y_j\right], \tag{10}$$

$$\underset{(N\times 3)}{L_3} = \left[t_j, J_{4j}, -J_{3j}\right], \qquad \underset{(N\times 1)}{R_3} = \left[\Delta Z_j\right].$$

Inversion of the corresponding matrices is possible if and only if the corresponding columns of matrices $L_1$, $L_2$, $L_3$ are linearly independent. Notations under matrices $L$ and $R$ denote their dimensions, for example, $(N\times 2)$ means that corresponding matrix has $N$ rows and two columns.

After determination of estimations of parameters $K_k = \bar{K}_k$, $(k = 1, 2, \ldots, 7)$ system (1) can be solved with new initial values $X(t = T) = X_N$, $Y(t = T) = Y_N$, $Z(t = T) = Z_N$ on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$. Existence of the solution is guaranteed by the corresponding general theorems of ordinary differential equations [9]. Hence, it is possible to realize continuation (prediction) of the solution on finite time interval. Of course, accuracy of parameters $K_k$ identification and hence, prediction of the solution depends on accuracy of calculation of integrals $J_{1j}$, $J_{2j}$, $J_{3j}$ and $J_{4j}$.

## 2.2 *State variable X is unknown and state variables Y, Z are known*

Let us consider one mechanical analogy for the case of known state variables $Y$ and $Z$. It is possible to solve the second equation of system (1) with respect to $X$, and, hence, obtaining

$$X(t) = \frac{1}{K_3}\left[\frac{dY(t)}{dt} - K_4 Y(t)\right] \tag{11}$$

and substituting this expression into the first equation of system (1) results in:

$$\begin{cases} \dfrac{d^2Y(t)}{dt^2} + 2\delta\dfrac{dY(t)}{dt} + \omega^2 Y(t) = bZ(t), \\ \dfrac{dZ(t)}{dt} + cZ(t) = f + g\left[\dfrac{dY(t)}{dt} + 2\delta Y(t)\right]Z(t), \\ X(t) = h\left[\dfrac{dY(t)}{dt} - K_4 Y(t)\right], \end{cases} \tag{12}$$

where $2\delta = -K_4$, $\omega^2 = K_1K_3$, $b = -K_2K_3$, $c = K_7$, $f = K_5$, $g = \dfrac{K_6}{K_3}$, $h = \dfrac{1}{K_3}$. In this system the viscous damping factor $\delta$ is negative if $K_4 > 0$ and inertial parameter $c$ is positive, if $K_7 > 0$. First two equations of this dynamical system characterize self-oscillatory system with linear oscillatory part and inertial nonlinear feedback [10–12]. The most remarkable fact is that model (12) can behave as either regular or chaotic system depending on parameters $\delta$, $\omega$, $b$, $c$, $f$, $g$. Hence, estimation of these parameters is of crucial importance for prediction of regimes of its behavior. The mentioned analogy between system (1) and corresponding mechanical (electro-mechanical, mechatronic, etc.) system (12) can be considered from the viewpoint of parameters identification of system (1) instead of the corresponding parameter identification of system (12). Advantage of consideration of mechanical model (12) in terms of the Rőssler attractor is that it is possible to perform the systematic topological characterization of the system [21].

It follows from the first and second Equations of (12) that without knowledge of $X(t)$ it is possible to individually estimate only $K_4$, $K_5$ and $K_7$ parameters. Other parameters can be determined in groups $K_1K_3$, $K_2K_3$ and $\dfrac{K_6}{K_3}$. It also follows from the third equation of system (3) that additional information about particular values state variable $X$ is necessary to estimate parameter $K_3$. After that it will be possible to individually estimate parameters $K_1$, $K_2$ and $K_6$, thus estimating all unknown parameters. The following Lemma is valid:

**Lemma 2** If state variables $Y$ and $Z$ are known in $N+1 >> 7$ points at $t_i = \dfrac{T}{N}i$, $(i = 0, 1, \ldots, N)$ and state variable $X$ is unknown parameters $K_4$, $K_5$, $K_7$ and group of parameters $K_1K_3$, $K_2K_3$, $\dfrac{K_6}{K_3}$ can be identified (it means that $K_1$, $K_2$, $K_3$ and $K_6$ cannot be identified individually).

**Proof.** Let us assume that state variables $Y$, $Z$ are known in $N+1 >> 7$ points at $t_i = \dfrac{T}{N}i$, $(i = 0, 1, \ldots, N)$ so that $Y_i = Y\left(t_i = \dfrac{T}{N}i\right)$, $Z_i = Z(t_i)$. We demonstrate constructive proof which illustrates algorithm of estimation of unknown parameters and groups of the parameters. First, we solve second equation of system (1) with respect to $X(t)$ using (11) and substitute it in the first and third equations of system (1) to obtain the following system of the first and second equations in (12) which is convenient to rewrite as follows:

$$\begin{cases} a_1\left[\dfrac{dY(t)}{dt}\right] + a_2[-Y(t)] + a_3[-Z(t)] - \dfrac{d^2Y(t)}{dt^2} = 0, \\ a_5[1] + a_6\left[Z(t)\dfrac{dY(t)}{dt} - a_1Y(t)Z(t)\right] + a_7[-Z(t)] - \dfrac{dZ(t)}{dt} = 0, \end{cases} \tag{13}$$

where

$$a_1 = K_4, \quad a_2 = K_1K_3, \quad a_3 = K_2K_3, \quad a_5 = K_5, \quad a_6 = \frac{K_6}{K_3}, \quad a_7 = K_7 \tag{14}$$

are new unknown parameters. New auxiliary parameter $a_4$ will be introduced later. Hence, only parameters $K_4$, $K_5$, $K_7$ and groups of parameters $K_1K_3$, $K_2K_3$, $\frac{K_6}{K_3}$ can be evaluated (and, of course, $K_1K_6 = K_1K_3\frac{K_6}{K_3}$, $K_2K_6 = K_2K_3\frac{K_6}{K_3}$, $\frac{K_1}{K_2} = \frac{K_1K_3}{K_2K_3}$, which follow from (14). Let us show how to calculate them. Integration of the first equation of system (13) yields:

$$a_1\left[\Delta Y(t)\right] + a_2\left[-J_2(t)\right] + a_3\left[-J_3(t)\right] + a_4\left[1\right] - \frac{dY(t)}{dt} = 0 \tag{15}$$

where

$$\begin{aligned} &\Delta Y(t) = Y(t) - Y_0, \quad Y_0 = Y(t=0), \quad J_2(t) = \int_0^t Y(\tau)\, d\tau, \\ &J_3(t) = \int_0^t Z(\tau)\, d\tau, \quad a_4 = \dot{Y}_0 = \left.\frac{dY(t)}{dt}\right|_{t=0}. \end{aligned} \tag{16}$$

□

Keep in mind that we introduced new unknown parameter $a_4 = \dot{Y}_0$ in (15) to eliminate numerical differentiation of array $Y_i = Y(t_i)$ at $t = 0$. After subsequent integration of Equation (16) and second equation of system (13) we obtain the following system:

$$\begin{cases} a_1\left[J_4(t)\right] + a_2\left[J_5(t)\right] + a_3\left[J_6(t)\right] + a_4\left[t\right] - \Delta Y(t) = 0, \\ a_5\left[t\right] + a_6\left[J_7(t) - a_1 J_8(t)\right] + a_7\left[-J_3(t)\right] - \Delta Z(t) = 0, \end{cases} \tag{17}$$

where

$$\begin{aligned} &J_4(t) = \int_0^t \Delta Y(\tau)\, d\tau, \quad J_5(t) = -\int_0^t J_2(\tau)\, d\tau, \quad J_6(t) = -\int_0^t J_3(\tau)\, d\tau, \\ &J_7(t) = \int_0^t Z(\tau)\frac{dY(\tau)}{d\tau}\, d\tau, \quad J_8(t) = \int_0^t Y(\tau) Z(\tau)\, d\tau, \quad \Delta Z(t) = Z(t) - Z_0. \end{aligned} \tag{18}$$

