Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Almost Golden Structures, Nonlinear Connections and Sprays

Nabil L. Youssef*, Amr M. Sid-Ahmed, Kamal A. Tawfik

Department of Mathematics, Faculty of Science, Cairo University, Egypt
E-mail: nlyoussef@sci.cu.edu.eg

Received: 26 August 2024; **Revised:** 20 November 2024; **Accepted:** 3 December 2024

Abstract: We introduce and investigate a special class of almost golden structures on the tangent bundle TM of a smooth manifold M. It is shown that there is a one-to-one correspondence between the nonlinear connection and the special almost golden structure on the tangent bundle. Moreover, there is a one-to-one correspondence between sprays and homogeneous special almost golden strictures. Finally, the question of integrability of almost golden structures is discussed and certain characterizations of integrability are found out.

***Keywords*:** almost golden structure, spray, nonlinear connection, Nijenhuis tensor

MSC: 53C15

1. Introduction

Being a source of fascination to many scholars, the golden ratio (also known as: golden number, golden section or golden mean) has been highly considered in many disciplines: geometry, architecture, music, art, ... , etc, due to its harmonic nature and proportionality.

The foundation of golden ratio concept is usually attributed to Pythagoras or his followers. However, the first known written definition of the golden ratio has been set by Euclid of Alexandria 300 BC in his famous book the Elements. He defined the term golden ratio as a proportion derived from a division of a line into what he calls its "extreme and mean ratio" [1].

In the last two decades, the study of golden structures in differential geometry was highly concerned by many authors, cf. for example, [2, 3]. Crâşmăreanua and Hreţcanu in their pioneering paper [2] demonstrated that an almost golden structure induces an almost product and vice versa. They used the corresponding almost product structure to investigate the geometry of the golden structure on a manifold.

As known, a polynomial structure is integrable if and only if it is possible to introduce a torsion-free linear connection with respect to which the structure tensor is covariantly constant. So, the issue of integrability for almost golden structures attracted many scholars. Etayo et al. [3] found the family of connections that are adapted to an almost golden structure. Moreover, Gezer et al. [4] introduced another condition for the integrability of almost golden Riemannian structures.

Moreover, depending on lifting techniques introduced by Yano and Ishihara [5] in 1970s, prolongation of almost golden structures from a manifold M to its tangent bundle TM is studied by many authers. However, the tangent bundle is characterized by the existence of a nonlinear connection which defines an almost product structure on TM. This motivates

DOI: https://doi.org/10.37256/cm.6520255569

us to search for necessary and sufficient conditions for an almost golden structure defined on TM to induce a nonlinear connection. On the other hand, because of the existence of a strong relationship between sprays and nonlinear connections, we investigate the relation between sprays and almost golden structures on TM. It is to be noted that we have followed a different approach, we do not lift objects from a manifold M to its tangent bundle TM but instead we place ourselves on TM itself.

2. Preliminaries

In 1970 Goldberg and Yano [6] introduced the notion of a polynomial structure on a smooth manifold M, which is a $(1,1)$-tensor field of constant rank on M satisfying the algebraic equation

$$Q(f) = f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 I = 0,$$

where $a_{n-1},\ \cdots,\ a_1,\ a_0$ are real numbers and I is the identity tensor field of type $(1,\ 1)$. In fact, the almost product structure P and the almost complex structure F are for example special cases of the polynomial structure, which satisfy

$$P^2 - I = 0 \text{ and } F^2 + I = 0,$$

In these cases $(M,\ P)$ and $(M,\ F)$ are called almost product manifold and almost complex manifold respectively.

Recall that, a polynomial structure $F \in \mathscr{T}_1{}^1(M)$ on a smooth manifold M is integrable if its Nijenhuis tensor N_F vanishes, where

$$N_F(X,\ Y) = [FX,\ FY] + F^2[X,\ Y] - F[FX,\ Y] - F[X,\ FY],\ \forall X,\ Y \in \mathfrak{X}(M).$$

In this context, an almost product structure P is said to be product structure if it is integrable, i.e. $N_P = \frac{1}{2}[P,\ P] = 0$.

