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Research Article

On a Model for Solving Mixed Fractional Integro Differential Equation

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Abstract: In this work, the mixed fractional integro differential equation (MfrIo-DE) of the second kind, under certain condition is considered, in the space $L_2(-1,\ 1)\times\ C\ [0,\ T]$; $T<1$ T is the time. The position kernel $k\left(|x-y|\right)$ of IE has a singularity. After integrating and using the properties of fractional integral, we have a MIE in position and time, where the kernel of position takes the singular form $k\left(|x-y|\right)$, and the kernel of time takes the singular Abel form $(t-\tau)^{\alpha-1}$, $0<\alpha<1$. Then, using separation of variable method, under certain substitution, we obtain FIE in position, with variable fractional coefficients in time. Using the Toeplitz matrix method (TMM), we have a nonlinear algebraic system (NAS). Moreover, numerical results are obtained and discussed, especially when $0<\alpha<1$. Also, the solutions of the mixed equation are considered when $\alpha=0,\ \alpha=1$. Finally, the error estimate, in each case, is computed.

Keywords: nonlinear algebraic system, mixed integral equation, mixed fractional integro differential equation, toeplitz matrix method

MSC: 65L05, 34K06, 34K28

1. Introduction

Over the past few years, the significance of frIo-DEs containing time-dependent coefficients has been continuously increasing due to their wide range of applications in physics, engineering, and other scientific disciplines. These equations capture both the non-local and local behavior of many complex systems, making them perfect for accurately representing a wide range of real-world events. Some applications of fractional calculus to physics were first presented by Hermann [1]. Using the usual least squares method, Oyedepo et al. [2] provided a numerical solution to the linear frIo-DE problem. Using Laguerre polynomials, approximation solutions to frIo-DEs were discovered in Daşcıoğlu and Bayram [3]. frIo-DE. Nanware et al. [4] used the Bernstein polynomial to solve frIo-DE with Caputo derivative and find the numerical solution. Oyedepo et al. [5] presented the homotopy perturbation approach and the least squares technique to address the frIo-DE problem's solution. Using the Toeplitz matrix approach in combination with the Product Nystrom method, Basseem and Alayani [6] solved a nonlinear quadratic mixed integral equation (MIE) of the second class with singular kernel. Katani [7] developed a quadrature scheme for the numerical results of the second kind of Fredholm integral (FI) model. Al-Bugami [8] performed numerical representations based on an integral model that made use of 2D surface crack layers by using the Simpson and Trapezoidal methods. Brezinski and Zalglia [9] used the extrapolation method to obtain numerical computing results for the second type of nonlinear integral model with a continuous kernel. The extended cubic B-spline was used

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by the authors in [10] to understand the collocation approach used to solve the frIo-DEs. The fundamental framework of the exponential Euler difference form for Caputo-Fabrizio fractional-order differential equations with numerous delays is established by Zhang and Li in [11]. Research of this nature offers a thorough comprehension of behavior for changing systems. the applications of the Lerch polynomial method is used in [12, 13] to solve frIo-DEs. with continuous kernels. Jan [14] solved MIE in position and time using the Chebyshev Polynomials approach with a weakly singular kernel in position. In order to solve hypersingular linear and nonlinear IE in automatic control problems, Boykov et al. [15] developed a novel iterative technique. A modified hat function was employed by Biazar and Ebrahimi [16] to solve a class of nonlinear FVIE of the second kind. In order to solve Cauchy singular IE of the second kind, which has numerous applications in the domains of physics and engineering, Seifi [17] employed the collocation technique method. While Jan [18] solved a nonlinear MIE using the collocation approach, which is based on orthogonal polynomials. A technique of separating method is used in Abdou et al. [19] to discuss the solution of V-FIE with a discontinuous kernel. The separating variables method is used in Alhazmi [20] to discuss the solution of a MIE of the first kind with logarithmic and Carleman singular kernels. The spectral computation of highly oscillatory systems is applied in Gao et al. [21], to solve IEs in laser theory. Lienert and Tumulka [22] investigated from relativistic quantum physics and computed its solution of the same integral equation numerically.

More information for solving the IEs with continuous or discontinuous using different methods, can be found in [23–26]. The importance of time fractional came from the work of Youssri and Atta [27] by introducing a novel spectral algorithm utilizing Fibonacci polynomials to numerically solve both linear and nonlinear integro-differential equations with fractional-order derivatives. In addition, Hafez, et al. [28], employed a rational Jacobi collocation technique to effectively address linear time fractional sub- diffusion and reaction sub-diffusion equations. Finally, Magdy et al. [29] presented a numerical strategy for solving the nonlinear time fractional Burgers's equation to obtain approximate solutions of time fractional Burgers's equation

