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Research Article

Ovals of Constant Width in Polar Coordinates

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Abstract: We explore ovals of constant width in polar coordinates in this paper. Conversion of a parametric function defined on a rectangular domain of angles, to a polar representation defined on a domain of polar angles is introduced, and the relationship between the rectangular angles and the polar angles is discussed. The length of the parametric curve in polar coordinates between opposite points and from one vertex point to the next can be determined using the oval's vertices. A new verification of Barbier's theorem in polar coordinates is presented. We show that the extreme values of the radial coordinate of the discussed polar oval are obtained at both its vertices and opposite points. Ovals and specific circles with the origin at the center are compared, and we demonstrate that every given oval is analytically and geometrically enclosed between those two specific circles. Intersection points between a polar oval and any circle related to it, centered at the origin, are formulated. Simulation and numerical examples are presented to support the analytical and theoretical results.

***Keywords*:** ovals, convex curve, constant width, support function, barbier's theorem, polar coordinates

MSC: 52A10, 53A04, 53C80

1. Introduction

The concept of constant width (CW) represents a tool for building new geometrical shapes in the Euclidean plane that are generalizations of the usual circle. While the work on ovals of CW is concentrated on building analytic parametrizations in Cartesian coordinates, it is reasonable to have a mathematical partner (polar coordinates) that aims to build new parametrizations and new geometrical formulas, of course by using the radial coordinate and the angular coordinate of a point in a plane. In this case, the object under consideration is the polar oval.

Polar ovals, or ovals in polar coordinates, provide a wealth of opportunities for research in physics and mathematics because they capture intricate relationships between atmospheric dynamics and the planet's magnetic field. Regardless of the direction in which they are measured, polar ovals of CW are intriguing geometric structures in which the oval maintains a constant distance across all of their parallel lines. Their unique characteristic sets them apart from regular ellipses and other oval shapes, which typically lack this kind of consistency.

The concept of CW is closely related to mathematical structures such as Reuleaux polygons, which are shapes of constant width other than the circle. In the context of planetary atmospheres. These polar ovals could theoretically originate from uniform distributions of forces or fields, such as consistent magnetic field strength around the poles or

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uniform gradients of atmospheric pressure. This uniformity may have implications for the stability and symmetry of atmospheric phenomena, potentially influencing the distribution of auroral emissions and how charged particles behave in these regions.

Closed and convex curves are fundamental concepts in the study of geometry and optimization. A closed curve is a continuous loop that encloses a particular area of the plane and goes back to its beginning point without crossing itself. Convex curves, on the other hand, are those where, the segment of a straight line joining any two points within the enclosed region lies entirely within the curve. Convexity is a crucial quality in many applications, including finding the shortest pathways, optimizing functions, and analyzing geometric structures, because it guarantees that the curve has no inward indentations. In both theoretical and applied mathematics, the relationship between closure and convexity is essential to comprehending the characteristics and behavior of shapes.

A curve of CW is regarded as a closed curve in $\mathbb{R}^2$ that, when it rotated in a square, has continuous point contact with each of the four sides. Any curve with a constant width is convex, meaning that no line may cross its boundary more than twice. A polar curve is a curve written in terms of a radial coordinate and an angular coordinate, where the radial coordinate is the measured distance or the length of the line segment connecting the origin and any point on the curve, and the angular coordinate (the polar angle) is the angle the line makes with the positive x-axis (the polar axis). An oval is regarded as a smooth convex curve in $\mathbb{R}^2$. For each point on the oval there exists a unique corresponding point on the oval at which the tangent is parallel to the tangent at the original point, and in this sense, we say that the two points are opposite to each other. The width of the oval is the shortest path between the tangent line at one point on the oval and the tangent at the corresponding opposite point. An oval, in polar coordinates, is regarded as a curve of CW when the measured perpendicular distance between the tangent lines at any two opposite points of the oval is constant [1]. While the maximum of the curvature represents the vertices on the oval, we also conclude the same results using polar coordinates with new proofs.