The tangent bundle TM of a manifold M carries a canonical integrable almost tangent structure J $(J^2 = 0)$. The almost tangent structure J is a vector 1-form on TM with constant rank n having the properties [7, 8]:

$$J^2 = 0,\ \ [J,\ J] = 0 \text{ and } \operatorname{Ker} J = \operatorname{Im} J = V(TM),$$

where $V(TM)$ is the vertical sub-bundle of TM. If the local coordinates on TM are $(x,\ y) = (x^i,\ y^i)_{1\leq i\leq n}$, then $J = \frac{tial}{tialy^i} \otimes dx^i$. Equivalently,

$$J\left(\frac{tial}{tialx^i}\right) = \frac{tial}{tialy^i} \text{ and } J\left(\frac{tial}{tialy^i}\right) = 0.$$

Definition 1 A nonlinear connection Γ on a manifold M is a vector 1-form on TM, C^∞ on $TM \setminus (0)$ such that:
(i) $J\Gamma = J$,
(ii)$\Gamma J = -J$.

The connection Γ is an almost product structure, which means that $\Gamma^2 = \mathrm{I}$, the identity endomorphism of TM. Further, Γ gives rise to the two projectors

$$v := \frac{1}{2}(\mathrm{I} - \Gamma), \; h := \frac{1}{2}(\mathrm{I} + \Gamma).$$

It is known that the projector v generates the vertical distribution $V(TM)$, while h induces a horizontal distribution $H(TM)$ supplementary to the vertical distribution such that;

$$T_z TM = V_z(TM) \oplus H_z(TM) \; \forall z \in TM,$$

or,

$$TTM = V(TM) \oplus H(TM).$$

Definition 2 The curvature of a nonlinear connection Γ is a 2-form Ω, C^∞ on $TM \setminus (0)$ defined by

$$\Omega = -\frac{1}{2}[h, \; h], \tag{1}$$

where h is the horizontal projector associated with Γ.

Definition 3 A vector valued ℓ-form L on TM is homogeneous of degree r in the fiber coordinates y^i if

$$[\mathscr{C}, \; L] = (r-1)L,$$

where $\mathscr{C} = y^i \dfrac{tial}{tialy^i}$ is the Liouville vector filed.

Definition 4 A semispray on M is a globally defined vector field S on TM such that

$$JS = \mathscr{C}.$$

A semispray is a spray if it satisfies $[\mathscr{C}, \; S] = S$; that is, S is homogeneous of degree 2.

In canonical coordinates, S is given by

$$S = y^i \frac{tial}{tialx^i} - 2G^i(x, \; y) \frac{tial}{tialy^i},$$

for some functions $G^i(x, \; y)$ called coefficients of S. If S is a spray the functions $G^i(x, \; y)$ are homogeneous of degree 2 in the variable y^i.

To any nonlinear connection Γ a semispray S is associated, given by $S = hS'$ where h is the horizontal projector of Γ and S' is an arbitrary semispray (note that S does not depend on the choice of S'). If Γ is homogeneous, then S is a spray.

Conversely, to each semispray S a nonlinear connection $\Gamma = [J,\ S]$ is associated. The semispray associated to this Γ is $\frac{1}{2}(S + [\mathscr{C},\ S])$. If S is a spray, then Γ is homogeneous whose associated semispray is the spray S itself.

Definition 5 [3] An almost golden structure φ on a manifold M is a $(1,\ 1)$-tensor field on M satisfying the equation $\varphi^2 = \varphi + I$. In this case $(M,\ \varphi)$ is an almost golden manifold. An almost golden structure is a golden structure if its Nijenhuis tensor vanishes, i.e. $[\varphi,\ \varphi] = 0$.

Let $\lambda_1 := \frac{1+\sqrt{5}}{2}$, $\lambda_2 := \frac{1-\sqrt{5}}{2}$ be the positive and negative eigenvalues of φ, respectively. We have: $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \lambda_2 = -1$.

3. Almost golden structure and nonlinear connections

In [2], the authors proved that each almost golden structure induces an almost product structure and vice versa. As a nonlinear connection defines an almost product structure [7], it is expected that there may exist some relation between nonlinear connections and almost golden structures on the tangent bundle TM.

Proposition 1 Let φ be an almost golden structure on TM. The necessary and sufficient conditions for the almost product structure Γ induced by φ,

$$\Gamma = \frac{1}{\sqrt{5}}(2\varphi - I), \tag{2}$$

to be a nonlinear connection are that

$$J\varphi = \lambda_1 J,\ \varphi J = \lambda_2 J. \tag{3}$$

(In this case, Γ may also be denoted by Γ_φ to indicate that it is induced by φ).

