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Research Article

Decision Making on Determinant and Adjoint of a Square Neutrosophic Fuzzy Economic Model

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Abstract: Matrices have significant contribution in the field of science and technology. It is often seen that fuzzy matrix theory theory cann't address all types of uncertainty. In order to solve real life problems containing the neutrosophic phenomena, determinant and adjoint of a neutrosophic fuzzy matrix are required. The article introduces the notions of determinant and adjoint of neutrosophic fuzzy matrices. We shall define the idea of the neutrosophic matrix in which every element will appear with the degree of membership values of independent functions, namely membership, indeterminacy, and non-membership functions. Each degree of these membership values will be taken from the interval [0, 1]. With the help of this notion, we shall introduce the determinant and adjoint of a square neutrosophic fuzzy matrix. We shall investigate some basic properties, theorems and results of the determinant and adjointof a square neutrosophic fuzzy matrix. We shall justify these concepts by providing proof. Some properties and results shall be verified by providing suitable numerical examples. We shall give some definitions based on the newly defined concepts. The article reveals an innovative idea which contributes to find determiant and adjoint of neutrosophic fuzzy matrices generated from real-life problems. These newly introduced definitions will have applications in relevant science and technology research fields such as to solve linear equations containing neutrosophic fuzzy matrices.

***Keywords*:** neutrosophic set, neutrosophicfuzzy matrix, determinant and adjoint of neutrosophic fuzzy matrices

MSC: 03E72, 54A05, 54A40, 54J05

1. Introduction

Zadeh [1] inventeda fuzzy set as a suitable mathematical tool to describe uncertainty that prevailed in nature. We have found intensive acceptability in various fields of fuzzy sets since its inception. Traditionally, membership value characterizesthe fuzzy set. After that, interval-valued fuzzy sets were defined [2] to deal with the ambiguous situation. Sometimes, it is found that the truth membership and the falsity-membership values are considered to describe an object

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properly, such as a belief system, expert system, information fusion, and so on. Such an ambiguous situation cannot be described properly with the notions of these concepts. To resolve such a situation, Atanassov [3] introduced the notion of intuitionistic fuzzy sets where membership and non-membership values are considered simultaneously. It fails to give inconsistent and indeterminate information aboutthe belief system.

After that, to deal with real-life problems of imprecise, indeterminacy, and inconsistent data, Smarandache [4] revealedneutrosophic set. Each element of the neutrosophic set hasthree independent functions, namely, the membership function (α), indeterminacy function (β), and non-membership (γ) function. Out of these, β denotes the non-non-deterministic part of the ambiguous situation. We can find the applications of NS in statistical physics, financial markets, risk management, mathematical biology, neutrosophic linguistic variables, neutrosophic decision making and preference structures, neutrosophic expert systems, neutrosophic reliability theory, neutrosophic logic to robotics etc. In recent years, several researchers [5–13] contributed themselves to investigate the applications of neutrosophic sets.

Matrices have an important role in science and technology. Due to its formulation of a mathematical formula, a matrix gives an extra advantage to solve the problem. It is a fact that the classical matrix theory is not sufficient to solve the ambiguous situation occurring in an imprecise environment where problems are solved by using a fuzzy matrix. Thomas [14] introduced the notion of fuzzy matrices and discussed their convergence of powers. Dhar [15] studied thedeterminant and adjoint of the fuzzy square matrix.Fuzzy and neutrosophic relational maps have been investigated by Kandasamy and Smarandache [16]. We can find the importance of matrices in the theory of vector spaces in classical algebra. These concepts have been generalized to neutrosophic matrices [17, 18]. The neutrosophic fuzzy matrices have been defined and studied by Dhar et al. [19]. Das et al. [20] investigated algebraic operations on neutrosophic fuzzy matrices. Abobala et al. [21] investigated the algebraic structure in neutrosophic square matrices. Kim and Baartmans [22] investigated determinant theory for fuzzy matrices.