In the remainder work of this paper in Section 2, we consider the MfrIo-DE, under certain condition, then after integrating and using the principal of fractional integral, we obtain a MIE in position and time. In the end of this section, we explain the physical phenomena of the fractional integral. In Section 3, Preliminaries and basic definitions of fractional calculus and Banach fixed point theorem are stated. In Section 4, we consider the MIE in the integral operator form. Then, using Banach fixed point theorem, the existence and uniqueness solution of the mixed integral operator in $L_2[-1,\ 1] \times C[0,\ T]$-space, under certain conditions, is be considered and proved. In Section 5, the convergence of solution and the stability of the error are discussed and proved, respectability. In section6 after using the technique of separation, with some special polynomials, we have a nonlinear system of FIEs in position. The coefficients of the FIEs take the integral operators form in time. In Section 7, the solution of the FIEs with continuous kernel, using Picard method is discussed, and the physical meaning of the parameter α and the time is explained. The effect of fractional time on the position kernel was studied. It was also proven that fractional time affects the original position kernel and thus affects the existence and uniqueness of the solution. Also, some examples of an integral equation with a continuous kernel and its coefficients are solved numerically. Numerical calculations of fractional time were done in the three cases (when $\alpha = 0$, when $= \alpha$ 1, when $0 < \alpha < 1$). In Section 8, we use TMM, as the best numerical method in solving the singular IEs to obtain LAS. Moreover, the local truncation error is founded. In Section 9, Numerical results are considered, when the kernel takes a logarithmic function and Carlemain kernel form. Finally, the error estimate, in each case, is computed. The model used gives a convergence error for all values of $0 < \alpha < 1$.

2. Principal integral equation and fractional time phenomenon

The aim of this section is to present a problem for a differential-integral equation in position and time, then transform it into a mixed equation and clarify the importance of fractional time in integral equations. For this, consider the MfrIo-DE in following form

$$\lambda \frac{\partial^{\alpha}}{\partial t^{\alpha}}\Phi(x;\, t)+\mu\Phi(x;\, t)-\delta\int_{-1}^{1}k(|x-y|)\Phi(y;\, t)\,dy=f(x;\, t);\ \{\Phi(x,\, 0)=\Psi(x),\ 0<\alpha<1\}. \tag{1}$$

The formula (1) represents the MfrIo-DE with singular kernel $k(|x-y|)$. μ, λ, δ are constants. The known function $f(x;\, t)\in L_2(-1,\, 1)\times C[0,\, T]$, $T<1$. While, $\Phi(x;\, t)$ is the unknown function.

Equation (1) is equivalent to the phase- lag equation, which can be written as follows:

$$\mu\Phi(x;\, t+q)-\delta\int_{-1}^{1}k(|x-y|)\Phi(y;\, t)\,dy=f(x;\, t),\ q<1 \tag{2}$$

This equation has special importance in mathematical physics, genetic engineering, and most thermoelasticity problems, see [30].

Integrating Equation (1) and using the initial condition, we get

$$\lambda\Phi(x;\, t)+\frac{\mu}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\Phi(x;\, \tau)\,d\tau-\frac{\delta}{\Gamma(\alpha)}\int_{0}^{t}\int_{-1}^{1}(t-\tau)^{\alpha-1}k(|x-y|)\Phi(y;\, \tau)\,dyd\tau=\mathrm{H}(x;\, t),$$

$$H(x,\, t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}f(x;\, \tau)\,d\tau+\psi(x). \tag{3}$$

The formula (3) is called MIE in position and time. The second term of (3) takes the Volterra integral (VI) with Abel kernel and the third term takes the Volterra- Fredholm integral (V-FI) with two singular kernels. Also, the free term contains after integrating Abel's kernel.

Fractional time is a key factor in the explanation of particle interactions, both thermal and electrical. Numerous physical phenomena, like the rate of heat transmission and how quickly it rises or falls depending on the application, can be explained on the fractional time scale. What previous techniques of explanation have failed to do is explain the phenomenon of quantum effects on heat transfer and the enormous delays in the response of heat flow caused by phonon and electron interactions. This is made possible by contemporary quantum mechanics. This phenomenon has aided in the solution of integral equations describing non-equilibrium systems, helping to explain the occurrence of dynamic stability and examine the changes related to the problem's original source.

3. Preliminaries and basic definitions of fractional calculus

Integral equations with fractional integrals or derivatives are known as fractional integral equations. By extending the idea of classical integral equations to fractional orders, these equations make it possible to represent systems with memory and hereditary characteristics. A wide range of disciplines, including physics, engineering, biology, and finance, heavily rely on fractional integral equations.

Definition 1 Let $(X,\, d)$ A contraction of X (also called a contraction mapping on X) is a function $f: X\to Y$ that satisfies $\forall x,\, x'\in X: d(f(x),\, f(x'))\leq\beta d(x,\, x')$ for some real number $\beta<1$. Such β is called a contraction modulus of f

Definition 2 Banach Fixed Point Theorem: Every contraction mapping on a complete metric space has a unique fixed point. (This is also called the Contraction Mapping Theorem).

Theorem 1 (Banach fixed Point Theorem): Let K be a contraction operator on Banach space B then, the integral equation $\boldsymbol{K}\,\phi = \phi$ has a unique solution.