Proof. Conditions are sufficient. Suppose that the almost golden structure φ satisfies the conditions $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$, then

$$J\Gamma = \frac{1}{\sqrt{5}}(2J\varphi - J) = \frac{1}{\sqrt{5}}(2\lambda_1 - 1)J = J.$$

Similarly, one can show that $\Gamma J = -J$.

Condition are necessary. Let Γ be a nonlinear connection, then

$$J = J\Gamma = \frac{1}{\sqrt{5}}(2J\varphi - J).$$

Consequently, $(\sqrt{5}+1)J = 2J\varphi$, from which $J\varphi = \lambda_1 J$.

Similarly, one can show that $\varphi J = \lambda_2 J$. This completes the proof. □

Now, let us consider the eigenspace of φ corresponding to λ_2:

$$E_{\lambda_2}(TM) := \{X \in \mathfrak{X}(TM) : \varphi X = \lambda_2 X\}.$$

The next proposition declares the coincidence of $E_{\lambda_2}(TM)$ and the vertical space $V(TM)$ (Both spaces are defined at the same point z, for all $z \in TM$).

Proposition 2 Let φ be an almost golden structure on TM satisfying $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$. Then $E_{\lambda_2}(TM) = V(TM)$.

Proof. Since $\varphi J = \lambda_2 J$, then we have $\varphi JX = \lambda_2 JX$, $\forall X \in \mathfrak{X}(TM)$. Hence $JX \in E_{\lambda_2}(TM)$, $\forall X \in \mathfrak{X}(TM)$. So, $V(TM) \subseteq E_{\lambda_2}(TM)$.

Conversely, let $X \in E_{\lambda_2}(TM)$, then $\varphi X = \lambda_2 X$. Applying J to this equation, we get $J\varphi X = \lambda_2 JX$. Consequently, $\lambda_1 JX = \lambda_2 JX$. Hence, $(\lambda_1 - \lambda_2)JX = 0$, from which $JX = 0$, i.e. $X \in V(TM)$. So, $E_{\lambda_2}(TM) \subseteq V(TM)$. □

Concerning the converse of the above proposition, we have

Proposition 3 Let φ be an almost golden structure on TM such that

$$E_{\lambda_2}(TM) = V(TM).$$

Then, $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$.

Consequently, $\Gamma = \frac{1}{\sqrt{5}}(2\varphi - I)$ is a nonlinear connection.

Proof. Let $X \in \mathfrak{X}(TM)$, then JX is vertical, and so $JX \in E_{\lambda_2}(TM)$. Therefore, $\varphi JX = \lambda_2 JX$, for all $X \in \mathfrak{X}(TM)$. This means that $\varphi J = \lambda_2 J$.

On the other hand, as φ is an almost golden structure, it induces an almost product structure $\Gamma = \frac{1}{\sqrt{5}}(2\varphi - I)$. Now, $\Gamma^2 = I \Longleftrightarrow \Gamma^2 - I = 0 \Longleftrightarrow (\Gamma + I)(\Gamma - I) = 0$. Consequently, $(\varphi - \lambda_2 I)(\varphi - \lambda_1 I) = 0$, which implies that $\mathrm{Im}(\varphi - \lambda_1 I) \subseteq \mathrm{Ker}(\varphi - \lambda_2 I) = E_{\lambda_2}(TM)$. Hence, $(\varphi - \lambda_1 I)X$ is a vertical vector field for all $X \in \mathfrak{X}(TM)$. That is, $J(\varphi - \lambda_1 I)X = 0$ for all $X \in \mathfrak{X}(TM)$. Therefore, $J\varphi = \lambda_1 J$. □

The above three Propositions can be combined together to give .

Theorem 1 Let φ be an almost golden structure on TM. The following assertions are equivalent:

(i) The vector form $\Gamma = \frac{1}{\sqrt{5}}(2\varphi - I)$ on TM is a nonlinear connection.

(ii) $E_{\lambda_2}(TM) = V(TM)$.

(iii) $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$.