Necesscity and objective of the article: The main objectives of the article are to define the notions of determinant and adjoint of neutrosophic fuzzy matrices and to investigate the properties, of these newly defined concepts. The articles also invents theorems and results based on the newly defined concepts.

The contribution of the article: We have invented notions of adjoint and determinant of neutrosophic fuzzy matrices which are quite different from usual determinat and adjoint of other matrices of real or complex entries. We have discussed basic properties, theorems and results of them in neutrosophic fuzzy matrices. We have also discussed suitable examples to justify the introduction of the notions.

We frame the article in different sections. The next section briefly briefly defines neutrosophic setsand fuzzy and neutrosophic matrices. In Section 3, we shall investigate some properties of determinants of neutrosophic fuzzy matrices. In Section 4, we shall focus on some properties of adjoint of neutrosophic fuzzy matrices. Conclusion appears in Section 5.

2. Preliminaries

Here, we recall a few relevant concepts and results for this article.

Definition 2.1 [4] The neutrosophic set A of a universal set U is a set such that $A = \{<x: \alpha_A(x), \ \beta_A(x), \gamma_A(x)>, x \in U\}$, where the functions $\alpha, \beta, \gamma: U \to [-0, 1+]$ denote respectively the degree of membership, degree of indeterminacy and degree of non-membership of an element x of U to the set A having the condition

$$^{-}0 \leq \alpha_A(x) + \beta_A(x) + \gamma_A(x) \leq 3^{+}.$$

From a philosophical point of view, it takes the value from real standard or non-standard subsets of [−0, 1 +]. We consider [0, 1] in science and engineering problems.

Example 2.2 Let $U = \{s_1, \ s_2, \ s_3\}$, where s_1, s_2 and s_3 describes the quality, reliability and price of the components. It isalso considered that the membership values of $\{s_1, \ s, \ s_3\}$ areobtained fromsome experts' investigations from [0, 1].

To explain the characteristics of the objects, the experts may provide observationson three components, viz; the degree of goodness, degree of indeterminacy, and degree of poorness. Consider N is a neutrosophic set (NS) of U, where $N = \{(s_1, 0.6, 0.4, 0.2), (s_2, 0.5, 0.3, 0.2), (s_3, 0.7, 0.4, 0.3)\}$. Here, 0.6 is the degreeof goodness, 0.4 is the degree of indeterminacy, and 0.2 is the degree of falsity of quality of s_1, etc.

Definition 2.3 [15] A fuzzy matrix isa matrix whosecomponentsare taken from the unit fuzzy interval [0, 1]. If the number of rows and columns of a fuzzy matrix are equal and objects of the matrix belong to the unit interval [0, 1], it is called a fuzzy square matrix.

A fuzzy square matrix of order two is expressed in the following way $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Where $a, b, c, d \in [0, 1]$.

Example 2.4 $M = \begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.3 \end{bmatrix}$ is a fuzzy matrix of order 2.

Definition 2.5 [19] Let A aneutrosophic fuzzy matrix of order $m \times n$ and it is defined as $A = (a_{ij})_{m \times n}$ where $(a_{ij}) = (a^{\alpha}_{ij}, a^{\beta}_{ij}, a^{\gamma}_{ij})$ is the ij-th element of A. Here a^{α}_{ij}, a^{β}_{ij} and a^{γ}_{ij} denote the degree of membership, indeterminacy, and non-membership functions, respectively, and each degree of membership value of these functions is taken from the interval [0, 1].

A neutrosophicfuzzy matrix for which the number of rows is equal to the number of columns and whose of degree of membership value of the three independent functions takes from the interval [0, 1] is called a fuzzy square matrix of order n.

Example 2.6 $M = \begin{bmatrix} (0.4, 0.2, 0.3) & (0.6, 0.7, 0.4) \\ (0.5, 0.3, 0.2) & (0.4, 0.3, 0.2) \end{bmatrix}$ is a neutrosophic fuzzy matrix of order 2×2.