Definition 3 [30] The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $\varphi : (0, \infty) \longrightarrow \mathbb{R}$ is given by $I^{\beta}_{0+}\varphi(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\varphi(\tau)d\tau$. Provided that the R.H.S is pointwise defined on$(0, \infty)$.

Definition 4 [30] The Caputo fractional derivative of order $\beta > 0$ of a function $\psi : (0, \infty) \longrightarrow \mathbb{R}$ is given by $D^{\beta}_{0+}\psi(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t \frac{\psi^n(\tau)}{(t-\tau)^{\beta-n+1}}d\tau$.

4. Existence and uniqueness solution of the mixed integral equation

For this aim, write (3) in the integral operator form

$$\overline{W}\Phi(x;\, t) = \frac{1}{\lambda}H(x;\, t) - \frac{\mu}{\lambda}W_1\Phi(x;\, t) + \frac{\delta}{\lambda}W_2\Phi(x;\, t),\ W_1\Phi(x;\, t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Phi(x;\, \tau)d\tau,$$

$$W_2\Phi(x;\, t) = \frac{1}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}k(|x-y|)\Phi(y;\, \tau)\,dyd\tau. \tag{4}$$

Then, assume the following conditions:

(i) The kernel of position satisfies

$\left\{\int_{-1}^{1}\int_{-1}^{1}|k(|x-y|)|^2dxdy\right\}^{\frac{1}{2}} = C$ (C-constant).

(ii) The kernel $(t-s)^{\alpha-1}$ $\forall t,\ \tau \in [0,\ T],\ 0 \le \tau \le t \le T < 1$, satisfies for every continuous function $h(\tau)$ and all $0 \le \tau_1 \le \tau_2 \le t$ that for any constant M > 0 the integrals,

$$\int_0^t (t-\tau)^{\alpha-1}h(x;\, \tau)d\tau,\ \max_{0\le t\le T}\int_0^t (t-\tau)^{\alpha-1}d\tau$$

$$\int_{\tau_1}^{\tau_2}(t-\tau)^{\alpha-1}h(x;\, \tau)d\tau$$

are continuous functions of t.

(iii) The given function *H(x, y: t)* with its partial derivatives with respect to position x and time t are continuous in the space $L_2(-1,\ 1)\times C[0,\ T]$, and its norm is defined as

$$\|H(x;\, t)\| = \max_{0\le t\le T}\left|\int_0^t\left\{\int_{-a}^{a}|H(x;\, \tau)|^2dx\right\}^{\frac{1}{2}}d\tau\right| = G$$

(G is a constant).

Then, we state the Banach fixed Point Theorem without proof:

Theorem 2 (Principal Theorem 2) The MIE (3), with the aid of conditions (i)-(iii), has a unique solution in the space $L_2(-1,\ 1)\times C[0,\ T]$ under the assumption

$$\frac{T^{\alpha}(|\mu|+|\delta|C)}{|\lambda|\Gamma(\alpha+1)}<1 \tag{5}$$

Proof. To prove the theorem, we must proof the following two lemmas: □

Lemma 1 Under the conditions (i)-(iii), the integral operator $\overline{W}$ of Equation (4) maps the space $L_2(-1,\ 1)\times C[0,\ T]$ into it self

Proof. In the light of conditions (i)-(iii), the integral operator of Equation (4), yields

$$\left\|\overline{W}\Phi(x,\ t)\right\| \leq \frac{G}{|\lambda|}+\frac{|\mu|}{|\lambda|\Gamma(\alpha)}\cdot\max_{0\leq t\leq T}\left|\int_0^t (t-\tau)^{\alpha-1}\left\{\int_{-1}^{1}|\Phi(x;\ \tau)|^2dx\right\}^{\frac{1}{2}}d\tau\right|$$

$$+\frac{|\delta|}{|\lambda|\Gamma(\alpha)}\max_{0\leq t\leq T}\left|\int_0^t (t-\tau)^{\alpha-1}\left[\int_{-1}^{1}\int_{-1}^{1}k^2(|x-y|)dxdy\right]^{-2}\left\{\int_{-1}^{1}|\Phi(x;\ \tau)|^2dx\right\}^{\frac{1}{2}}d\tau\right|.$$

Hence, we follow

$$\left\|\overline{W}\Phi(x,\ t)\right\| \leq \frac{G}{|\lambda|}+\frac{|\mu|}{|\lambda|}\cdot\frac{T^{\alpha}}{\Gamma(\alpha+1)}\left\|\Phi(x;\ t)\right\|+\frac{|\delta|}{|\lambda|}\cdot\frac{T^{\alpha}C}{\Gamma(\alpha+1)}\left\|\Phi(x;\ t)\right\|.$$

Adapting the above inequality to have

$$\left\|\overline{W}\Phi(x,\ t)\right\| \leq \frac{G}{|\lambda|}+\sigma\left\|\Phi(x;\ t)\right\|,\ \sigma=\frac{T^{\alpha}(|\mu|+|\delta|C)}{|\lambda|\Gamma(\alpha+1)} \tag{6}$$

Inequality (6) shows that, the operator $\bar{W}$ is bounded and maps the ball S_ρ into itself. □

Lemma 2 Assume that, the conditions (i), (ii) are verified, then $\overline{W}$ is a contraction operator in the space $L_2(-1,\ 1)\times C[0,\ T]$.