Expression of φ in local coordinates

An almost golden structure φ on TM can be written locally, in the natural basis $\{\frac{tial}{tialx^i}, \frac{tial}{tialy^i}\}$ of TM, in the form

$$\varphi = \alpha^i_j \frac{tial}{tialx^i} \otimes dx^j + \beta^i_j \frac{tial}{tialx^i} \otimes dy^j + \varphi^i_j \frac{tial}{tialy^i} \otimes dx^j + \gamma^i_j \frac{tial}{tialy^i} \otimes dy^j, \tag{4}$$

where α^i_j, β^i_j, φ^i_j and γ^i_j are functions in x and y. Assume that the golden structure φ satisfies the conditions $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$.

Now, $\varphi J\left(\frac{tial}{tialx^k}\right) = \lambda_2 J\left(\frac{tial}{tialx^k}\right)$, which gives $\varphi\left(\frac{tial}{tialy^k}\right) = \lambda_2 \frac{tial}{tialy^k}$. But from (4), we have $\varphi\left(\frac{tial}{tialy^k}\right) = \beta^i_k \frac{tial}{tialx^i} + \gamma^i_k \frac{tial}{tialy^i}$. Hence, $\beta^i_k = 0$ and $\gamma^i_k = \lambda_2 \delta^i_k$. A similar argument using (4) leads to $\varphi\left(\frac{tial}{tialx^k}\right) = \alpha^i_k \frac{tial}{tialx^i} + \varphi^i_k \frac{tial}{tialy^i}$ and the second condition $J\varphi = \lambda_1 J$ implies that $\alpha^i_k = \lambda_1 \delta^i_k$. Consequently, an almost golden structure φ satisfying $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$ has the matrix form

$$\varphi = \begin{bmatrix} \lambda_1 \delta^i_j & 0 \\ \varphi^i_j & \lambda_2 \delta^i_j \end{bmatrix} \tag{5}$$

where $\varphi^i_j(x,\ y)$ are functions in x and y.

Remark 1 Let φ be an almost golden structure on TM satisfying $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$. The vertical and horizontal projectors of φ are defined to be the vertical and horizontal projectors of the nonlinear connection Γ_φ induced by φ, respectively,

$$v_\varphi := \frac{1}{2}(I - \Gamma_\varphi) = \frac{1}{\sqrt{5}}(-\varphi + \lambda_1 I),\ h_\varphi := \frac{1}{2}(I + \Gamma_\varphi) = \frac{1}{\sqrt{5}}(\varphi - \lambda_2 I). \tag{6}$$

One can easily show that

$$\text{Im}\ v_\varphi = \text{Ker}\ h_\varphi = V(TM) = E_{\lambda_2}(TM),$$

$$\text{Im}\ h_\varphi = \text{Ker}\ v_\varphi = H(TM) = E_{\lambda_1}(TM),$$

where $V(TM)$ and $H(TM)$ are the vertical and horizontal sub-bundles associated with the nonlinear connection Γ_φ. Consequently,

$$TTM = V(TM) \oplus H(TM) = E_{\lambda_2}(TM) \oplus E_{\lambda_1}(TM).$$

As a nonlinear connection defines an almost product structure on TM [7] and the latter induces an almost golden structure on TM [2], hence we have

Proposition 4 Given a nonlinear connection Γ, there is associated a unique almost golden structure φ on TM such that $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$. It is given by:

$$\varphi = \frac{1}{2}(I + \sqrt{5}\,\Gamma).$$

(In this case, φ may also be denoted by φ_Γ to indicate that it induced by Γ).

Having the two sequences of implications:

$$\Gamma \overset{\text{Prop. 3.}}{\Longrightarrow} \varphi_\Gamma \overset{\text{Prop. 3.}}{\Longrightarrow} \Gamma_{\varphi_\Gamma},$$

$$\varphi \overset{\text{Prop. 3.}}{\Longrightarrow} \Gamma_\varphi \overset{\text{Prop. 3.}}{\Longrightarrow} \varphi_{\Gamma_\varphi},$$

we then conclude the present section by the following fundamental result.

Theorem 2 There is a one-to-one correspondence between the nonlinear connections and the almost golden structure on TM satisfying $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$, and we have

$$\Gamma_{\varphi_\Gamma} = \Gamma,\ \varphi_{\Gamma_\varphi} = \varphi.$$

4. Almost golden structures and semisprays

Definition 6 An almost golden structure φ on TM satisfying the conditions $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$ will be referred to as a special almost golden structure.

Definition 7 An almost golden structure φ on TM is homogeneous if

$$[\mathscr{C},\ \varphi] = 0.$$

In this case, the components φ^i_j of a special almost golden structure are homogeneous of degree 1 in y^i (cf. Eqn. (5)).