Definition 2.7 In case of classical matrix, the two matrices P and Q are said to be comfortable for product if the number of column of matrix P is equal to the number of rows im the matrix Q. In the same way, in order to find the product of two two neutrosophic fuzzy matrices A and B, the number of colun of matrix A is equal to the number rows in the matrix B. Thus the producf AB can be defined in the following way:

Let us consider two neutrosophic fuzzy matrices of order $m \times n$ and $n \times p$ respectively where $A = ((a^{\alpha}_{ij}, a^{\beta}_{ij}, a^{\gamma}_{ij})) = (a_{ij})$ and $B = ((b^{\alpha}_{ij}, b^{\beta}_{ij}, b^{\gamma}_{ij})) = (b_{ij})$. Then the product of A and B is defined as $AB = ()$.

The order of AB will be $m \times p$. If the number of columns of A is same as the number of rows of B, then product AB is defined. In this case A and B are said to be comfortable for multiplication.

3. Determinant of a neutrosophic fuzzy matrix

Here, the concept of the determinant of a neutrosophic fuzzy matrix and some of its basic properties are investigated.

Definition 3.1 Let $M = ((m^{\alpha}_{ij}, m^{\beta}_{ij}, m^{\gamma}_{ij})) = (m_{ij})$ be a neutrosophic fuzzy matrix of order $n \times n$. Now the determinant of M, denoted by $|M|$, is as follows:

$|M| = ()$.

Here S_n means the symmetric group of all permutations of the indices $(1, 2, 3, \ldots.., n)$.

Example 3.2 Consider $M = ((m^{\alpha}_{ij}, m^{\beta}_{ij}, m^{\gamma}_{ij}))$ be a neutrosophic fuzzy matrix of order 2×2 such that

$M = \begin{bmatrix} (0.4, 0.2, 0.3) & (0.6, 0.7, 0.4) \\ (0.5, 0.3, 0.2) & (0.4, 0.3, 0.2) \end{bmatrix}$

$|M| = () \vee ($

$=$

$= (0.5, 0.3, 0.3)$.

Theorem 3.3 If a neutrosophicfuzzy matrix P is obtained from an $n \times n$ neutrosophic fuzzy matrix $M = (m^{\alpha}_{ij}, m^{\beta}_{ij}, m^{\gamma}_{ij})$ of order $n \times n$ by multiplying the i-th row (or i-th column) of M by $k \in [0, 1]$, then $\det(P) = k \det(M)$.

Proof. Suppose that $P = (p^{\alpha}_{ij}, p^{\beta}_{ij}, p^{\gamma}_{ij})$.Then

$\det(P) = |P| = ()$

= ()
= ()
$= k((m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}))$
$= k|M|$.

Definition 3.4 Let $M = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma})$ be a neutrosophic fuzzy matrix of order $m \times n$. Now M^T, the transpose of M, is a matrix of order $n \times m$ and is defined by

$$M^T = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}).$$

Theorem 3.5 Let $M = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma})$ be a neutrosophic fuzzy matrix of order $n \times n$. Then det (M) = det (M^T).

Proof. Consider $M^T = (p_{ij}^{\alpha}, p_{ij}^{\beta}, p_{ij}^{\gamma})$. Permutation σ is a one-to-one function. Thus,
$|M^T| = ()$
= ()
= ()
$= |A|$.
Here, the permutation ζ is induced by the rearrangement of each σ in S_n.