Providing that $Y_i = Y(t_i)$, $Z_i = Z(t_i)$ are known at $t_i = \frac{T}{N} i$, $(i = 0, 1, \ldots, N)$ we calculate $J_{2i} = J_2(t_i), \ldots, J_{8i} = J_8(t_i)$, $\Delta Y_i = \Delta Y(t_i)$, $\Delta Z_i = \Delta Z(t_i)$ and compose first objective function:

$$G_1 = G_1(a_1, a_2, a_3, a_4) = \frac{1}{2}\sum_{j=1}^{N}\left\{a_1\left[J_{4j}\right] + a_2\left[J_{5j}\right] + a_3\left[J_{6j}\right] + a_4\left[t_j\right] - \Delta Y_j\right\}^2 \tag{19}$$

which is subjected to minimization. Solution of system of equations $\frac{\partial G_1}{\partial a_1} = \frac{\partial G_1}{\partial a_2} = \frac{\partial G_1}{\partial a_3} = \frac{\partial G_1}{\partial a_4} = 0$ is as follows:

$$[\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4]^T = \left(L_1^T L_1\right)^{-1} \left(L_1^T R_1\right), \tag{20}$$

where

$$\underset{(N\times 4)}{L_1} = \left[J_{4j}, J_{5j}, J_{6j}, t_j\right], \quad \underset{(N\times 1)}{R_1} = [\Delta Y_j], \quad (j = 1, 2, \ldots, N) \tag{21}$$

Next, we compose the second objective function:

$$G_2 = G_2(a_5, a_6, a_7) = \frac{1}{2}\sum_{j=1}^{N} \left\{a_5[t_j] + a_6\left[J_{7j} - \bar{a}_1 J_{8j}\right] + a_7\left[-J_{3j}\right] - \Delta Z_j\right\}^2 \tag{22}$$

which is subjected to minimization. Solution of system of equations $\frac{\partial G_2}{\partial a_5} = \frac{\partial G_2}{\partial a_6} = \frac{\partial G_2}{\partial a_7} = 0$ is:

$$[\bar{a}_5, \bar{a}_6, \bar{a}_7]^T = \left(L_2^T L_2\right)^{-1} \left(L_2^T R_2\right) \tag{23}$$

where

$$\underset{(N\times 3)}{L_2} = \left[t_j, J_{7j} - \bar{a}_1 J_{8j}, -J_{3j}\right], \quad \underset{(N\times 1)}{R_2} = [\Delta Z_j], \quad (j = 1, 2, \ldots, N) \tag{24}$$

Keep in mind that in (22) and (24) we use estimation of parameter $\bar{a}_1$ obtained in (20).

Next lemma gives sufficient condition for individual evaluation of parameters $K_1$, $K_2$, $K_3$, $K_6$ and hence prediction of further behavior of state variables $X$, $Y$, $Z$ on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$.

**Lemma 3** If in addition to conditions of Lemma 2 initial value of state variable $X$ is known, i.e., $X_0 = X(t = 0)$ is available, then all parameters of system (1) can be evaluated, unknown state variable $X$ can be recovered, and behavior of state variables $X$, $Y$, $Z$ can be predicted on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$.

**Proof.** It follows from the third expression of system (12) at initial time instant $(t = 0)$:

$$\overline{K_3} = \frac{\left.\frac{dY(t)}{dt}\right|_{t=0} - \overline{K_4} Y(0)}{X(0)} = \frac{\overline{a_4} - \overline{a_1} Y(0)}{X(0)} \tag{25}$$

where $\overline{a_1}$, $\overline{a_4}$ are parameters estimated in (20). Hence, from the estimated groups of unknown parameters (14) it follows that

$$\overline{K_1} = \frac{\overline{a_2}}{\overline{K_3}}, \quad \overline{K_2} = \frac{\overline{a_3}}{\overline{K_3}}, \quad \overline{K_4} = \overline{a_1}, \quad \overline{K_5} = \overline{a_5}, \quad \overline{K_6} = \overline{a_6}\overline{K_3}, \quad \overline{K_7} = \overline{a_7} \tag{26}$$

□

Hence, all unknown parameters of system (1) are estimated by formulas (25) and (26). Knowledge of initial values $X(0)$, $Y(0)$, $Z(0)$ enables to formulate initial condition for system (1) and restore the unknown state variable $X$ for $t \in [0, T]$.

**Remark 1** Instead of solution of the above mentioned initial value problem, it is possible to solve the first equation in system (1) and obtain solution:

$$\overline{X(t)} = X(0) - \int_0^t \left[\overline{K_1}Y(\tau) + \overline{K_2}Z(\tau)\right] d\tau \tag{27}$$

Obtaining terminal value $\overline{X_N} = \overline{X(T)}$ from (27) and considering it with terminal values $Y_N$, $Z_N$ of other state variables we formulate new initial value problem for prediction of behavior of state variables $X, Y, Z$ on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$ (see Lemma 1).

**Remark 2** In Lemma 3 the initial value of state variable $X$ is used, thus any value of the state variable $X$ from the time interval $t \in [0, T]$ can be used for solution of the problem. Moreover, if several known values from the interval we can improve estimations of the unknown $K$-parameters and $X$ state variable. Let us demonstrate alternative approach to the problem in the case when both initial and terminal values of state variable $X$ are available from Lemma 4: If in addition to conditions of Lemma 2 both initial and terminal values of state variable $X$, $X_0 = X(0)$ and $X_N = X(T)$, are known then all parameters of system (1) can be evaluated, unknown state variable $X$ can be recovered, and behavior of all state variables $X, Y, Z$ can be predicted on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$.

**Proof.** Integrating first equation of system (1) with respect to time we obtain:

$$X(t) = X_0 - K_1 \int_0^t Y(\tau)\, d\tau - K_2 \int_0^t Z(\tau)\, d\tau \tag{28}$$

□

Hence,

$$K_1 \int_0^T Y(\tau)\, d\tau + K_2 \int_0^T Z(\tau)\, d\tau = X_0 - X(T) = X_0 - X_N = -\Delta X_N \tag{29}$$

Moreover, $K_1K_3 = \bar{a}_2$, $K_2K_3 = \bar{a}_3$, and hence

$$\bar{a}_3K_1 - \bar{a}_2K_2 = 0 \tag{30}$$

From (29), (30) it is possible to find parameters $K_1$, $K_2$ and hence, find $K_3$, $K_6$ from (14). Estimated parameters are as follows:

$$\bar{K}_1 = -\frac{\bar{a}_2 \Delta X_N}{\int\limits_0^T [\bar{a}_2 Y(\tau) + \bar{a}_3 Z(\tau)]\, d\tau}, \quad \bar{K}_2 = -\frac{\bar{a}_3 \Delta X_N}{\int\limits_0^T [\bar{a}_2 Y(\tau) + \bar{a}_3 Z(\tau)]\, d\tau},$$

$$\bar{K}_3 = -\frac{\int\limits_0^T [\bar{a}_2 Y(\tau) + \bar{a}_3 Z(\tau)]\, d\tau}{\Delta X_N}, \quad \bar{K}_4 = \bar{a}_2, \quad \bar{K}_5 = \bar{a}_4, \quad \bar{K}_6 = \bar{a}_5 \bar{K}_3, \quad \bar{K}_7 = \bar{a}_6. \tag{31}$$

Using estimations of parameters $\bar{K}_1$, $\bar{K}_2$ and formula (28) unknown state variable $X$ is estimated.

### 2.3 *State variable Y is unknown and state variables X, Z are known*

In this case one can imagine another mechanical analogy. Solving the first equation of system (1) with respect to $Y$, and, hence, obtaining

$$Y(t) = -\frac{1}{K_1}\left[\frac{dX(t)}{dt} + K_2 Z(t)\right] \tag{32}$$

and substituting this expression in the second equation of system (1) gives the following system:

$$\begin{cases} \dfrac{d^2X(t)}{dt^2} + 2\delta \dfrac{dX(t)}{dt} + \omega^2 X(t) = -b\left[\dfrac{dZ(t)}{dt} + 2\delta Z(t)\right], \\ \dfrac{dZ(t)}{dt} + cZ(t) = f + gX(t)Z(t), \\ Y(t) = -h\left[\dfrac{dX(t)}{dt} + bZ(t)\right], \end{cases} \tag{33}$$

where $2\delta = -K_4$, $\omega^2 = K_1K_3$, $b = K_2$, $c = K_7$, $f = K_5$, $g = K_6$, $h = \dfrac{1}{K_1}$. In this system the viscous "damping" factor $\delta$ is negative if $K_4 > 0$ and inertial parameter $c$ is positive, if $K_7 > 0$. First two equations of this dynamical system characterize self-oscillatory system with linear oscillations and inertial non-linear excitation [10–12]. Comparison of the left-hand sides of the first and second equations of systems (12) and (33) demonstrates their identity, but the right-hand sides are different.