Theorem 3 To each semispray (resp. spray), there exists a unique special (resp. homogeneous special) almost golden structure φ on TM. Conversely, to each special (resp. homogeneous special) almost golden structure on TM, there exists a unique semispray (resp. spray).

Proof. Let S be a semispray on M, then $[J,\ S]$ is a nonlinear connection on M. Hence, by Proposition 4,

$$\varphi = \frac{1}{2}\left(I + \sqrt{5}\,[J,\ S]\right) \tag{7}$$

is the unique special almost golden structure induced by the connection $[J,\ S]$. Moreover, if S is a spray, then $[J,\ S]$ is a homogeneous nonlinear connection. Consequently, by the relation $[C,\ \varphi] = \frac{\sqrt{5}}{2}\ [C,\ [J,\ S]]$ and the Jacobi identity, φ is homogeneous.

Conversely, let φ be a special almost golden structure on TM. Then, by Proposition 1, φ induces a unique nonlinear connection Γ given by (2), which induces in turn a unique semispray $S = hS'$, where h is the horizontal projector of Γ and S' is an arbitrary semispray [7]. Moreover, if φ is homogeneous, then so are Γ and h. Consequently, by the relation $[C.S] = [C,\ hS'] = [C,\ h]S' + h[C,\ S']$, S is homogeneous, i.e., S is a spray. □

local expression of φ in terms of S

Let S be a semispray on M. It is expressed, in the natural basis $(\frac{tial}{tialx^i}, \frac{tial}{tialy^i})$ of TM, as $S = y^i\frac{tial}{tialx^i} - 2G^i(x,\ y)\frac{tial}{tialy^i}$. The unique special almost golden structure φ on TM induced by the semispray S is given locally by (5). Now, it follows from (7) that

$$\varphi\left(\frac{tial}{tialx^k}\right) = \frac{1}{2}(I + \sqrt{5}[J,\ S])\left(\frac{tial}{tialx^k}\right).$$

As

$$[J,\ S]\left(\frac{tial}{tialx^k}\right) = \left[J\left(\frac{tial}{tialx^k}\right),\ S\right] - J\left[\frac{tial}{tialx^k},\ S\right] = \left[\frac{tial}{tialy^k},\ S\right] = \frac{tial}{tialx^k} - 2\frac{tialG^i}{tialy^k}\frac{tial}{tialy^i},$$

then

$$\varphi\left(\frac{tial}{tialx^k}\right) = \frac{1}{2}\left(\frac{tial}{tialx^k} + \sqrt{5}\left(\frac{tial}{tialx^k} - 2\frac{tialG^i}{tialy^k}\frac{tial}{tialy^i}\right)\right) = \lambda_1\frac{tial}{tialx^k} - \sqrt{5}\,\frac{tialG^i}{tialy^k}\frac{tial}{tialy^i}.$$

On the other hand, it follows from (5), that

$$\varphi\left(\frac{tial}{tialx^k}\right)=\lambda_1\frac{tial}{tialx^k}+\varphi^i_k\frac{tial}{tialy^i}.$$

The above two equation give rise to

$$\varphi^i_k=-\sqrt{5}\,\frac{tialG^i}{tialy^k}. \tag{8}$$

Therefore,

$$\varphi=\begin{bmatrix}\lambda_1\delta^i_j & 0\\ -\sqrt{5}\dfrac{tialG^i}{tialy^j} & \lambda_2\delta^i_j\end{bmatrix}$$

where G^i are coefficients of the semispray S. Note that $\varphi^i_k(x,\ y)$ are homogeneous of degree 1 in y if and only if $G^i(x,\ y)$ are homogeneous of degree 2 in y, that is, φ is homogeneous if and only if S is a spray.

The following example is inspired by [9].