In differential geometry (DG), the support function is an important tool, and plays a crucial role in describing the geometry of a curve by encoding how it interacts with external directions. The distance, measured in the direction of the normal vector, between the origin and the tangent line at a certain location on a smooth convex curve is represented by the support function. Our work depends highly on the support function in polar coordinates to produce new proofs of many results deduced in rectangular coordinates. The use of the support function was done by Al-Banawi in [2] with measuring the perpendicular length of the segment drawn from the origin to a point on the tangent of a convex curve, such a measurement helped in producing a formula for an oval in $\mathbb{R}^2$.

Al-Banawi and Al-Btoush [3] made an analytic study for the support function of an oval of CW aiming to study optimization and area regarding ovals of constant width in $\mathbb{R}^2$. Since the support function is periodic and has continuous derivatives, they managed to count vertices and gave a formula for the region enclosed within a convex curve. In [4], Resnikoff concluded different results by working on the average width of an oval with different degrees.

Al-Banawi and Jaradat [5] showed that properties of ovals of constant width can be derived by solving ordinary differential equations (ODEs) that are linear with constant coefficients. While Al-Banawi in [2] managed to work with linearly ODEs of the first order, the work in [5] was mostly on second linear ODEs.

The work here ensures that continuous ovals of CW have infinitely continuous approximations as proved by Tanno in [6]. While Tanno showed that the form of an oval is a continuous parametrization of a differential curve in $\mathbb{R}^2$, Wegner in [7] gave an analytic approximation for such parametrization of convex curves in $\mathbb{R}^2$, and in particular, for those of constant width. Tanno demonstrated in [6] that a C^∞-oval with the same CW exists for any positive neighborhood of a continuous oval with CW. Wegner in [7] demonstrated that by employing the support function to simplify Tanno's proof, such a C^∞-oval is real analytic.

In [8], Fu and Zhou analyzed a smooth oval of constant width by implementing the time delay system's characteristic equation. They showed that a circle is the only smooth convex curve in $\mathbb{R}^2$ that can work as an oval of CW whose parametric form in rectangular coordinates is still differentiable finitely many times. Fu and Zhou built their simulation findings according to the derivative orders influence on smooth noncircular ovals of CW.

Fillmore presented a parametric formula for a curve with three vertices of CW in [9]. Al-rabtah and Al-Banawi [10] extended this formulation and analysis to an odd natural number of vertices that is greater than three, taking into consideration that the shadow concept [11] was included to Fillmore's formulation.

The results in [10] were derived analytically in Cartesian coordinates, and in this work we reproduce these results in addition to several totally new results, but in a polar coordinates system.

In [12], Mozgawa examined a curve's curvature formula in polar coordinates, and showed that the only possibility for that curve is to be a portion of a circle. Through the use of a curve representation by an appropriate support function, Leichtweiss [13] examined the global DG of a closed polar convex curve in the hyperbolic and spherical plane geometry. In order to describe the black hole shadow, Farah et al. [14] explored simple polar approximations such as an ellipse and a limacon. Resnikoff [4] determined the area enclosed by an oval using polar representation.

Our work is structured as follows: In Section 2, we reformulate the representation of a general form of an oval into a polar coordinates representation, we discuss the relationship between the rectangular angles and the polar angles, we locate the vertices and their corresponding opposite points on an oval of CW, and measure the curve length of the polar oval curve between any two opposite points, and from any vertex point to the next one. A new form for the verification of Barbier's Theorem is presented in polar coordinates. We also show that the extremum values are attained at the vertices and their opposite points of the polar oval. In Section 3, we compare polar ovals with two particular circles related to these ovals, and deduce that all considered types of polar ovals are enclosed between the two specified circles. Furthermore, we determine the intersection points between a polar oval and a circle related to it centered at the origin. In Section 4, our obtained theoretical results are endorsed by numerical examples accompanied by figures, and we clarify the relationship between the polar angles and the rectangular angles corresponding to the parametric curve. We compare ovals with CW to a specific origin-centered circle corresponding to the investigated oval, and find the differences in polar coordinates. Additionally, comparisons with two concentric circles at the origin associated with the polar curve show that all types of discussed polar ovals are enclosed between these particular circles. Section 5 is the conclusion.