Proof. For two different solutions $\Phi_1(x;\ t),\ \Phi_2(x;\ t)$ of Equation (3) , the integral operator (4) becomes

$$\left\|\overline{W}(\Phi_1(x,\ t)-\Phi_2(x,\ t))\right\|$$

$$\leq \frac{|\mu|}{|\lambda|\Gamma(\alpha)}\cdot \max_{0\leq t\leq T}\left|\int_0^t (t-\tau)^{\alpha-1}\left\{\int_{-1}^{1}|(\Phi_1(x,\ t)-\Phi_2(x,\ t)|^2 dx\right\}^{\frac{1}{2}} d\tau\right|$$

$$+\frac{|\delta|}{|\lambda|\Gamma(\alpha)}\max_{0\leq t\leq T}\left|\int_0^t (t-\tau)^{\alpha-1}\left[\int_{-1}^{1}\int_{-1}^{1}k^2(|x-y|)dxdy\right]^{-2}\left\{\int_{-1}^{1}|\Phi_1(x;\ \tau)-\Phi_2(x;\ \tau)|^2 dx\right\}^{\frac{1}{2}} d\tau\right|.$$

Hence, we have

$$\left\|\overline{W}(\Phi_1(x,\ t)-\Phi_2(x,\ t))\right\|\leq \sigma\left\|\Phi_1(x,\ t)-\Phi_2(x,\ t)\right\|,\ \sigma=\frac{T^{\alpha}(|\mu|+|\delta|C)}{|\lambda|\Gamma(\alpha+1)} \tag{7}$$

□

The continuity of the integral operator is proved. Moreover, under the condition of (4) $\sigma < 1$ $\overline{W}$ is a contraction operator. Then, by Banach fixed point theorem, we have a unique solution.

5. The convergence of solution and the error stability

The series is constructed in order to examine the behavior solution of Equation (7). So, consider the following sequence $\{\Phi_1(x,\ t),\ \Phi_2(x,\ t),\ ...,\ \Phi_{n-1}(x,\ t),\ \Phi_n(x,\ t),\ ...\}\subset\{\Phi_i(x,\ t)\}_{i=0}^{\infty}$.

From (3), we have

$$\lambda\Phi_n(x;\ t)+\frac{\mu}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Phi_{n-1}(x;\ \tau)d\tau-\frac{\delta}{\Gamma(\alpha)}\int_0^t\int_{-1}^{1}(t-\tau)^{\alpha-1}k(|x-y|)\Phi_{n-1}(y;\ \tau)dyd\tau=\mathrm{H}(x;\ t),$$

$$\lambda\Phi_{n-1}(x;\ t)+\frac{\mu}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Phi_{n-2}(x;\ \tau)d\tau-\frac{\delta}{\Gamma(\alpha)}\int_0^t\int_{-1}^{1}(t-\tau)^{\alpha-1}k(|x-y|)\Phi_{n-2}(y;\ \tau)dyd\tau=\mathrm{H}(x;\ t). \tag{8}$$

Subtracting the above equation, and using the assumption

$$\mathrm{Z}_n(x,\ t)=\Phi_n(x,\ t)-\Phi_{n-1}(x,\ t),\ \Phi_n(x,\ t)=\sum_{i=0}^{n}\mathrm{Z}_i(x,\ t),\ \mathrm{Z}_0(x,\ t)=H(x,\ t) \tag{9}$$

We have

$$Z_n(x;\ t)=\frac{-\mu}{\lambda\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}Z_{n-1}(x;\ \tau)\,d\tau+\frac{\delta}{\lambda\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}k(|x-y|)Z_{n-1}(y;\ \tau)\,dyd\tau.$$

Taking the norm of both sides and applying the conditions (i), (ii), we follow

$$\|Z_n(x;\ t)\|\leq\sigma\,\|Z_{n-1}(x;\ t)\|,\ \left[\sigma=\frac{T^{\alpha}(|\mu|+|\delta|C)}{|\lambda|\Gamma(\alpha+1)}\right]. \tag{10}$$

Using the mathematical induction and condition (iii), inequality (10) becomes

$$\|Z_n(x;\ t)\|\leq\sigma^n G \tag{11}$$

Hence, the solution is convergence, as $n\to\infty$ to $\Phi(x,\ t)$.

To discuss the error stability, we consider $\Phi_n(x,\ t)$ as a numerical solution of equation (3).