It follows from the first and second Equations of (33) that without knowledge of $Y(t)$ it is possible to individually estimate $K_2$, $K_4$, $K_5$, $K_6$ and $K_7$ parameters. Other parameters can be determined in group $K_1K_3$. It also follows from the third equation of system (33) that additional information about partial values state variable $Y$ is necessary to estimate parameter $K_1$. After that it will be possible to estimate parameter $K_3$, thus estimating all unknown parameters. Hence, the following *Lemma* is valid:

**Lemma 4** In the case of known state variables $X$, $Z$ in $N+1 >> 7$ points at $t_i = \dfrac{T}{N}i$, $(i = 0, 1, \ldots, N)$ and unknown state variable $Y$ parameters $K_2$, $K_4$, $K_5$, $K_6$, $K_7$ and group of parameters $K_1K_3$ can be identified (it means that $K_1$ and $K_3$ cannot be identified individually).

**Proof.** Let us calculate state variable $Y$ from the first equation of system (1) (using formula (32)) and substitute it in the second equation of system (1). Considering the obtained equation with the third equation of (1) we obtain the following system:

$$\begin{cases} a_1[1]+a_2[X(t)Z(t)]+a_3[-Z(t)]-\dfrac{dZ(t)}{dt}=0, \\ a_4[X(t)]+a_6\left[\dfrac{dZ(t)}{dt}\right]+a_7\left[-\dfrac{dX(t)}{dt}\right]+a_8[-Z(t)]+\dfrac{d^2X(t)}{dt^2}=0, \end{cases} \tag{34}$$

where new unknown parameters $a_1, a_2, \ldots, a_8$ are introduced as follows:

$$\begin{aligned} &a_1=K_5, \quad a_2=K_6, \quad a_3=K_7, \quad a_4=K_1K_3, \\ &a_6=K_2, \quad a_7=K_4, \quad a_8=K_2K_4, \end{aligned} \tag{35}$$

(parameter $a_5$ will be introduced further). From (31) it follows that there is the constraint between parameters $a_6$, $a_7$ and $a_8$:

$$a_6a_7-a_8=0 \tag{36}$$

□

After integration of first equation and double integration of the second equation of system (33) with respect to time we obtain:

$$\begin{cases} a_1[t]+a_2[J_4(t)]+a_3[-J_3(t)]-\Delta Z(t)=0, \\ a_4[J_5(t)]+a_5[t]+a_6[J_6(t)]+a_7[J_7(t)]+a_8[J_8(t)]+\Delta X(t)=0, \end{cases} \tag{37}$$

where

$$\begin{aligned} &J_1(t)=\int_0^t X(\tau)d\tau, \quad J_3(t)=\int_0^t Z(\tau)d\tau, \quad J_4(t)=\int_0^t X(\tau)Z(\tau)d\tau, \quad J_5(t)=\int_0^t J_1(\tau)d\tau, \\ &J_6(t)=\int_0^t \Delta Z(\tau)d\tau, \quad J_7(t)=-\int_0^t \Delta X(\tau)d\tau, \quad J_8(t)=-\int_0^t J_3(\tau)d\tau, \\ &\Delta X(t)=X(t)-X_0, \quad \Delta Z(t)=Z(t)-Z_0. \end{aligned} \tag{38}$$

Next, we compose the first and second objective functions which will be subjected to minimization:

$$G_1 = G_1(a_1, a_2, a_3) = \frac{1}{2}\sum_{j=1}^{N}\left\{a_1[t_j] + a_2[J_{4j}] + a_3[-J_{3j}] - \Delta Z_j\right\}^2,$$

$$G_2 = G_2(a_4, a_5, a_6, a_7, a_8) = \frac{1}{2}\sum_{j=1}^{N}\left\{a_4[J_{5j}] + a_5[t_j] + a_6[J_{6j}] + a_7[J_{7j}] + [a_8[J_{8j}] + \Delta X_j]\right\}^2. \tag{39}$$

In objective function $G_2 = G_2(a_4, a_5, a_6, a_7, a_8)$ unknown parameter $a_5 = \left.\frac{dX(t)}{dt}\right|_{t=0}$ is initial time derivative of unknown state variable $X$. In objective function $G_1 = G_1(a_1, a_2, a_3)$ parameters $a_1, a_2, a_3$ are independent and hence its minimization means solution of system of equations $\frac{\partial G_1}{\partial a_1} = \frac{\partial G_1}{\partial a_2} = \frac{\partial G_1}{\partial a_3} = 0$. Solution of this system is:

$$[\bar{a}_1, \bar{a}_2, \bar{a}_3]^T = \left(L_1^T L_1\right)^{-1}\left(L_1^T R_1\right), \tag{40}$$

where

$$\underset{(N\times 3)}{L_1} = [t_j, J_{4j}, -J_{3j}], \quad \underset{(N\times 1)}{R_1} = [\Delta Z_j] \tag{41}$$

Vice versa, parameters $a_6, a_7, a_8$ in objective function $G_2 = G_2(a_4, a_5, a_6, a_7, a_8)$ are not independent but connected by constraint (31). Let us select parameter $a_8$ as independent one and calculate other parameters as functions of this parameter: $a_4 = a_4(a_8)$, $a_5 = a_5(a_8)$, $a_6 = a_6(a_8)$, $a_7 = a_7(a_8)$ as follows:

$$[a_4(a_8), a_5(a_8), a_6(a_8), a_7(a_8)]^T = a_8\left(L_2^T L_2\right)^{-1}\left(L_2^T R_2\right) + \left(L_2^T L_2\right)^{-1}\left(L_2^T R_3\right), \tag{42}$$

where

$$\underset{(N\times 4)}{L_2} = [J_{5j}, t_j, J_{6j}, J_{7j}], \quad \underset{(N\times 1)}{R_2} = [-J_{8j}], \quad \underset{(N\times 1)}{R_3} = [-\Delta X_j] \tag{43}$$

Parameter $a_8$ is determined from solution of equation (see (35)):

$$Constr(a_8) = a_6(a_8) \cdot a_7(a_8) - a_8 = 0 \tag{44}$$

If there are several solutions of nonlinear Equation (44) we select only that which guarantees global minimum of objective function $G_2$. Let us denote this solution as $\bar{a}_8$, $\bar{a}_4 = \bar{a}_4(\bar{a}_8)$, $\bar{a}_5 = \bar{a}_5(\bar{a}_8)$, $\bar{a}_6 = \bar{a}_6(\bar{a}_8)$, $\bar{a}_7 = \bar{a}_7(\bar{a}_8)$. In this case original parameters are:

$$\bar{K}_2 = \bar{a}_6, \bar{K}_4 = \bar{a}_7, \bar{K}_5 = \bar{a}_1, \bar{K}_6 = \bar{a}_2, \bar{K}_7 = \bar{a}_3 \tag{45}$$

and $\overline{\left.\dfrac{dX(t)}{dt}\right|_{t=0}} = \overline{a_5}$. Parameters $K_1$ and $K_3$ are not evaluated at this stage and only their product is known: $K_1K_3 = \bar{a}_4$. Next Lemma gives us sufficient condition for determination of parameters $K_1$ and $K_3$ and hence, estimation of unknown state variable $Y$.

**Lemma 5** If in addition to conditions of Lemma 5 initial value of state variable $Y$ at $t = 0$ ($Y_0 = Y(0)$) is known then all parameters of system (1) can be estimated, unknown state variable $Y$ can be recovered at any $t \in [0, T]$, and behavior of all state variables $X, Y, Z$ can be predicted on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$.