Example 1 Let $M=\{(x^1,\ x^2)\in\mathbb{R}^2:\ x^2>0\}$ and S be the spray given by the coefficients

$$G^1:=\alpha\, y^1+\frac{y^1y^2}{2x^2},\ G^2:=\alpha\, y^2-\frac{(y^1)^2}{4},$$

where $\alpha:=\left(x^2(y^1)^2+(y^2)^2\right)^{1/2}$. Then the coefficients of the almost golden structure φ are given by

$$\varphi^1_1=-\sqrt{5}\left[\frac{y^2}{2x^2}+\alpha+\frac{x^2(y^1)^2}{\alpha}\right],\ \varphi^1_2=-\sqrt{5}\left[\frac{y^1}{2x^2}+\frac{y^1y^2}{\alpha}\right],$$

$$\varphi^2_1=-\sqrt{5}\left[-\frac{y^1}{2}+\frac{x^2y^1y^2}{\alpha}\right],\ \varphi^2_2=-\sqrt{5}\left[\alpha+\frac{(y^2)^2}{\alpha}\right].$$

These coefficients determine completely the special almost golden structure according to Theorem 3.

We use the notations:

φ_S: the special almost golden structure induced by a semispray S,

S_φ: the semispray induced by a special almost golden structure φ. Recall that a special almost golden structure φ defines a horizontal projector h_φ given by (6). Locally h_φ has the form

$$h_\varphi=\frac{tial}{tialx^i}\otimes dx^i+\frac{1}{\sqrt{5}}\,\varphi^i_j\frac{tial}{tialy^i}\otimes dx^j.$$

Hence, for an arbitrary semipspray $S'=y^i\dfrac{tial}{tialx^i}-2\overline{G}^i\dfrac{tial}{tialy^i}$, we have

$$S_\varphi = h_\varphi\, S' = y^i \frac{tial}{tialx^i} - 2\left(-\frac{1}{2\sqrt{5}}\varphi^i_j y^j\right)\frac{tial}{tialy^i}.$$

That is, the functions G^i of the semispray induced by φ are give by

$$G^i = -\frac{1}{2\sqrt{5}}\varphi^i_j y^j \tag{9}$$

Now, consider the following sequence of implications

$$S \Longrightarrow \varphi_S \Longrightarrow S_{\varphi_S}$$

$$G^i \overset{(8)}{\Longrightarrow} -\sqrt{5}\,\frac{tialG^i}{tialy^j} \overset{(9)}{\Longrightarrow} -\frac{1}{2\sqrt{5}}\left(-\sqrt{5}\frac{tialG^i}{tialy^j}\right)y^j$$

Therefore, $S_{\varphi_S} = S$ if and only if $y^j\dfrac{tialG^i}{tialy^j} = 2G^i$, that is, the functions G^i are homogeneous of degree 2 in y^i.

Corollary 1 Let S be a semispray and φ_S the associated almost golden structure. Then the semispray S_{φ_S} is precisely S if and only if S is a spray.

Again, consider the sequence of implications

$$\varphi \Longrightarrow S_\varphi \Longrightarrow \varphi_{S_\varphi}$$

$$\varphi^i_j \overset{(9)}{\Longrightarrow} -\frac{1}{2\sqrt{5}}\varphi^i_j y^j \overset{(8)}{\Longrightarrow} -\sqrt{5}\frac{tial}{tialy^j}\left(\frac{-1}{2\sqrt{5}}\varphi^i_k y^k\right)$$

Hence, $\varphi_{S_\varphi} = \varphi$ if and only if $\varphi^i_j = \dfrac{1}{2}\dfrac{tial}{tialy^j}\left(\varphi^i_k y^k\right)$ or $2\varphi^i_j = \varphi^i_j + \dfrac{tial\varphi^i_k}{tialy^j}y^k$. Equivalently, $\varphi^i_j = \dfrac{tial\varphi^i_k}{tialy^j}y^k$.

Corollary 2 Let φ be a special almost golden structure on TM and S_φ the associated semispray. Then, φ_{S_φ} is precisely φ if and only if

$$\varphi^i_j = \frac{tial\varphi^i_k}{tialy^j}y^k.$$

In this case, S_φ is a spray and φ is homogeneous.

Proof. We prove the last part. As the coefficients of S_φ are given by (9), then

$$\frac{tialG^i}{tialy^j} = \frac{-1}{2\sqrt{5}}\left(\frac{tial\varphi^i_k}{tialy^j}y^k + \varphi^i_j\right) = \frac{-1}{\sqrt{5}}\varphi^i_j.$$

From which, using again (9), we get

$$y^j \frac{tialG^i}{tialy^j} = \frac{-1}{\sqrt{5}} \varphi^i_j y^j = 2G^i.$$

Hence, S_φ is a spray and, by Theorem 3, φ is homogeneous. □

Theorem 4 There is a one-to-one correspondence between the sprays and the homogeneous special almost golden structures on TM, and we have

$$S_{\varphi_S} = S \text{ and } \varphi_{S_\varphi} = \varphi.$$

5. Integrability of golden structures on tangent bundle

In this section we are going to discuss the integrability of a golden structure φ on the tangent bundle TM of a smooth manifold M. As we have mentioned before, an almost golden structure φ is said to be a golden structure if it is integrable, i.e., if its Nijenhuis tensor N_φ vanishes. Moreover, a golden structure on TM induces an almost product structure P on TM and vice versa. By using Equation (2), we can find the relation between Nijenhuis tensors of P and φ.