2. Ovals of constant width in polar coordinates

We start with $\mathscr{C}$ as a smooth closed convex parametric curve in the rectangular plane. We can assume that the coordinate system origin denoted by O lies in the interior of $\mathscr{C}$. Then consider the ray from the origin that makes an angle t (rectangular angle) with the positive x-axis, where $t \in [0, 2\pi]$. We draw a tangent on the curve perpendicular to the ray with angle t to obtain $P(x, y) = P(x(t), y(t))$ as the point of tangent contact with the given curve. The point $P(x, y)$ gives a parametrization for the curve $\mathscr{C}$.

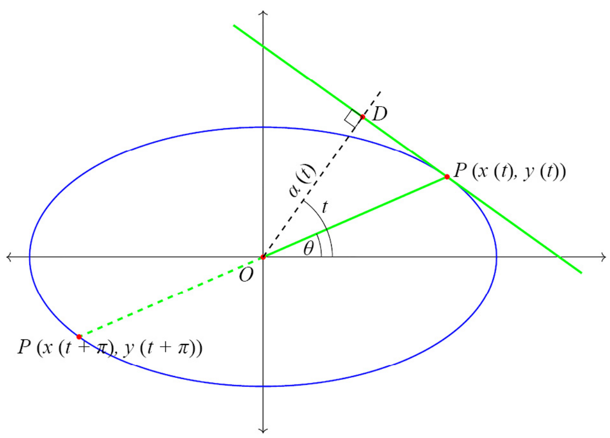


Figure 1. Oval and the support function

Assume that D represents the point of intersection between the ray and the tangent line as in Figure 1. The two points $P(x(t),\ y(t))$ and $P(x(t+\pi),\ y(t+\pi))$ are opposite points. If α measures the distance from the origin O to the point D, then $\alpha = \alpha(t)$, and D has the coordinates $(\alpha(t)\cos t,\ \alpha(t)\sin t)$. The function $\alpha(t)$ is called the support function.

The equation of the tangent line is

$$\frac{y(t) - \alpha(t)\sin t}{x(t) - \alpha(t)\cos t} = -\cot t\ ,$$

here, $x(t)$ and $y(t)$ are differentiable functions of the real variable $t \in [0,\ 2]$.

Hence,

$$\alpha(t) = x(t)\cos t + y(t)\sin t, \tag{1}$$

and so

$$\alpha'(t) = -x(t)\sin t + y(t)\cos t + x'(t)\cos t + y'(t)\sin t.$$

Since $(x'(t),\ y'(t))$ represents the tangent along the curve $(x(t),\ y(t))$, and $(\cos t,\ \sin t)$ is a unit normal, we have

$$(x'(t),\ y'(t)) \cdot (\cos t,\ \sin t) = 0,$$

where $(\cdot)$ is the usual Euclidean or (vector) dot product. Hence,

$$x'(t)\cos t + y'(t)\sin t = 0.$$

Thus,

$$\alpha'(t) = -x(t)\sin t + y(t)\cos t. \tag{2}$$

Solving the two equations (1) and (2) for $x(t)$ and $y(t)$ gives

$$x(t) = \alpha(t)\cos t - \alpha'(t)\sin t, \text{ and}$$

$$y(t) = \alpha(t)\sin t + \alpha'(t)\cos t.$$

Hence, the parametric curve $\mathscr{C}$ is defined by:

$$P(t) = \big(\alpha(t)\cos t - \alpha'(t)\sin t,\ \alpha(t)\sin t + \alpha'(t)\cos t\big)\,. \tag{3}$$

In [10], Al-rabtah and Al-Banawi addressed the parametric curve

$$\begin{aligned} P(t) =&(x(t),\ y(t)) \\ =&(a\cos t + \cos t\cos nt + n\sin t\sin nt, \\ &a\sin t + \sin t\cos nt - n\cos t\sin nt), \end{aligned} \tag{4}$$

where $t \in [0,\ 2\pi]$, the coefficient a is a positive real number, and the parameter n is an odd positive integer.