**Proof.** From the second equation of system (1) at $t = 0$:

$$\overline{K_1} = -\frac{\overline{\left.\dfrac{dX(t)}{dt}\right|_{t=0}} + \overline{K_2}Z(0)}{Y(0)} = -\frac{\overline{a_5} + \overline{a_6}Z(0)}{Y(0)} \tag{46}$$

if $Y(0) \neq 0$, and hence,

$$\overline{K_3} = \frac{\overline{a_4}}{\overline{K_1}} \tag{47}$$

□

Solution of initial value problem with the second ODE of system (1) with initial condition $Y(0) = Y_0$ is as follows:

$$\overline{Y\left(t, \overline{K_3}\right)} = Y_0 e^{\bar{K}_4 t} + \overline{K_3}\int_0^t X(\tau)\, e^{\bar{K}_4(t-\tau)} d\tau \tag{48}$$

and estimation of unknown state variable $Y$ is found for any $t \in [0, T]$.

Prediction of behavior of state variables $X, Y, Z$ on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$, can be done as in Lemma 1, i.e., by solving of initial value problem with system (1) and new initial values $X_N$, $\overline{Y\left(T, \overline{K_3}\right)}$, $Z_N$.

**Remark 3** Analogous to the Section 2.2, it is also possible to use several values of state variable $Y$ on time interval $t \in [0, T]$, and formulate the following lemma.

**Lemma 6** If in addition to conditions of Lemma 4 the initial and terminal values of state variable $Y$, namely $Y_0 = Y(0)$ and $Y_N = Y(T)$, are known then all parameters of system (1) can be evaluated, unknown state variable $Y$ can be estimated on time interval $t \in [0, T]$, and behavior of state variables $X, Y, Z$ can be predicted on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$.

**Proof.** Solution of initial value problem of second ODE of system (1) with initial condition $Y(0) = Y_0$ is as follows:

$$Y(t, K_3) = Y_0 e^{\bar{K}_4 t} + K_3 \int_0^t X(\tau)\, e^{\bar{K}_4(t-\tau)} d\tau \tag{49}$$

where $K_3$ is considered as parameter. After substitution of the terminal value $Y(T) = Y_N$ in (49) we obtain equation with respect to $K_3$ solution of which is:

$$\bar{K}_3 = \frac{Y_N - Y_0 e^{\bar{K}_4 T}}{\int_0^T X(\tau) e^{\bar{K}_4 (T-\tau)} d\tau} \tag{50}$$

□

Hence, from (35):

$$\bar{K}_1 = \frac{\bar{a}_4}{\bar{K}_3} = \frac{\bar{a}_4 \int_0^T X(\tau) e^{\bar{K}_4 (T-\tau)} d\tau}{Y_N - Y_0 e^{\bar{K}_4 t}} \tag{51}$$

if $Y_N - Y_0 e^{\bar{K}_4 t} \neq 0$. Unknown state variable $Y$ is estimated by expression (48).

### 2.4 *Function Z(t) is unknown and functions X(t), Y(t) are known*

It follows from the first equation of system (1) that:

$$Z(t) = -\frac{1}{K_2}\left[\frac{dX(t)}{dt} + K_1 Y(t)\right] \tag{52}$$

Substituting (52) in the third equation of system (1) and taking into consideration the second equation of (1):

$$\begin{cases} \dfrac{d^2 X(t)}{dt^2} + K_7 \dfrac{dX(t)}{dt} = -K_2 K_5 + K_6 X(t) \dfrac{dX(t)}{dt} - K_1 \left[\dfrac{dY(t)}{dt} + K_7 Y(t)\right] + K_1 K_6 X(t) Y(t), \\ \dfrac{dY(t)}{dt} - K_4 Y(t) = K_3 X(t). \end{cases} \tag{53}$$

This system can be transformed to more convenient model after substitution of the second equation of (53) into the first equation of this system:

$$\begin{cases} \dfrac{d^2 X(t)}{dt^2} + 2\delta \dfrac{dX(t)}{dt} + \omega^2 X(t) = -b + c \cdot X(t) \dfrac{dX(t)}{dt} - d \cdot Y(t) + f \cdot X(t) Y(t), \\ \dfrac{dY(t)}{dt} - g \cdot Y(t) = h \cdot X(t), \end{cases} \tag{54}$$

where $2\delta = K_7$, $\omega^2 = K_1 K_3$, $b = K_2 K_5$, $c = K_6$, $d = K_1 (K_4 + K_7)$, $f = K_1 K_6$, $g = K_4$, $h = K_3$. System (54) describes mechanical, electro-mechanical or mechatronic nonlinear oscillatory or aperiodic system with linear inertial feedback. The oscillatory (or aperiodic) part has positive damping factor $\delta$ (if $K_7 > 0$) and the inertial part has negative inertial parameter $g$ (if $K_4 > 0$).

**Lemma 7** In the case of known state variables $X$, $Y$ in $N+1>>7$ points at $t_i=\frac{T}{N}i$, $(i=0,\,1,\,\ldots,\,N)$ and unknown state variable $Z$ parameters $K_1$, $K_3$, $K_4$, $K_6$, $K_7$ and group of parameters $K_2K_5$ can be identified (i.e., $K_2$ and $K_5$ cannot be identified individually).

**Proof.** Let us rewrite system (52) as:

$$\begin{cases} a_1\left[X\left(t\right)\right]+a_2\left[Y\left(t\right)\right]-\dfrac{dY\left(t\right)}{dt}=0, \\ a_3\left[-1\right]+a_5\left[-\dfrac{dY\left(t\right)}{dt}\right]+a_6\left[\dfrac{1}{2}\dfrac{dX^2\left(t\right)}{dt}\right]+a_7\left[-\dfrac{dX\left(t\right)}{dt}\right] \\ +\left\{a_8\left[X\left(t\right)Y\left(t\right)\right]+a_9\left[-Y\left(t\right)\right]-\dfrac{d^2X\left(t\right)}{dt^2}\right\}=0, \end{cases} \tag{55}$$

where parameters $a_1$, $a_2$, $\ldots$, $a_9$ are as follows:

$$\begin{aligned} &a_1=K_3, \quad a_2=K_4, \quad a_3=K_2K_5, \quad a_5=K_1, \\ &a_6=K_6, \quad a_7=K_7, \quad a_8=K_1K_6, \quad a_9=K_1K_7. \end{aligned} \tag{56}$$

□

After integration of both equations of system (55) with respect to time we obtain:

$$\begin{cases} a_1\left[J_1\left(t\right)\right]+a_2\left[J_2\left(t\right)\right]-\Delta Y\left(t\right)=0, \\ a_3\left[-t\right]+a_4\left[1\right]+a_5\left[-\Delta Y\left(t\right)\right]+a_6\left[\Delta X^2\left(t\right)\right]+a_7\left[-\Delta X\left(t\right)\right] \\ +\left\{a_8\left[J_4\left(t\right)\right]+a_9\left[-J_2\left(t\right)\right]-\dfrac{dX\left(t\right)}{dt}\right\}=0, \end{cases} \tag{57}$$

where $a_4=\dot{X}_0=\left.\dfrac{dX\left(t\right)}{dt}\right|_{t=0}$ is initial time derivative of state variable $X$, which is artificially introduced as new unknown parameter so to eliminate numerical differentiation of $X$ and

$$\begin{aligned} &J_1\left(t\right)=\int_0^t X\left(\tau\right)d\tau, \quad J_2\left(t\right)=\int_0^t Y\left(\tau\right)d\tau, \quad J_4\left(t\right)=\int_0^t X\left(\tau\right)Y\left(\tau\right)d\tau, \\ &\Delta X\left(t\right)=X\left(t\right)-X_0, \quad \Delta Y\left(t\right)=Y\left(t\right)-Y_0, \quad \Delta^2 X\left(t\right)=\frac{1}{2}\left(X^2\left(t\right)-X_0^2\right). \end{aligned} \tag{58}$$

Integrating the second equation of system (57) with respect to time again we obtain the following system:

$$\begin{cases} a_1\left[J_1(t)\right]+a_2\left[J_2(t)\right]-\Delta Y(t)=0, \\ a_3\left[-\dfrac{t^2}{2}\right]+a_4\left[t\right]+a_5\left[J_5(t)\right]+a_6\left[J_6(t)\right]+a_7\left[J_7(t)\right] \\ +\left\{a_8\left[J_8(t)\right]+a_9\left[J_9(t)\right]-\Delta X(t)\right\}=0, \end{cases} \tag{59}$$

where

$$J_5(t)=-\int_0^t \Delta Y(\tau)\,d\tau, \quad J_6(t)=\int_0^t \Delta^2 X(\tau)\,d\tau, \quad J_7(t)=-\int_0^t \Delta X(\tau)\,d\tau,$$
$$J_8(t)=\int_0^t J_4(\tau)\,d\tau, \quad J_9(t)=-\int_0^t J_2(\tau)\,d\tau. \tag{60}$$

There are two constraints between parameters (61):

$$a_5a_6-a_8=0, \quad a_5a_7-a_9=0, \tag{61}$$

and hence, set of parameters (56) cannot be considered as independent. It also follows from (56) that parameters $K_2$ and $K_5$ cannot be individually determined in the case of knowledge of only state variables $X$ and $Y$. The same is true for parameters $K_1$ and $K_7$ and hence, it is necessary to have additional information about state variable $Z$ (preferably in more than two points) to individually estimate parameters $K_1$, $K_2$, $K_5$ and $K_7$.