Proposition 5 [2] Let $(TM,\ \varphi)$ be an almost golden manifold. Then

$$N_P(X,\ Y) = \frac{4}{5} N_\varphi(X,\ Y) \tag{10}$$

for all $X,\ Y \in \mathfrak{X}(TM)$.

Corollary 3 Let $(TM,\ \varphi)$ be an almost golden manifold and let P be the almost product structure on TM induced by φ. Then φ is integrable if and only if P is integrable.

For the almost product structure P on TM induced by φ, we define

$$r := \frac{1}{2}(I+P),\ s := \frac{1}{2}(I-P). \tag{11}$$

Then we have

$$r+s = I,\ r-s = P,\ rs = sr = 0,\ r^2 = r,\ s^2 = s. \tag{12}$$

We set $R :=$ Im (r) and $S :=$ Im (s), then R and S are complementary distributions with r and s their corresponding projector tensors of type $(1,\ 1)$, respectively.

By using $P = \frac{1}{\sqrt{5}}(2\varphi - I)$, straightforward computations give

$$r = \frac{1}{\sqrt{5}}(\varphi - \lambda_2 I),\ s = \frac{1}{\sqrt{5}}(-\varphi + \lambda_1 I), \tag{13}$$

from which

$$\varphi r = r\varphi = \lambda_1 r, \tag{14}$$

$$\varphi s = s\varphi = \lambda_2 s. \tag{15}$$

Recall that a distribution R is integrable if for every $X,\ Y \in \mathfrak{X}(TM)$, $[rX,\ rY] \in R$, or equivalently $s[rX,\ rY] = 0$. Similarly a distribution S is integrable if $r[sX,\ sY] = 0$.

Now, we are going to give a characterization of integrability of an almost golden structure φ in terms of the Nijenhuis tensors N_φ, N_P, N_r and N_s.

Theorem 5 The following assertions are equivalent:

1. The almost golden structure φ is integrable.
2. $N_\varphi = 0$.
3. $N_P = 0$.
4. $N_r = 0$.
5. $N_s = 0$.

Proof. We shall show that $\frac{4}{5}N_\varphi = N_P = 4N_r = -4N_s$. Proposition 5 shows that $\frac{4}{5}N_\varphi = N_P$. On the other hand, the Nijenhuis tensor for P is given by

$$N_P(X,\ Y) = [X,\ Y] + [PX,\ PY] - P[PX,\ Y] - P[X,\ PY],$$

Using Equations (12), we can write

$$[X,\ Y] + [PX,\ PY] = [rX + sX,\ rY + sY] + [rX - sX,\ rY - sY]$$

$$= 2[rX,\ rY] + 2[sX,\ sY],$$

also,

$$[PX,\ Y] + [X,\ PY] = [rX - sX,\ rY + sY] + [rX + sX,\ rY - sY]$$

$$= 2[rX,\ rY] - 2[sX,\ sY],$$

Consequently,

$$\frac{1}{2}N_P(X,\ Y) = [rX,\ rY] + [sX,\ sY] - P[rX,\ rY] + P[sX,\ sY]$$

$$= (I - P)[rX,\ rY] + (I + P)[sX,\ sY]$$

$$= 2s[rX,\ rY] + 2r[sX,\ sY].$$

Hence,

$$\frac{1}{4}N_P(X,\ Y) = r[sX,\ sY] + s[rX,\ rY]. \tag{16}$$

Moreover, considering $r+s=I$, the Nijenhuis tensor for r is

$$\begin{aligned} N_r(X,\ Y) &= r[X,\ Y] + [rX,\ rY] - r[rX,\ Y] - r[X,\ rY] \\ &= r([X,\ Y] - [rX,\ Y]) + [rX,\ rY] - r[X,\ rY] \\ &= r[sX,\ Y] + [rX,\ rY] - r[X,\ rY] \\ &= r[sX,\ rY+sY] + (r+s)[rX,\ rY] - r[rX+sX,\ rY] \\ &= r[sX,\ sY] + s[rX,\ rY]. \end{aligned} \tag{17}$$