By substituting $x(t)$ and $y(t)$ from Formula (4) into Formula (1), $\alpha(t)$ becomes

$$\alpha(t) = a + \cos nt. \tag{5}$$

In this study, we consider the parametric curve given by the Formula (3), with $\alpha(t)$ defined by the Formula (5). In order to present our analysis in polar coordinates, we consider the representation of the convex curve as $(r,\ \theta)$ where the radial coordinate r represents the distance from the origin (the pole) to any point on the polar curve, and the angular coordinate θ stands for the polar angle from the polar axis to the line segment that connecting the pole and that point on the polar curve.

In Theorem 1 from [10], Al-rabtah and Al-Banawi proved that the image of the parametric function (4) represents an oval with CW of $2a$, where $a - (n^2 - 1)\cos nt > 0$, and therefore, the values of the odd natural number n, when a is determined, can be chosen with $n < \sqrt{a+1}$.

Theorem 1 Suppose the parametric function P is defined by (3) with $\alpha(t)$ defined by (5). Then, the oval that represents the image of P is of constant width $2a$, with $P(t)$ and $P(t+\pi)$ being opposite points on the oval.

Proof. Since $P(t) = (\alpha(t)\cos t - \alpha'(t)\sin t,\ \alpha(t)\sin t + \alpha'(t)\cos t)$, then

$$\begin{aligned} P(t+\pi) =&(-\alpha(t+\pi)\cos t + \alpha'(t+\pi)\sin t, \\ &-\alpha(t+\pi)\sin t - \alpha'(t+\pi)\cos t). \end{aligned}$$

The distance between $P(t)$ and $P(t+\pi)$ is

$$\begin{aligned} d(P(t),\ P(t+\pi)) =&\left\{ \Big((\alpha(t) + \alpha(t+\pi))\cos t - (\alpha'(t) + \alpha'(t+\pi))\sin t \Big)^2 \right. \\ &\left. + \Big((\alpha(t) + \alpha(t+\pi))\sin t + (\alpha'(t) + \alpha'(t+\pi))\cos t \Big)^2 \right\}^{1/2}. \end{aligned}$$

Since $\alpha(t) = a + \cos nt$, and n is an odd natural number, then

$$\alpha(t+\pi) = a + \cos(nt + n\pi) = a - \cos(nt),$$

and therefore,

$$\alpha(t)+\alpha(t+\pi)=2a.$$

By a derivation, we get

$$\alpha'(t)+\alpha'(t+\pi)=0.$$

Thus,

$$d(P(t),\ P(t+\pi))=\sqrt{(2a\cos t)^2+(2a\sin t)^2}=2a.$$

□

Theorem 2 Consider the oval $P(t)=(x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$. Then, the radial coordinate r that represents the distance from a point on the polar curve to the pole, at the rectangular angle t, is

$$r(t)=\sqrt{\alpha(t)^2+(\alpha'(t))^2}, \tag{6}$$

where $\alpha(t)=a+\cos nt$, as stated in Formula (5).

Proof. The distance from the point on the polar curve to the pole, using Pythagorean theorem, is formulated by

$$\begin{aligned}
r(t) &= \sqrt{x^2(t)+y^2(t)}\\
&= \sqrt{(\alpha(t)\cos t-\alpha'(t)\sin t)^2+(\alpha(t)\sin t+\alpha'(t)\cos t)^2}\\
&= \Big\{\alpha^2(t)\cos^2 t-2\alpha(t)\alpha'(t)\cos t\sin t+(\alpha'(t))^2\sin^2 t\\
&\quad +\alpha^2(t)\sin^2 t+2\alpha(t)\alpha'(t)\sin t\cos t+(\alpha'(t))^2\cos^2 t\Big\}^{1/2}\\
&= \sqrt{\alpha^2(t)(\cos^2 t+\sin^2 t)+(\alpha'(t))^2(\sin^2 t+\cos^2 t)}\\
&= \sqrt{\alpha^2(t)+(\alpha'(t))^2}.
\end{aligned}$$

□

Theorem 3 Consider the oval $P(t)=(x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$. Then, the angular coordinate θ that represents the polar angle from the polar axis to the line segment which connecting the pole and a point on the polar curve, is

$$\theta(t) = t + \tan^{-1}\frac{\alpha'(t)}{\alpha(t)}, \tag{7}$$

where $\alpha(t) = a + \cos nt$ as in Formula (5), and $t \in [0,\ 2\pi]$.