Example 3.6 Let $M = ((m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}))$ be a neutrosophic fuzzy matrix of order 2×2 such that

$$M = \begin{bmatrix} (0.3,\ 0.4,\ 0.5) & (0.4,\ 0.2,\ 0.6) \\ (0.7,\ 0.4,\ 0.5) & (0.6,\ 0.4,\ 0.5) \end{bmatrix}$$

$|M| = () \vee ($
=
= (0.4, 0.4, 0.5)

$$\text{Now } M^T = \begin{bmatrix} (0.3,\ 0.4,\ 0.5) & (0.7,\ 0.4,\ 0.5) \\ (0.4,\ 0.2,\ 0.6) & (0.6,\ 0.4,\ 0.5) \end{bmatrix}$$

$|M^T| = () \vee ($
=
= (0.4, 0.4, 0.5)
Thus $|M| = |M^T|$.

Theorem 3.7 Let $M = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma})$ be a neutrosophic fuzzy matrix of order $n \times n$. We consider M contains a zero row or a zero column.Then $|M|$ = (0, 0, 1).

Proof. Here, each term in $|M|$ contains a factor of each row or column. Thus, it contains a factor of zero rowor zero column. Hence, each term of $|M|$ is equal to zero and consequently $|M|$ = (0, 0, 1). Here, zero means the component of the form (0, 0, 1).

Theorem 3.8 Consider $M = ((m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}))$ is a neutrosophic fuzzy matrix of order $n \times n$. If we consider M is triangular, then the determinant of M is given by $|M|$ = ().

Proof. Suppose that $M = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma})$ is lower triangular. We consider the term of $|M|$where $t_{m^{\alpha}} =$, $t_{m^{\beta}} =$, $t_{m^{\gamma}} =$.
Consider $\sigma(1) \neq 1$. Then $1 < \sigma(1)$ and so $m_{1\sigma(1)}^{\alpha} = 0$, $m_{1\sigma(1)}^{\beta} = 0$, $m_{1\sigma(1)}^{\gamma} = 1$.
It implies that $t_{m^{\alpha}} = 0$, $t_{m^{\beta}} = 0$, $t_{m^{\gamma}} = 1$.

Now, let $\sigma(1) = 1$ and $\sigma(2) \neq 2$. Then $2 < \sigma(2) \neq 2$, $2 < \sigma(2)$ and $m_{2\sigma(2)}^{\alpha} = 0$, $m_{2\sigma(2)}^{\beta} = 0$, $m_{2\sigma(2)}^{\gamma} = 1$ and $t_{m^{\alpha}} = 0$, $t_{m^{\beta}} = 0$, $t_{m^{\gamma}} = 1$. That means that $t_{\alpha^{\alpha}} = 0$, $t_{m^{\beta}} = 0$, $t_m\gamma = 1$, if $\sigma(1) \neq 1$ or $\sigma(2) \neq 2$.

Here, we see that each term $t_{m^{\alpha}}$, $t_{m^{\beta}}$, $t_{m^{\gamma}}$, for all $\sigma(1) \neq 1$, $\sigma(2) \neq 2$, ….., $\sigma(n) \neq n$ must be zero, zero, one respectively. Thus, $|M|$ = ().

4. Adjoint of a neutrosophic fuzzy matrix

Here, we define the adjoint of a neutrosophic fuzzy matrix and investigate some of its basic properties.

Definition 4.1 Let $M = ((m_{\text{ij}}^{\alpha},\ m_{\text{ij}}^{\beta},\ m_{\text{ij}}^{\gamma}))$ be a neutrosophic fuzzy matrix of order $n \times n$. Let $(p_{\text{ij}}^{\alpha},\ p_{\text{ij}}^{\beta},\ p_{\text{ij}}^{\gamma}) = |M_{ji}|$ be the value of determinant of the $(n - - 1) \times (n - -1)$ neutrosophic fuzzy matrix obtained by deleting row j and column i from M. Then the neutrosophic fuzzy matrix $P = \text{adj}M = ((p_{\text{ij}}^{\alpha},\ p_{\text{ij}}^{\beta},\ p_{\text{ij}}^{\gamma}))$ is called the adjoint of the neutrosophic fuzzy matrix.

The order of adjointof a neutrosophic fuzzy matrix is the same as that of the given one.