Similarly, we can show that

$$N_s(X,\ Y) = -r[sX,\ sY] - s[rX,\ rY]. \tag{18}$$

The identities to be proved follow directly from Equations (15-18). □

If the almost golden structure φ on TM is special, i.e., $J\varphi = \lambda_1 J$ and $\varphi J = \lambda_2 J$, then the almost product structue $\Gamma = \frac{1}{\sqrt{5}}(2\varphi - I)$ induced by φ is a nonlinear connection. Consequently, in view of Theorem 5, we have the following

Proposition 6 A special almost golden structure φ on TM is integrable if and only if the curvature (1) of the nonlinear connection $\Gamma = \frac{1}{\sqrt{5}}(2\varphi - I)$ induced by φ vanishes.

Proposition 7 For distribution R and S in M, we have

$$s[rX,\ rY] = \frac{1}{5}sN_\varphi(rX,\ rY)$$

$$r[sX,\ sY] = \frac{1}{5}rN_\varphi(sX,\ sY),$$

for every $X,\ Y \in \mathfrak{X}(TM)$.

From the previous discussion we conclude that, the distribution R is integrable if and only if $sN_\varphi(rX,\ rY) = 0$ and the distribution S is integrable if and only if $rN_\varphi(sX,\ sY) = 0$. Finally if the golden structure φ is integrable then both distributions R and S are integrable.

6. Concluding remarks

We conclude the present article by the following comments and remarks:

In view of Theorem 2, there is a one-to-one correspondence between the nonlinear connections and the special almost golden structure on TM, and we have

$$\Gamma_{\varphi_\Gamma} = \Gamma, \ \varphi_{\Gamma_\varphi} = \varphi. \tag{19}$$

In view of Theorem 4, there is a one-to-one correspondence between the sprays and the homogeneous special almost golden structures on TM, and we have

$$S_{\varphi_S} = S, \ \varphi_{S_\varphi} = \varphi. \tag{20}$$

According to [7], there is a one-to-one correspondence between the sprays and the homogeneous nonlinear connections, and we have

$$\Gamma_{S_\Gamma} = \Gamma, \ S_{\Gamma_S} = S. \tag{21}$$

Conclusion. There are one-to-one correspondences between the following geometric objects:

(a) homogeneous nonlinear connections,

(b) homogeneous special almost golden structures on TM,

(c) sprays.

and we have (19), (20) and (21).

The question of integrability of almost golden structures is discussed and certain characterizations of integrability are found out.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Hrețcanu C, Crăsmăreanu M. Applications of the golden ratio on Riemannian manifolds. *Turkish Journal of Mathematics*. 2009; 33(4): 179-191.

[2] Crăsmăreanu M, Hrețcanu C. Golden differential geometry. *Chaos, Solitons and Fractals*. 2008; 38: 1229-1238.

[3] Etayo F, Santamaria R, Upadhyay A. On the geometry of almost golden Riemannian manifolds. *Mediterranean Journal of Mathematics*. 2017; 14(5): 187-200.

[4] Gezer A, Cengiz N, Salimov A. On integrability of Golden Riemannian structures. *Turkish Journal of Mathematics*. 2013; 37(4): 693-703.

[5] Yano K, Ishihara S. *Tangent and Cotangent Bundle*. New York: Marcel Dekker Inc.; 1973.

[6] Goldberg S, Yano K. Polynomial structures on manifolds. *Kodai Mathematical Seminar Reports*. 1970; 22(2): 199-218.

[7] Grifone J. Structureprésque-tangente et connexions, I. *Annales de l'Institut Fourier, Grenoble [Grenoble Fourier Institute Yearbook]*. 1972; 22(1): 287-334.

[8] Grifone J, Muzsnay Z. *Variational Principles for Second-order Differential Equations, Application of the Spencer Theory of*. World Scientific; 2000.
[9] Elgendi S, Muzsnay Z. Freedom of $h(2)$-variationality and metrizability of sprays. *Differential Geometry and Its Applications*. 2017; 54: 194-207.