Now let us introduce two objective functions which will be subjected to minimization:

$$G_1=G_1(a_1, a_2)=\frac{1}{2}\sum_{j=1}^{N}\left\{a_1\left[J_{1j}\right]+a_2\left[J_{2j}\right]-\Delta Y_j\right\}^2 \tag{62}$$

and

$$G_2=G_2(a_3, a_4, a_5, a_6, a_7; a_8, a_9)=\frac{1}{2}\sum_{j=1}^{N}\left\{a_3\left[-\frac{t_j^2}{2}\right]+a_4\left[t_j\right]+a_5\left[J_{5j}\right]\right.$$
$$\left.+a_6\left[J_{6j}\right]+a_7\left[J_{7j}\right]+\left[a_8\left[J_{8j}\right]+a_9\left[J_{9j}\right]-\Delta X_j\right]\right\}^2. \tag{63}$$

Solution of the minimization problem for objective function (62) is:

$$[\bar{a}_1, \bar{a}_2]^T = \left(L_1^T L_1\right)^{-1}\left(L_1^T R_1\right) \tag{64}$$

where

$$\underset{(N\times 2)}{L_1} = \left[J_{1j}, J_{2j}\right], \ \underset{(N\times 1)}{R_1} = [\Delta Y_j]. \tag{65}$$

In objective function (63) parameters $a_8$, $a_9$ are considered as auxiliary free parameters and other parameters are considered as functions of them: $a_3(a_8, a_9), a_4(a_8, a_9), a_5(a_8, a_9), a_6(a_8, a_9), \ a_7(a_8, a_9)$. In this case solution is:

$$\begin{aligned} &[a_3(a_8, a_9), a_4(a_8, a_9), a_5(a_8, a_9), a_6(a_8, a_9), a_7(a_8, a_9)]^T \\ &= a_8\left(L_2^T L_2\right)^{-1}\left(L_2^T R_2\right) + a_9\left(L_2^T L_2\right)^{-1}\left(L_2^T R_3\right) + \left(L_2^T L_2\right)^{-1}\left(L_2^T R_4\right), \end{aligned} \tag{66}$$

where

$$\underset{(N\times 5)}{L_2} = \left[-\frac{t_j^2}{2}, t_j, J_{5j}, J_{6j}, J_{7j}\right], \quad \underset{(N\times 1)}{R_2} = \left[J_{8j}\right], \quad \underset{(N\times 1)}{R_3} = \left[J_{9j}\right], \quad \underset{(N\times 1)}{R_4} = [-\Delta X_j] \tag{67}$$

Using solution (66) and taking into consideration constraints (61) we minimize objective function:

$$G_3 = G_3(a_8, a_9) = \frac{1}{2}\left\{[a_5(a_8, a_9)\, a_6(a_8, a_9) - a_8]^2 + [a_5(a_8, a_9)\, a_7(a_8, a_9) - a_9]^2\right\}. \tag{68}$$

Denoting solution of minimization problem of objective function (54), corresponding to global minimum of (68) as $\bar{a}_8$, $\bar{a}_9$, we obtain:

$$\begin{aligned} &\bar{K}_1 = a_5(\bar{a}_8, \bar{a}_9), \quad \bar{K}_3 = a_1(\bar{a}_8, \bar{a}_9), \quad \bar{K}_4 = a_2(\bar{a}_8, \bar{a}_9), \\ &\bar{K}_6 = a_6(\bar{a}_8, \bar{a}_9), \quad \bar{K}_7 = a_7(\bar{a}_8, \bar{a}_9), \quad \overline{K_2 K_5} = a_3(\bar{a}_8, \bar{a}_9). \end{aligned} \tag{69}$$

The next lemma shows how to estimate parameters $\bar{K}_2$ and $\bar{K}_5$ and evaluate unknown state variable $Z$ for $t \in [0, T]$.

**Lemma 8** If in addition to conditions of Lemma 6 boundary values of state variable $Z$, $Z_0 = Z(t = 0)$ and $Z_N = Z(t = T)$, are known then all parameters of system (1) can be evaluated, unknown state variable $Z$ can be recovered, and behavior of state variables $X, Y, Z$ can be predicted on finite time interval $t \in \left[T, \tilde{T}\right]$, where $\tilde{T} > T$.

**Proof.** Assuming that estimations of parameters $\bar{K}_6$ and $\bar{K}_7$ are known let us derive solution of third equation of system (1) depending on unknown parameter $K_5$ (using, for example, the method of integrating factor):

$$Z(t, K_5) = Z_0 \exp\left(\bar{K}_6 \int_0^t X(\tau)\, d\tau - \bar{K}_7 t\right) + K_5 \int_0^t \exp\left(\bar{K}_6 \int_\tau^t X(\eta)\, d\eta - \bar{K}_7 (t-\tau)\right) d\tau \tag{70}$$

Solving equation $Z(t = T, K_5) = Z_N$, we obtain estimation of parameter $K_5$ as follows:

$$\bar{K}_5 = \frac{Z_N - Z_0 \exp\left(\bar{K}_6 \int_0^T X(\tau)\, d\tau - \bar{K}_7 T\right)}{\int_0^T \exp\left(\bar{K}_6 \int_\tau^T X(\eta)\, d\eta - \bar{K}_7 (T-\tau)\right) d\tau} \tag{71}$$

Hence, estimation of unknown state variable $Z$ on time interval $t \in [0, T]$ is:

$$\bar{Z}(t) = Z(t, \bar{K}_5) \tag{72}$$

and parameter $K_2$ is estimated as

$$\bar{K}_2 = \frac{\overline{K_2 K_5}}{\bar{K}_5} = \frac{a_3(\bar{a}_8, \bar{a}_9)}{\bar{K}_5} = \frac{a_3(\bar{a}_8, \bar{a}_9) \int_0^T \exp\left(\bar{K}_6 \int_\tau^T X(\eta)\, d\eta - \bar{K}_7 (T-\tau)\right) d\tau}{Z_N - Z_0 \exp\left(\bar{K}_6 \int_0^T X(\tau)\, d\tau - \bar{K}_7 T\right)} \tag{73}$$

providing that $Z_N - Z_0 \exp\left(\bar{K}_6 \int_0^T X(\tau)\, d\tau - \bar{K}_7 T\right) \neq 0.$ □

Prediction of behavior of state variables $X, Y, Z$ on finite time interval $t \in [T, \tilde{T}]$, where $\tilde{T} > T$, can be done as in Lemma 1.

## 3. Numerical examples

In this section we consider numerical simulation of two cases: situation when information about all functions is available and situation with known $X, Z$ and unknown $Y$ state variables. First, we assume that parameters $K_1, K_2, \ldots, K_7$ as well as initial conditions are given and make direct calculation of state variables $X, Y, Z$ on small initial time interval $t \in [0, T]$. Next, assuming some state variables are known, we solve inverse problem of parametric identification and compare the estimated parameters with original ones.

Please note that all numerical results in this paper are obtained using Mathcad 15, which contains powerful routines for manipulating numerical arrays and performing linear algebra operations. All the numerical manipulations used in this paper are standard and explained in numerous numerical methods books such as in [23].

Now assume that parameters of the Rőssler systems (1), (2) are given as follows:

$$K_1 = 1,\, K_2 = 1,\, K_3 = 1,\, K_4 = a = 0.2,\, K_5 = b = 0.2,\, K_6 = 1,\, K_7 = c = 5.7. \tag{74}$$

At these parameters the Rőssler system demonstrates its chaotic behavior at a "substantially long" time interval [1, 2]. Assuming that initial conditions are $X_0 = X(0) = 0.1$, $Y_0 = Y(0) = 0.1$, $Z_0 = Z(0) = 0.035$, we calculate solution of system (1) by the adaptive Runge-Kutta method at time interval $t \in [0,\, T = 20]$ in $N + 1 = 51$ points (see Figures 1-3).