Hence, we have

$$\lambda\Phi_n(x;\ t)+\frac{\mu}{\lambda\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Phi_n(x;\ \tau)\,d\tau-\frac{\delta}{\lambda\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}k(|x-y|)\Phi_n(y;\ \tau)\,dyd\tau=\frac{\mathrm{H}_n(x;\ t)}{\lambda} \tag{12}$$

If we assume $R_n(x;\ t)$ the error such that

$$R_n(x;\ t)=\Phi(x;\ t)-\Phi_n(x;\ t).$$

Then, we have

$$R_n(x;\ t)+\frac{\mu}{\lambda\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}R_n(x;\ \tau)\,d\tau-\frac{\delta}{\lambda\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}k(|x-y|)R_n(y;\ \tau)\,dyd\tau=\frac{\mathrm{E}_n(x;\ t)}{\lambda},$$

$$(\mathrm{E}_n(x;\ t)=\mathrm{H}(x;\ t)-\mathrm{H}_n(x;\ t)). \tag{13}$$

The formula (13) of the error represents a MIE in time and position. Using the normality properties, and fixed point theorem, and following the same way of Section 4, we can prove that.

Equation (13) has a unique solution under the condition $\frac{T^{\alpha}(|\mu|+|\delta|C)}{|\lambda|\Gamma(\alpha+1)}<1$.

6. Technique of separation of variables and system of fredholm integral equations

In this section, we use separation of variables method in new technique to obtain a system of Fredholm integral equations where its coefficients are variable in time. For this assume the solution of Equation (3), in the form:

$$\Phi(x;\ t) \simeq \phi_n(x)t^{\alpha n},\ f(x;\ t) \simeq f_n(x)t^{\alpha n};\ t \in [0,\ T];\ T < 1,\ n = 1,\ 2,\ \ldots \tag{14}$$

The series in Equation (14) is absolutely convergent and when $n \to \infty$ the two functions $\Phi(x,\ t) = \dfrac{\varphi_\infty(x).}{1-t^\alpha}$, $f(x,\ t) = \dfrac{f_\infty(x).}{1-t^\alpha}$, $0 \le t < 1$. Therefore, we assume n is finite numbers.

After substituting (14) into (3) and using some relations of Gamma and Beta functions, we have

$$t^{\alpha n}[\frac{\lambda\Gamma(\alpha+\alpha n+1)+\mu\Gamma(\alpha n+1)t^\alpha}{\Gamma(\alpha+\alpha n+1)}]\phi_n(x) - \delta\frac{t^{\alpha n+\alpha}\Gamma(n\alpha+1)}{\Gamma(\alpha+\alpha n+1)}\int_{-1}^{1} k|x-y|\phi_n(y)dy = \frac{t^{\alpha n+\alpha}\Gamma(n\alpha+1)}{\Gamma(\alpha+\alpha n+1)}f_n(x)+\psi(x).$$

Adapting the above equation to obtain

$$\phi_n(x) - \chi(\alpha,\ n,\ t)\int_{-1}^{1} k|x-y|\phi_n(y)dy = h_n(x;\ \alpha,\ n,\ t)(h_n(x;\ \alpha,\ n,\ t) = \zeta(\alpha,\ n,\ t)f_n(x)+\xi(\alpha,\ n,\ t)\psi(x)), \tag{15}$$

where

$$\chi(\alpha,\ n,\ t) = \frac{\delta t^\alpha\Gamma(n\alpha+1)}{[\lambda\Gamma(\alpha+\alpha n+1)+\mu\Gamma(\alpha n+1)t^\alpha]},$$

$$\zeta(\alpha,\ n,\ t) = (\frac{t^\alpha\Gamma(n\alpha+1)}{[\lambda\Gamma(\alpha+\alpha n+1)+\mu\Gamma(\alpha n+1)t^\alpha]},$$

$$\xi(\alpha,\ n,\ t) = \frac{\Gamma(\alpha+\alpha n+1).}{[\lambda\Gamma(\alpha+\alpha n+1)+\mu\Gamma(\alpha n+1)t^\alpha]}t^{-\alpha n}.$$

The formula (15) represents a system of nonlinear FIEs with discontinuous kernel and variable coefficients in $(\alpha,\ n,\ t)$.

It is clear, from (15) that: the system of nonlinear FIE has a unique solution under the following important condition

$$\|k|x-y|\|\,|\chi(\alpha,\ n,\ t)| < 1 \tag{16}$$

It is clear that, the uniqueness of the solution of (15) depends on the following parameters: time t, α, δ, μ, λ, t, p and the degree of approximation n. Many different numerical methods can be used to solve (15). More information for these methods can be found in [27, 28].

7. The effect of time on the problem and its physical meaning

In mathematical physical problems the parameters play an important role in describing the problems. So, in (15) we consider the term

$$\chi(\alpha,\ n,\ t)=\frac{\delta t^{\alpha}\Gamma(n\alpha+1)}{[\lambda\Gamma(\alpha+\alpha n+1)+\mu\Gamma(\alpha n+1)t^{\alpha}]} \tag{17}$$

The parameters of the term $\chi(\alpha,\ n,\ t,\ p)$ play an important role in the existence and uniqueness of the solution of (15). Moreover, it plays an important role in determining the maximum and minimum values of the error. To explain this, we consider Equation (15) with continuous kernel and the solution of it will be discussed, using Picard method when $k(x,\ y)=x^2y^2$ and the second example when $k(x,\ y)=e^{xy}$.