Remark 4.2 The element p_{ij} of adj$M = P = (p_{ij}) = ((p_{\text{ij}}^{\alpha},\ p_{\text{ij}}^{\beta},\ p_{\text{ij}}^{\gamma}))$ is expressed as given below:

$$p_{ij} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \left(m_{\text{t}\pi(\text{t})}^{\alpha},\ m_{\text{t}\pi(\text{t})}^{\beta},\ m_{\text{t}\pi(\text{t})}^{\gamma} \right)$$

where $n_j = \{1,\ 2,\ 3,\ \ldots..,\ n\} \backslash \{j\}$ and $S_{n_j n_i}$ is the set of all permutations of the set n_j over the set n_i.

Example 4.3 Let $M = \begin{bmatrix} (0.3,\ 0.4,\ 0.1) & (0.7,\ 0.2,\ 0.5) \\ (0.6,\ 0.7,\ 0.5) & (0.1,\ 0.5,\ 0.3) \end{bmatrix}$.

Then adj$M = \begin{bmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{bmatrix}$.

Here $M_{11} = (0.1,\ 0.5,\ 0.3)$

$M_{12} = (0.6,\ 0.7,\ 0.5)$

$M_{21} = (0.7,\ 0.2,\ 0.5)$

$M_{22} = (0.3,\ 0.4,\ 0.1)$.

Therefore, adj$M = \begin{bmatrix} (0.1,\ 0.5,\ 0.3) & 0.7,\ 0.2,\ 0.5 \\ (0.6,\ 0.7,\ 0.5) & (0.3, 0.4,\ 0.1) \end{bmatrix}$.

Proposition 4.4 Let $M = ((m_{\text{ij}}^{\alpha},\ m_{\text{ij}}^{\beta},\ m_{\text{ij}}^{\gamma}))$ and $P = ((p_{\text{ij}}^{\alpha},\ p_{\text{ij}}^{\beta},\ p_{\text{ij}}^{\gamma}))$ be two neutrosophic fuzzy matrices, each of order $n \times n$. Then the following hold:

(a) $M \le P$ implies adj$M \le$ adj$P$.

(b) adj$M +$ adj$P \le$ adj$(M + P)$.

(c) adj$M^T = (\text{adj}M)^T$.

Proof. (a) $C =$ adj$M$ and $D =$ adj$P$. Then

$$\text{c}_{\text{ij}} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \left(m_{\text{t}\pi(\text{t})}^{\alpha},\ m_{\text{t}\pi(\text{t})}^{\beta},\ m_{\text{t}\pi(\text{t})}^{\gamma} \right) \text{ and}$$

$$d_{\text{ij}} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \left(p_{\text{t}\pi(\text{t})}^{\alpha},\ p_{\text{t}\pi(\text{t})}^{\beta},\ p_{\text{t}\pi(\text{t})}^{\gamma} \right).$$

It is clear that $c_{ij} \le d_{ij}$ since

$m_{\text{t}\pi(\text{t})}^{\alpha} \le p_{\text{t}\pi(\text{t})}^{\alpha}$

$m_{\text{t}\pi(\text{t})}^{\beta} \le p_{\text{t}\pi(\text{t})}^{\beta}$

$m_{\text{t}\pi(\text{t})}^{\gamma} \ge p_{\text{t}\pi(\text{t})}^{\gamma}$ for each $t \ne j,\ \pi(t) \ne i$.

(b) Since $M,\ P \le M + P$, it is clear that adj$M$, adj$P \le$ adj $(M + P)$ and so adj$M$ + adj$P \le$ adj $(M + P)$.