As we can see from Figures 1-3, state variables $X$, $Y$, $Z$ demonstrate regular behavior with increasing amplitudes of vibration and do not manifest their chaotic behavior. So, time interval$_{\leftrightarrow}$2 $[0 \approx = 20]$ cannot be considered as "substantially long".

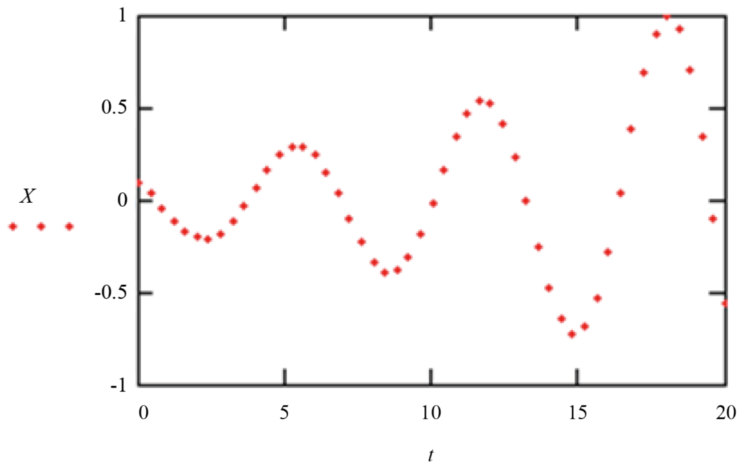


**Figure 1.** Solution of the Rőssler's system for state variable $X$ on time interval 2 [0 = 20]

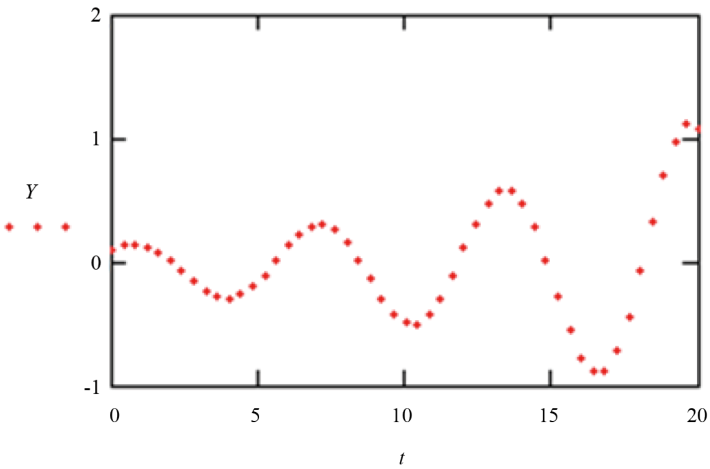


**Figure 2.** Solution of the Rőssler's system for state variable $Y$ on time interval 2 [0 = 20]

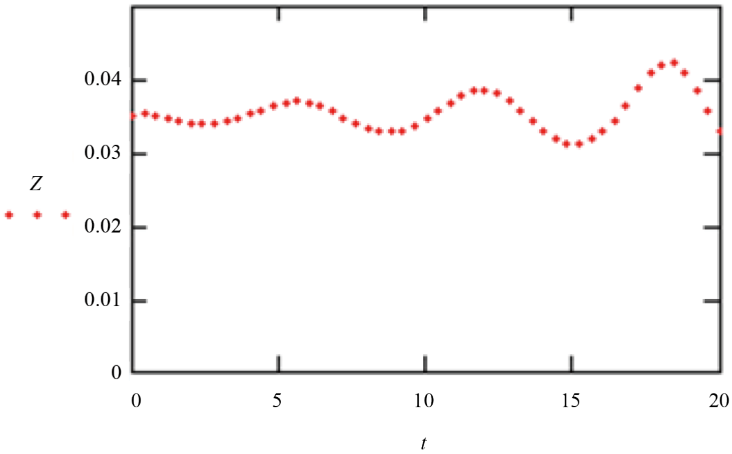


**Figure 3.** Solution of the Rőssler's system for state variable $Z$ on time interval 2 [0 = 20]

In all the algorithms discrete set of data is subjected to cubic spline interpolation with subsequent adaptive numerical integration with tolerance $10^{-7}$. As a result, the following parameters were obtained (see Table 1).

It follows from Table 1 that accuracy of estimation of the parameters is relatively high and, as it was observed, it is growing with increasing of number of data points.

**Table 1.** Original and estimated parameters, their absolute and percentage errors in the case of complete knowledge of state variables $X$, $Y$, $Z$

| Original parameter ($K$) | Estimated parameter ($\bar{K}$) | Absolute error ($\|K-\bar{K}\|$) | Percentage error $\left(\left\|\frac{K-\bar{K}}{K}\right\| 100\%\right)$ |
|---|---|---|---|
| $K_1 = 1$ | $\bar{K}_1 \approx 1.000022$ | $2.2 \cdot 10^{-5}$ | $2.2 \cdot 10^{-3}\%$ |
| $K_2 = 1$ | $\bar{K}_2 \approx 0.999995$ | $5 \cdot 10^{-6}$ | $5 \cdot 10^{-4}\%$ |
| $K_3 = 1$ | $\bar{K}_3 \approx 1.000032$ | $3.2 \cdot 10^{-5}$ | $3.2 \cdot 10^{-3}\%$ |
| $K_4 = 0.2$ | $\bar{K}_4 \approx 0.200029$ | $2.9 \cdot 10^{-5}$ | $1.4 \cdot 10^{-2}\%$ |
| $K_5 = 0.2$ | $\bar{K}_5 \approx 0.200107$ | $1.07 \cdot 10^{-4}$ | $5.0 \cdot 10^{-2}\%$ |
| $K_6 = 1$ | $\bar{K}_6 \approx 1.001249$ | $1.25 \cdot 10^{-3}$ | $1.25 \cdot 10^{-1}\%$ |
| $K_7 = 5.7$ | $\bar{K}_7 \approx 5.703228$ | $3.23 \cdot 10^{-3}$ | $5.7 \cdot 10^{-2}\%$ |

Next, we simulate situation with incomplete information about state variables $X$, $Y$, $Z$, namely we assume that function $Y(t)$ is unknown inside time interval $t \in (0, T)$ and only initial and terminal values, $Y_0 = Y(0)$ and $Y_N = Y(T)$, are known (see Section 3). In this situation we employ algorithms in (34)-(45) and (49)-(51). We obtain two roots of Equation (44), shown in Figure 4a as sharp negative spikes. Simultaneously graph of objective function (37), $G(a_8) = G_2(a_4(a_8), a_5(a_8), a_6(a_8), a_7(a_8), a_8)$, is shown in Figure 4b. In Figure 4a the first root, which corresponds to $a_{8,1} \approx 0.179355$ is spurious, because it does not correspond to the global minimum of the objective function. The second root, $a_{8,2} = \bar{a}_8 \approx 0.200013$, is the proper root which simultaneously corresponds to constraint (36) and guarantees global minimum of the objective function $G_2(a_8)$ in (39).