Example 1

$$\phi_n(x)-\chi(\alpha,\ n,\ t)\int_{-1}^{1}x^2y^2\phi_n(y)dy=h_n(x;\ \alpha,\ n,\ t) \tag{18}$$

The solution of (18) can be obtained by using successive approximation method (Picard method).

We will be interested in studying the effect of the coefficients associated with Equation (17) on each other, and the extent of their compatibility with the kernel in the presence of a single solution.

Table 1. The values of $\chi(\alpha,\ 2,\ t)$ for different times

α	$\chi(\alpha,\ 2,\ 0.001)$	$\chi(\alpha,\ 2,\ 0.01)$	$\chi(\alpha,\ 2,\ 0.1)$	$\chi(\alpha,\ 2,\ 0.8)$	$\chi(\alpha,\ 2,\ 0.9)$
0	0.00500	0.00500	0.00500	0.00500	0.00500
0.1	0.00339	0.00392	0.00448	0.00500	0.00503
0.3	0.00105	0.00189	0.00318	0.00465	0.00474
0.4	0.00051	0.00118	0.00252	0.00436	0.00448
0.5	0.00023	0.00070	0.00192	0.00402	0.00416
0.7	0.00004	0.00022	0.00101	0.00326	0.00344
0.8	0.00002	0.00012	0.00071	0.00286	0.00306
1.0	0.000003	0.00003	0.00032	0.00210	0.00231

Table 2. The error at $t=0.1,\ t=0.8,\ x=0.5$

α	Error at t = 0.1, x = 0.5	Error at t = 0.8 x = 0.5
0	0.0000760974	0.0000760974
0.1	0.0000265421	0.0000386283
0.2	0.0000076274	0.0000171253
0.4	0.0000004150	0.0000024288
0.5	0.0000000850	0.0000008084
0.6	0.0000000165	0.0000002538
0.8	0.0000000006	0.0000000220
0.9	0.0000000001	0.0000000062
1.0	0.00000000002	0.0000000017

Example 2

$$\phi_n(x) - \chi(\alpha,\ n,\ t) \int_{-1}^{1} e^{xy} \phi_n(y) dy = h_n(x;\ \alpha,\ n,\ t) \tag{19}$$

When the kernel takes the form $k(x,\ y) = e^{xy} \simeq 1 + xy + \frac{(xy)^2}{2} + \frac{(xy)^3}{6} + \frac{(xy)^4}{24}$, $n = p = 2$, and after using the Picard method, we obtain

Table 3. The values of $\chi(\alpha,\ 2,\ t)$ at different time and the error at $t = 0.1$, $t = 0.9$

α	$\chi(\alpha,\ 2,\ 0.8)$	$\chi(\alpha,\ 2,\ 0.9)$	Error at $t = 0.1$, $x = 0.5$	Error at $t = 0.9$, $x = 0.5$
0	0.00500	0.00500	0.0000760974	0.0000760974
0.1	0.00510	0.00513	0.0000555101	0.0000775038
0.3	0.00472	0.00479	0.0000226015	0.0000649339
0.4	0.00445	0.00449	0.0000129651	0.0000552772
0.5	0.00412	0.00419	0.0000070033	0.0000453132
0.7	0.00336	0.00348	0.0000017545	0.0000276032
0.8	0.00291	0.00315	0.0000008246	0.0000206088
1.0	0.00212	0.00238	0.0000003746	0.0000760974

8. Toeplitz matrix method, see [31, 32]

Here, we present TMM to transform the SFIEs of the second kind with singular kernel of Equation (15) into LAS. The idea of this method is to obtain system of $2N+1$ linear algebraic equation, where $2N+1$ is the number of the discretization points used. For this, consider

$$\int_{-1}^{1} k(|x-y|)\phi_n(y)dy = \sum_{m=-N}^{N-1} \int_{mh}^{(m+1)h} k(|x-y|)\phi_n(y)dy$$

$$= \sum_{m--N}^{m-N-1} \{A_{nm}(x)\phi_n(a) + B_{nm}(x)\phi_n(a+h)\} + R(x,\ y,\ n);\ \left(h = \frac{1}{N}, a = mh\right) \tag{20}$$

Here, $A_{nm}(x)$ and $B_{nm}(x)$ are arbitrary functions to be determined, and R is the estimate error. Thus, the IE (15) after putting $x = \ell h$ and using the following notations $\phi_n(mh) = \phi_{nm},\ D_n(mh) = D_{mn},\ h_n(mh) = h_{nm},\ \phi_n(\ell h) = \phi_{n\ell}$, leads to the following LASs

$$\varphi_{n,\ \ell} - \chi(\alpha,\ n,\ t) \sum_{m=-N}^{m=N} D_{mn,\ \ell}\varphi_{n,\ m} = h_{n,\ \ell} \tag{21}$$