(c) Let $B =$ adj$M$ and $C =$ adjM^T. Then

$$b_{\text{ij}} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \left(m^{\alpha}_{\text{t}\pi(\text{t})},\ m^{\beta}_{\text{t}\pi(\text{t})},\ m^{\gamma}_{\text{t}\pi(\text{t})} \right) \text{ and}$$

$$\text{c}_{\text{ij}} = \sum_{\pi \in S_{n_j n_i}} \prod_{\pi(t) \in n_j} \left(b^{\alpha}_{\text{t}\pi(\text{t})},\ b^{\beta}_{\text{t}\pi(\text{t})},\ b^{\gamma}_{\text{t}\pi(\text{t})} \right).$$

which is the element b_{ji}. Hence $(\text{adj}A)^T = \text{adj}A^T$.

Proposition 4.5 Consider M as aneutrosophic fuzzy matrix thathas a zero row. Then (adj$M$) M is a zero matrix i.e., (adj$M$) M = ((0, 0, 1)).

Proof. Consider D = (adj$M$) M. Then $d_{\text{ij}} = \sum_k |M_{ki}|(m^{\alpha}_{\text{kj}},\ m^{\beta}_{\text{kj}},\ m^{\gamma}_{\text{kj}})$. If the i-th row of M is ((0, 0, 1)), then M_{ki} contains a zero row where $k \neq i$. Thus, by the Theorem 3.7, $|A_{ki}|$ = (0, 0, 1) for every $k \neq i$ and if $k = i$, then m_{ij} = 0 for every j, and hence

$$\sum_k |M_{ki}|\,(m^{\alpha}_{\text{kj}},\ m^{\beta}_{\text{kj}},\ m^{\gamma}_{\text{kj}}) = ((0,0,1)).$$

Thus, (adj$M$)M = ((0, 0, 1)).

Theorem 4.6 Consider Masa neutrosophic fuzzy matrix. Then

$$|M| = |\text{adj}M|.$$

Proof. Since $\text{adj}M = (|M_{ji}|)$, so

$$|\text{adj}M| = \sum_{\pi \in S_n} |M_{1\pi(1)}||M_{2\pi(2)}| \ldots |M_{n\pi(n)}|$$

This can be expressed as:

$$= \sum_{\pi \in S_n} \prod_{i=1}^{n} |M_{i\pi(i)}|$$

Continuing, we have:

$$= \sum_{\pi \in S_n} \left[\coprod_{i=1}^{n} \left(\sum_{\pi \in S_{n_i}} n_{\pi(i)} \coprod_{t \in n_i} \left(m^{\alpha}_{t\theta(t)},\ m^{\beta}_{t\theta(t)},\ m^{\gamma}_{t\theta(t)} \right) \right) \right]$$

This can be broken down further as:

$$= \sum_{\pi \in S_n} \left[\left(\sum_{\pi \in S_{n_1}} n_{\pi(1)} \coprod_{t \in n_1} \left(m^{\alpha}_{t\theta(t)},\ m^{\beta}_{t\theta(t)},\ m^{\gamma}_{t\theta(t)} \right) \right) \left(\sum_{\pi \in S_{n_2}} n_{\pi(2)} \coprod_{t \in n_2} \left(m^{\alpha}_{t\theta(t)},\ m^{\beta}_{t\theta(t)},\ m^{\gamma}_{t\theta(t)} \right) \right) \cdots \right.$$

$$\left(\sum_{\pi \in S_{n_n}} n_{\pi(n)} \coprod_{t \in n_n} \left(m^{\alpha}_{t\theta(t)},\ m^{\beta}_{t\theta(t)},\ m^{\gamma}_{t\theta(t)} \right) \right)$$

This leads to:

$$= \sum_{\pi \in S_n} \left(\coprod_{t \in n_1} (m^{\alpha}_{t\theta_1(t)},\ m^{\beta}_{t\theta_1(t)},\ m^{\gamma}_{t\theta_1(t)}) \cdots \coprod_{t \in n_n} (m^{\alpha}_{t\theta_n(t)},\ m^{\beta}_{t\theta_n(t)},\ m^{\gamma}_{t\theta_n(t)}) \right)$$