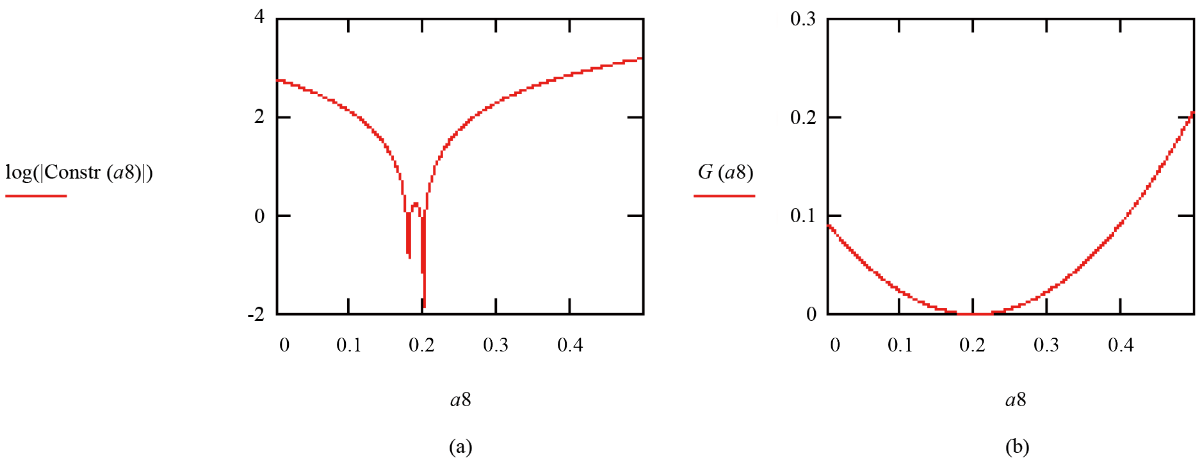


**Figure 4.** Roots of constraint Equation (42); Minimum of function $G(a_8)$ (37)

The estimated parameters and their comparison with the original ones are given in Table 2.

**Table 2.** Original and estimated parameters, their absolute and percentage errors in the case of known state variables $X$, $Z$ and unknown $Y$

| Original parameter $(K)$ | Estimated parameter $(\bar{K})$ | Absolute error $(\lvert K-\bar{K}\rvert)$ | Percentage error $\left(\left\lvert \frac{K-\bar{K}}{K} \right\rvert 100\%\right)$ |
|---|---|---|---|
| $K_1 = 1$ | $\bar{K}_1 \approx 0.999856$ | $1.44 \cdot 10^{-4}$ | $1.4 \cdot 10^{-2}\%$ |
| $K_2 = 1$ | $\bar{K}_2 \approx 0.999966$ | $3.4 \cdot 10^{-5}$ | $3.4 \cdot 10^{-3}\%$ |
| $K_3 = 1$ | $\bar{K}_3 \approx 1.000179$ | $1.8 \cdot 10^{-4}$ | $1.8 \cdot 10^{-2}\%$ |
| $K_4 = 0.2$ | $\bar{K}_4 \approx 0.200020$ | $2.0 \cdot 10^{-5}$ | $1.0 \cdot 10^{-2}\%$ |
| $K_5 = 0.2$ | $\bar{K}_5 \approx 0.200107$ | $1.07 \cdot 10^{-4}$ | $5.0 \cdot 10^{-2}\%$ |
| $K_6 = 1$ | $\bar{K}_6 \approx 1.001249$ | $1.25 \cdot 10^{-3}$ | $1.25 \cdot 10^{-1}\%$ |
| $K_7 = 5.7$ | $\bar{K}_7 \approx 5.703228$ | $3.23 \cdot 10^{-3}$ | $5.7 \cdot 10^{-2}\%$ |

Evaluated unknown function $\bar{Y}(t) = EY(t)$ in the interval $t \in [0, T]$ is shown as dashed graph in Figure 5 and compared with the originally simulated function $Y(t)$ (dotted graph shown in $N+1 = 51$ points, see Figure 2).

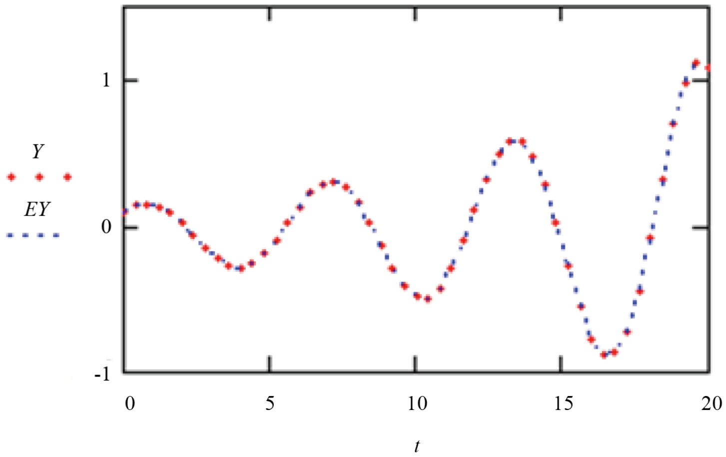


**Figure 5.** Evaluated function $\bar{Y}(t) = EY(t)$ (dashed graph) and originally simulated function $Y(t)$ (dotted graph shown in $N+1 = 51$ points)

Absolute error of the evaluated function $\bar{Y}(t)$: $\Delta EY(t) = |Y(t) - EY(t)|$ is shown in Figure 6.

Comparison of the predicted functions $EX(t)$, $EY(t)$, $EZ(t)$ calculated with the estimated parameters, given in Table 2, with state variables $X$, $Y$, $Z$ calculated with the original parameters (58) on time interval $t \in \left[T = 20, \tilde{T} = 200\right]$ is shown in Figures 7-9.

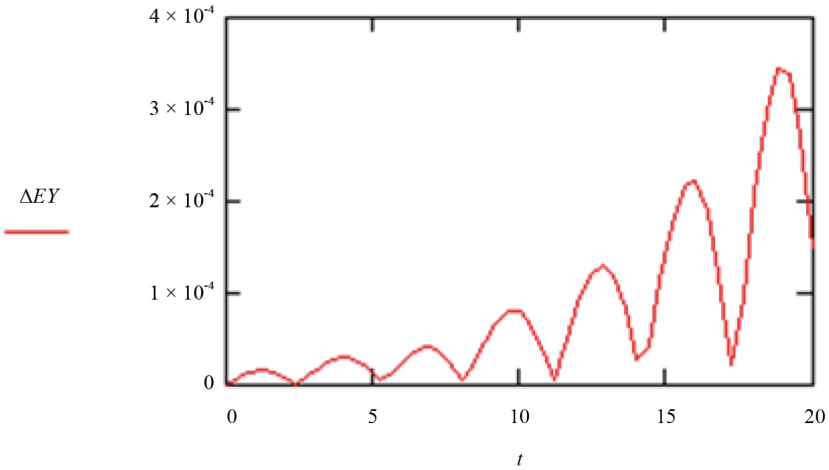


**Figure 6.** Absolute error of the evaluated function $\Delta EY(t) = |Y(t) - EY(t)|$

It follows from Figures 7-9 that in the interval $t \in \left[T = 20, \tilde{T} = 130\right]$ the state variables predictions properly describe extremums and spike's time instants and their magnitude. Further, in interval $t \in \left[T = 130, \tilde{T} = 200\right]$ the time instants of the extremums and spikes are predicted fairly accurately.

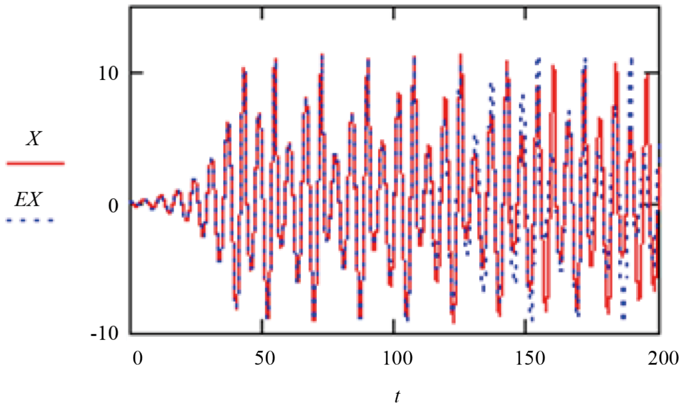


**Figure 7.** Comparison of the predicted function $EX(t)$ with state variable $X$, calculated with original parameters in (60) on time interval $t \in [T = 20, \tilde{T} = 200]$

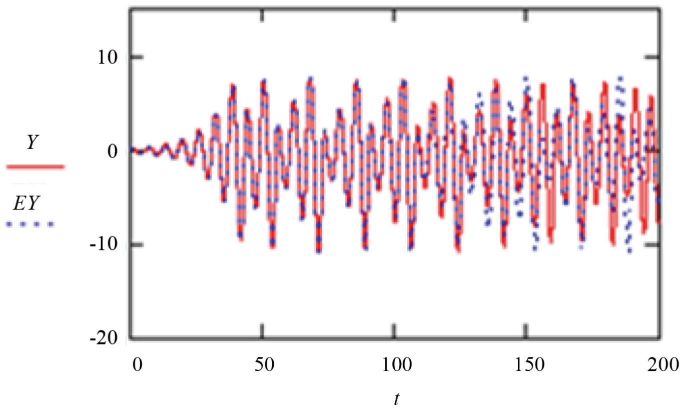


**Figure 8.** Comparison of the predicted function $EY(t)$ with state variable $Y$, calculated with original parameters (60) on time interval $t \in [T = 20, \tilde{T} = 200]$