Here, in (20), (20), we consider,

$$A_{n,\,m}(x)=\frac{1}{h}\left[(mh+h)I(x)-J(x)\right],\; B_{n,\,m}(x)=\frac{1}{h}\left[J(x)-mh\,I(x)\right],\; I(x)=\int_{mh}^{mh+h}k(|x-y|)dy.$$

$$J(x)=\int_{mh}^{mh+h}yk(|x-y|)dy;\; D_{m\,n,\,\ell}=\begin{cases}A_{-N,\,n}(\ell h) & m=-N\\ A_{n,\,m}(\ell h)+B_{n,\,(m-1)}(\ell h) & -N<m<N\\ B_{n,\,(N-1)}(\ell h), & m=N\end{cases} \tag{22}$$

The matrix $D_{m\,n,\,\ell}$& can be written in the TMM form of order (2N + 1)

$$D_{mn,\,\ell}=G_{mn,\,\ell}-E_{mn,\,\ell};\; G_{mn\ell}=A_{nm}(\ell h)+B_{n(m-1)}(\ell h),\; -N\le m,\;\ell\le N,$$

$$E_{mn}=\begin{cases}B_{n(-N-1)}(\ell h), & m=-N\\ 0 & -N<m<N\\ A_{nN}(\ell h), & m=N\end{cases} \tag{23}$$

The method is said to be convergent of order ρ if and only if for N sufficiently large there exists a constant $C > 0$ independent of N such that

$$\|\varphi_n(x)-\varphi_{nN}(\ell h)\|_\infty\le C_iN_i^{-\rho};\; n=1,\,2\ldots,\; j,\; (x=\ell \mathrm{h}) \tag{24}$$

Also, in general the local truncation error is defined by

$$t_{nV}\,(k,\;\varphi_n,\;x)=\left|\int_{-a}^{a}k(x,\;y)\varphi_n(y)dy-\sum_{m=-N}^{N}D_{mnl}\,\varphi_n(lh)\right|. \tag{25}$$

In order to guarantee the existence of a unique solution of the LAS (21) in the space ℓ^∞, we assume the following conditions:

$$\sup_m|h_{n,\,m}|\le\sigma;\;\sup_N\sum_{m=-N}^{N}\left|D_{mn,\,\ell}\right|\le\varsigma,\;(\sigma,\;\varsigma\text{ are constants})$$

Theorem 3 (without proof): The finite LAS of (21) has a unique solution under the condition: $|\chi(\alpha,\;n,\;t)|\left|\sum_{m=-N}^{N}\mathrm{D}_{m\,n,\,\ell}\right|<1,\; n=0,\;1,\;2,\;\ldots,\;L;\;\ell=0,\;1,\;\ldots K.$

Example 1 (Consider the mixed integral equation) $\lambda\Phi(x;\;t)+\frac{\mu}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\Phi(x;\;\tau)\,d\tau-\frac{\delta}{\Gamma(\alpha)}\int_0^t\int_{-1}^{1}(t-\tau)^{\alpha-1}k(|x-y|)\,\Phi(y;\;\tau)\,dyd\tau=H(x;\;t)$, With the exact solution is $\Phi(x;\ t) = x^2\left(1+t^2\right)$; and assume the kernel of position in the logarithmic form and Carleman function respectively

$$(i)\ k(|x-y|) = \ln(|x-y|),\ (ii)\ k(|x-y|) = |x-y|^{-\upsilon},\ (0<\upsilon<1).$$

Applying the TMM and assuming the following values of the parameters $\delta = \lambda = \mu = 1$; $m = 2$, and $n = 5$ at $t = 0.3$. The Tables 4 and 5 are obtained for different values of $0 < \alpha \leq 1$.

The numerical results and the corresponding error are computed in Tables 4-6.

Table 4. The numerical solution and the corresponding error when the kernel takes logarithmic form $k(|x-y|) = \ln(|x-y|)$

α	x	$\varphi_{numeric}$	R_{error}	α	x	$\varphi_{numeric}$	R_{error}
	−1	1.089857	0.000143		−1	1.089909	0.000091
	−0.5	0.272274	0.000226		−0.5	0.272356	0.000144
0.1	0	0.000007	0.000007	0.6	0	0.000004	0.000004
	0.5	0. 272274	0.000226		0.5	0.272356	0.000144
	1	1.089857	0.000143		1	1.089909	0.000091
	−1	1.089873	0.000127		−1	1.089937	0.000063
	−0.5	0.272299	0.000201		−0.5	0.272400	0.000100
0.3	0	0.000006	0.000006	0.8	0	0.000003	0.000003
	0.5	0.272299	0.000201		0.5	0.272400	0.000100
	1	1.089873	0.000127		1	1.089937	0.000063
	−1	1.089896	0.000104		−1	1.089961	0.000039
	−0.5	0.272335	0.000165		−0.5	0.272438	0.000061
0.5	0	0.000005	0.000005	1.0	0	0.000002	0.000002
	0.5	0.272335	0.000165		0.5	0.272438	0.000061
	1	1.089896	0.000104		1	1.089961	0.000039