Finally, we conclude with:

$$= \sum_{\pi \in S_n} \left[m_{1\theta_{f_1}(1)} m_{2\theta_{f_2}(2)} \cdots m_{n\theta_{f_n}(n)} \right]$$

for some $f_h \in \{1,\ 2,\ \ldots,\ n\} \setminus \{h\}$, $h = 1,\ 2,\ \ldots,\ n$. However, because $m_h\theta_{f_h}(h) \neq m_{h\sigma}(f_h)$, we can see that $m_h\theta_{f_h}(h) = m_{h\pi}(f_h)$. Therefore,

$$|\text{adj}M| = \sum_{\sigma \in S_n} m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{n\sigma(n)}$$

which is the expansion of $|M|$.

Hence, the theorem.

Example 4.7 Consider the Example 4.3

$|M|$ = ()∨(

=

= (0.6,   0.5,   0.3).

Again, we get

$|\text{adj}M|$ = ()∨(

=

= (0.6,   0.5,   0.3).

Hence $|M| = |\text{adj}M|$.

Definition 4.8 Consider M aneutrosophic fuzzy matrix of order $m \times n$. Itis said to be constant if $(m^{\alpha}_{\text{ij}},\ m^{\beta}_{\text{ij}},\ m^{\gamma}_{\text{ij}}) = (m^{\alpha}_{\text{jk}},\ m^{\beta}_{\text{jk}},\ m^{\gamma}_{\text{jk}})$ for all $i,\ j,\ k$.

Definition 4.9 Let $M = ((m^{\alpha}_{\text{ij}},\ m^{\beta}_{\text{ij}},\ m^{\gamma}_{\text{ij}}))$ be a constant neutrosophic fuzzy matrix of order $n \times n$. Then the following hold:

(a) $(\text{adj}M)^T$ is constant.

(b) $C = M(\text{adj}M)$ is constant and $C_{ij} = |M|$ which is the least element in M.

Proof. (a) Let $B = \text{adj}M$. Then

$$b_{ij} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \left(\left(m^{\alpha}_{\mathrm{t}\pi(\mathrm{t})},\ m^{\beta}_{\mathrm{t}\pi(\mathrm{t})},\ m^{\gamma}_{\mathrm{t}\pi(\mathrm{t})} \right) \right) \text{ and}$$

$$b_{ik} = \sum_{\pi \in S_{n_k n_i}} \prod_{t \in n_k} \left(\left(m^{\alpha}_{\mathrm{t}\pi(\mathrm{t})},\ m^{\beta}_{\mathrm{t}\pi(\mathrm{t})},\ m^{\gamma}_{\mathrm{t}\pi(\mathrm{t})} \right) \right).$$

We note that $b_{ij} = b_{ik}$, since the numbers $\pi(t)$ of columns cannot be changed in the two expansions of b_{ij} and b_{ik}. So that $(adjM)^T$ is constant.

(b) Since M is constant, we find that $M_{jk} = M_{ik}$ and so $|M_{jk}| = |M_{ik}|$ for every $i,\ j \in \{1,\ 2,\ \ldots,\ n\}$. Thus

$$c_{ij} = \sum_{k=1}^{n} \left(\left(m^{\alpha}_{\mathrm{ik}},\ m^{\beta}_{\mathrm{ik}},\ m^{\gamma}_{\mathrm{ik}} \right) \right) |M_{jk}|$$

$$= \sum_{k=1}^{n} \left(\left(m^{\alpha}_{\mathrm{ik}},\ m^{\beta}_{\mathrm{ik}},\ m^{\gamma}_{\mathrm{ik}} \right) \right) |M_{ik}|$$

$$= |M|.$$

Now, we have $|M| = \sum_{\pi \in S_n} m_{1\pi(1)} m_{2\pi(2)} \ldots\ldots\ldots m_{n\pi(n)} = m_{2\pi(2)} m_{3\pi(3)} \ldots\ldots. m_{n\pi(n)}$, for any $\pi \in S_n$ (since M is constant). Considering π as the identity permutation, we get

$|M| = m_{11} m_{22} \ldots\ldots. m_{nn}$, the least element of M.