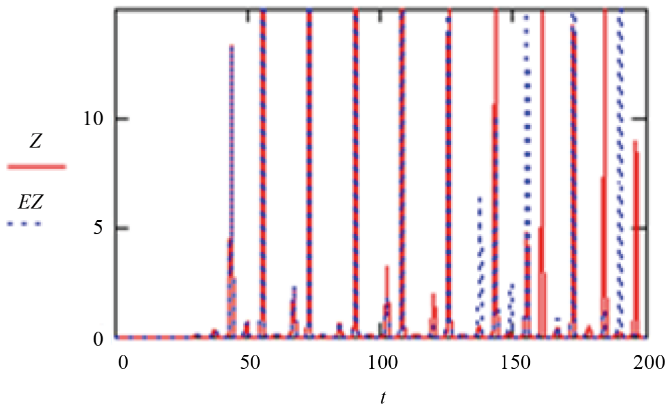


**Figure 9.** Comparison of the predicted function $EZ(t)$ with state variable $Z$, calculated with original parameters (60) on time interval $t \in [T = 20, \tilde{T} = 200]$

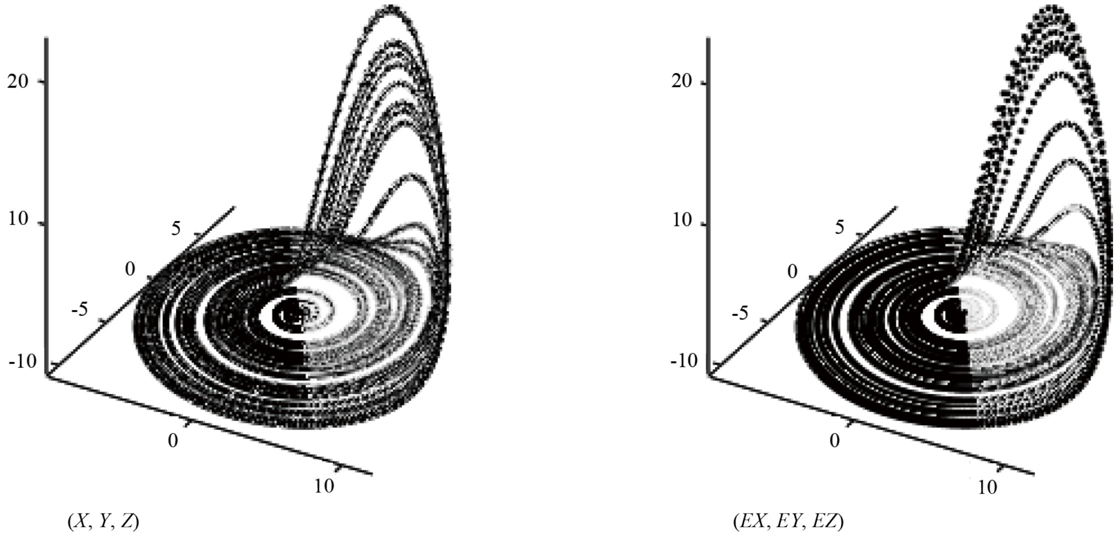


**Figure 10.** (a) Rőssler's attractor calculated with original parameters (60), (b) Rőssler's attractor calculated with estimated parameters (Table 2)

Three dimensional graphs of the Rőssler attractors calculated with the original and estimated parameters are shown in Figure 10(a, b). Similarity of these attractors follows from the graphs.

Hence, we conclude from Figure 10(a, b) that the evaluated parameters properly approximate the original Rőssler attractor.

## 4. Conclusions

Algorithms for identification of the Rőssler attractor's parameters were developed in the case of knowledge of either complete information about state variables $X$, $Y$, $Z$ or knowledge of only two functions. In the case of having complete information about the state variables, it is possible to apply the algorithms discussed to any system of linear or nonlinear ordinary differential equations of arbitrary order in the Cauchy form that linearly depends on the unknown parameters (or groups of unknown parameters). The problems of parameter identification in the case of incomplete information about the state variables must be solved individually, depending on the possibility (or impossibility) of eliminating unknown steady states from the system of equations. Most real-world problems in fields such as chemical kinetics, mathematical ecology, predator-prey dynamics in game reserves, and the spread of infectious diseases belong to this class of problems, in which the algorithms discussed demonstrate their applicability. The lemmas about full identification of all unknown parameters and unknown state variable were formulated and proven. The algorithms composed on the basis of the lemmas gave possibility of complete reconstruction of the set of unknown parameters and unknown state variables. Moreover, the algorithms helped to make prediction of the functional behavior of the attractor for a new finite time interval. Numerical simulations demonstrated the efficiency of the numerical algorithms.

It can be seen from Tables 1 and 2 that the absolute and percentage errors in the parameter identification in the case of incomplete information are larger than those in the case with complete information about the state variables, as expected. However, in the deterministic case, the errors in parameter identification are still reasonable, demonstrating the effectiveness of the proposed algorithms.

It worth noting that chaotic systems exhibit strong sensitivity to their initial conditions. The algorithms developed require accurate measurements of both initial and final values of the state variables as intermediate values are not available. It is recommended to use limited time intervals where the chaotic behavior has not been fully developed, typically with two to four almost-periods of oscillations as in examples discussed in Section 3. In the absence of chaotic behavior, the proposed algorithms are insensitive to initial conditions, in accordance with the fundamental theorem on the continuous dependence of ODE solutions with initial conditions. Thus, to implement the algorithm, it is essential to eliminate potential sources of errors in their numerical implementations.

The focus of this paper was systems with two known and one unknown state variable of the Rőssler attractor system in a deterministic case where random components are absent. In future research, we will analyse situations with one known and two unknown deterministic state variables and demonstrate corresponding algorithms for parameter identification. Additionally, we will investigate situations involving incomplete information about state variables that are perturbed by random noise.

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## Conflict of interest

There is no conflict of interest in this study.

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## Appendix

Conversion system (1) to the standard form can be obtained by the similarity transformation $(t, X, Y, Z) \rightarrow (T, x, y, z)$:

$$t = T_0 T, \quad X(t) = X_0 x(T), \quad Y(t) = Y_0 y(T), \quad Z(t) = Z_0 z(T) \tag{A.1}$$

In this case system (1) is transformed to the following one:

$$\begin{cases} \dfrac{dx(T)}{dT} + \dfrac{K_1 T_0 Y_0}{X_0} y(T) + \dfrac{K_2 T_0 Z_0}{X_0} z(T) = 0, \\ \dfrac{dy(T)}{dT} - \dfrac{K_3 T_0 X_0}{Y_0} x(T) - K_4 T_0 y(T) = 0, \\ \dfrac{dz(T)}{dT} - \dfrac{K_5 T_0}{Z_0} - K_6 T_0 X_0 x(T) z(T) + K_7 T_0 z(T) = 0. \end{cases} \tag{A.2}$$

Assuming that

$$T_0 = \frac{1}{\sqrt{K_1 K_3}}, \quad X_0 = \frac{\sqrt{K_1 K_3}}{K_6}, \quad Y_0 = \frac{K_3}{K_6}, \quad Z_0 = \frac{K_1 K_3}{K_2 K_6} \tag{A.3}$$

We obtain the Rőssler system in the standard form [2]:

$$\begin{cases} \dfrac{dx(T)}{dT} + y(T) + z(T) = 0, \\ \dfrac{dy(T)}{dT} - x(T) - a y(T) = 0, \\ \dfrac{dz(T)}{dT} - b - x(T) z(T) + c z(T) = 0, \end{cases} \tag{A.4}$$

where

$$a = \frac{K_4}{\sqrt{K_1 K_3}}, \quad b = \frac{K_2 K_5 K_6}{(K_1 K_3)^{3/2}}, \quad c = \frac{K_7}{\sqrt{K_1 K_3}} \tag{A.5}$$