Table 5. The numerical solution and the corresponding error when the kernel takes logarithmic form $k(|x-y|) = |x-y|^{-\upsilon}$

α	x	$\varphi_{numeric}$	R_{error}	α	x	$\varphi_{numeric}$	R_{error}
	-1	1.089808	0.000192		-1	1.089876	0.000124
	-0.5	0.275618	0.003118		-0.5	0.274469	0.001969
0.1	0	0.003094	0.003094	0.6	0	0.001954	0.001954
	0.5	0.275618	0.003118		0.5	0.274469	0.001969
	1	1.089808	0.000192		1	1.089876	0.000124
	-1	1.089828	0.000172		-1	1.089913,	0.000087
	-0.5	0.275270	0.002770		-0.5	0.273865	0.001365
0.3	0	0.002748	0.002748	0.8	0	0.001354	0.001354
	0.5	0.275270	0.002770		0.5	0.273865	0.001365
	1	1.089828	0.000172		1	1.089913	0.000087
	-1	1.089858	0.000142		-1	1.089946	0.000054
	-0.5	0.274764	0.002264		-0.5	0.273339	0.000839
0.5	0	0.002246	0.002246	1.0	0	0.000832	0.000832
	0.5	0.274764	0.002264		0.5	0.273339	0.000839
	1	1.089858	0.000142		1	1.089946	0.000054

Table 6. The numerical solution and the corrupting error when the kernel takes logarithmic and Carleman form at $\alpha = 0$

x	$\varphi_{numeric}\,\text{ln}(\|x-y\|)$	R_{error}	$\varphi_{numeric}\|x-y\|^{-\nu}$	R_{error}
-1	1.089852	0.000148	1.089801	0.000199
-0.5	0.272265	0.000235	0.275735	0.003235
0	0.000007	0.000007	0.003210	0.003210
0.5	0.272265	0.000235	0.275735	0.003235
1	1.089852	0.000148	1.089801	0.000199

9. Numerical results and conclusion

From the above numerical results, we can establish the following:

(I). In example (1) of Equation (18) and in the Tables 1-2, we discuss the values of $\chi(\alpha,\ n,\ t)$, at $n = 2$, according to different values of time $t,\ 0 \leq t < 1$, and $\alpha;\ 0 \leq \alpha \leq 1$. Thus, we can establish the following:

1-At $\alpha = 0$, the values of time have no effect in $\chi(\alpha,\ n,\ t)$.

2-$\chi(\alpha,\ n,\ t)$ tends to minimum value when $\alpha \to 1$.

3-When time increases the value of $\chi(\alpha,\ n,\ t)$ increases also.

Also, from the numerical results we deduce that

4-For the kernel $k(x,\ y) = x^2y^2$, the error takes minimum value $(1.77\ \times 10^{-10})$ when $\alpha = 1$ and $t \to 0$.

5-Also, for the kernel $k(x,\ y) = x^2y^2$, when $t \to 1$ and $\alpha = 1$ the maximum value of error takes $1.1\ \times\ 10^{-6}$.

(II). In example (2) of Equation (19), we deduce that: the error is computed at $x = 0.5$ in Tables 10-11. When $t \to 1$ and $\alpha \to 1$ the maximum value of error takes (8.9×10^{-5}). Also, the error takes minimum value (1.77×10^{-10}) when $\alpha \to 1$ and $t \to 0$.

This model converts all mixed integral equations with continuous and discontinuous kernels in time and positions to system of FIEs. Moreover, it gives the solutions directly and explains the physical parameters.

(III) In example (3), we deduce that: the new model enable us to discuss the solution at $\alpha = 0$, see Table 6. Also, from the results we can establish the following for logarithmic kernel:

(i) When m = 4, the Max. error 1.73×10^{-4} and Min. error about x= 0 is 1×10^{-6}.

(ii) When m = 8, the Max. error 4×10^{-4} and Min. error is 1×10^{-7}.

(iii) In the case of logarithmic kernel and Carleman function when n increases ($n \geq 10$), the error still stable, for example in logarithmic kernel, the Max. error 5×10^{-5} and Min. error is 3×10^{-6}.

Also, for the Carleman function

(i) When m = 4, the Max. error 1.0×10^{-3} and Min. error is 7×10^{-4}.

(ii) When m = 8, the Max. error 1.0×10^{-3} and Min. error is 9×10^{-5}.

10. Future work

In our future work, we will try to discuss the numerical solution of the following nonlinear fractional integral equation

$$\lambda \frac{\partial^\alpha}{\partial t^\alpha}\Phi(x;\, t) + \mu\Phi(x;\, t) - \delta \int_{-1}^{1} k(|x-y|)\Phi(y;\, t)\, dy = f(x;\, t);\ \{\Phi(x,\, 0) = \Psi(x),\ 0 < \alpha < 1\}.$$

Conflict of interest

The authors declare no competing financial interest.

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