Definition 4.10 Let $M = ((m^{\alpha}_{\mathrm{ij}},\ m^{\beta}_{\mathrm{ij}},\ m^{\gamma}_{\mathrm{ij}}))$ be a neutrosophic fuzzy matrix of order n. Then the following hold:

(a) If $M = M^2$, then M is called idempotent.

(b) If $M \geq I_n$, then M is called reflexive.

(c) If $M^2 \leq M$, then M is called transitive.

(d) If $M = M^T$, then M is called symmetric.

Example 4.11 Let $M = ((m^{\alpha}_{\mathrm{ij}},\ m^{\beta}_{\mathrm{ij}},\ m^{\gamma}_{\mathrm{ij}}))$ be a neutrosophic fuzzy matrix of order 2×2 such that

$$M = \begin{bmatrix} (1,\ 1,\ 0) & (0.3,\ 0.5,\ 0.4) \\ (0.1,\ 0.4,\ 0.3) & (1,\ 1,\ 0) \end{bmatrix}.$$

$$M^2 = \begin{bmatrix} (1,\ 1,\ 0) & (0.3,\ 0.5,\ 0.4) \\ (0.1,\ 0.4,\ 0.3) & (1,\ 1,\ 0) \end{bmatrix} \begin{bmatrix} (1,\ 1,\ 0) & (0.3,\ 0.5,\ 0.4) \\ (0.1,\ 0.4,\ 0.3) & (1,\ 1,\ 0) \end{bmatrix}$$

$$= \begin{bmatrix} (1,\ 1,\ 0) & (0.3,\ 0.5,\ 0.4) \\ (0.1,\ 0.4,\ 0.3) & (1,\ 1,\ 0) \end{bmatrix}$$

$$= A.$$

Therefore, M is idempotent.

Example 4.12 Let $M = ((m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}))$ be a neutrosophic fuzzy matrix of order 2×2 such that

$$M = \begin{bmatrix} (0.4,\ 0.3,\ 0.5) & (1,\ 1,\ 0) \\ (1,\ 1,\ 0) & (0.2,\ 0.4,\ 0.3) \end{bmatrix}.$$

$$M^T = \begin{bmatrix} (0.4,\ 0.3,\ 0.5) & (1,\ 1,\ 0) \\ (1,\ 1,\ 0) & (0.2,\ 0.4,\ 0.3) \end{bmatrix} = M.$$

Thus M is a symmetric matrix.

5. Conclusions

In this article, we introduced the notion of determinant and adjoint of neutrosophic fuzzy matrices. We investigated the basic properties, results, examples and theorem of a square neutrosophic fuzzy matrix. We verified some properties and results with the help of suitable examples. The investigation of newly defined concepts will be beneficial for further research in science and engineering.

6. Limitation of the article

The main disadvantage of the article lies in the fact that determinat and adjoint of neutrosophic fuzzy matrices can not be used to solve equations which have imaginary roots. The arithmetic operations like multiplication become very difficult when it involves neutrosophic fuzzy matrices. A neutrosophic matrix is not capable of depicting non-linear transformations since it is a linear structure.

7. Future direction of the article

It is expected that the work done will motivate in further investigation of the neutrosophic fuzzy matrices. It is hoped that the notions of finding determinant and adjoint of neutrosophic fuzzy matrices which have been discussed in the article can also be to find inverse of neutrosophic fuzzy matrices as well as to solve simultaneous neutrosophic linear equations.

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Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Significance statement

We have introduced the new concepts on the determinants and adjoints of neutrosophic fuzzy matrices and investigated the different basic characteristics, theorems, and results of these newly defined concepts. Some numerical examples have also been provided. It may be helpful for further investigation in different areas of science and technology research.

Conflict of interest

The authors declare no competing financial interest.

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