Proof. The polar angle θ, as shown in Figure 1, is the angle created by the line segment OP with the polar axis, while the rectangular angle t is the angle between the positive x-axis and the line segment OD, so

$$\tan(t-\theta) = \frac{\tan t - \tan\theta}{1+\tan t\ \tan\theta} = \frac{\tan t - \dfrac{y(t)}{x(t)}}{1+\dfrac{y(t)}{x(t)}\ \tan t}$$

$$= \frac{x(t)\tan t - y(t)}{x(t)+y(t)\tan t}$$

$$= \frac{(\alpha(t)\cos t - \alpha'(t)\sin t)\tan t - (\alpha(t)\sin t + \alpha'(t)\cos t)}{\alpha(t)\cos t - \alpha'(t)\sin t + \tan t\,(\alpha(t)\sin t + \alpha'(t)\cos t)}$$

$$= \frac{\alpha(t)\sin t - \alpha'(t)\dfrac{\sin^2 t}{\cos t} - \alpha(t)\sin t - \alpha'(t)\cos t}{\alpha(t)\cos t - \alpha'(t)\sin t + \alpha(t)\dfrac{\sin^2 t}{\cos t} + \alpha'(t)\sin t}$$

$$= \frac{-\alpha'(t)}{\alpha(t)}.$$

Hence,

$$\tan(\theta - t) = \frac{\alpha'(t)}{\alpha(t)} \Rightarrow \theta(t) = t + \tan^{-1}\frac{\alpha'(t)}{\alpha(t)}.$$

□

In [10], Al-rabtah and Al-Banawi defined a vertex as a point of the curve, parameterized by $P(t)$, with maximum curvature. And in Theorem 2, they showed that $P(t)$ has exactly n vertices at

$$t = \frac{2q\pi}{n},\ \ q = 0,\ 1,\ \ldots,\ n-1,$$

while the minimum values take place at the corresponding opposite points of the vertices, precisely at

$$t = (2q+1)\frac{\pi}{n},\ \text{ where } q = 0,\ 1,\ \ldots,\ n-1.$$

In the following theorem, we show that we achieve equivalent results considering the formulation in polar coordinates.

Theorem 4 Consider the oval $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$. If the polar angle θ and the rectangular angle t are related with Formula (7), then, the resulting oval, in polar coordinates, has exactly n vertices at

$$\theta = \frac{2q\pi}{n},\ \ q = 0,\ 1,\ \ldots,\ n-1.$$

Proof. We need to find the points with $\frac{dr}{d\theta} = 0$.

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta},$$

where, from (6),

$$\frac{dr}{dt} = \frac{2\alpha\alpha' + 2\alpha'\alpha''}{2\sqrt{\alpha^2 + (\alpha')^2}} = \frac{\alpha'(\alpha + \alpha'')}{\sqrt{\alpha^2 + (\alpha')^2}}.$$

Deriving all Formula (7) by $\frac{d}{d\theta}$, we get:

$$1 = \frac{dt}{d\theta} + \frac{1}{1 + (\alpha'/\alpha)^2}\frac{\alpha\alpha'' - (\alpha')^2}{\alpha^2}\frac{dt}{d\theta}$$

$$= \left(1 + \frac{\alpha\alpha'' - (\alpha')^2}{\alpha^2 + (\alpha')^2}\right)\frac{dt}{d\theta},$$

$$\Longrightarrow \frac{dt}{d\theta} = \frac{\alpha^2 + (\alpha')^2}{\alpha^2 + \alpha\alpha''}.$$

So,

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta} = \frac{\alpha'(\alpha + \alpha'')}{\sqrt{\alpha^2 + (\alpha')^2}}\frac{\alpha^2 + (\alpha')^2}{\alpha^2 + \alpha\alpha''} = \frac{\alpha'\sqrt{\alpha^2 + (\alpha')^2}}{\alpha}.$$

This is equal to zero when $\alpha'(t) = -n\sin nt = 0$, which is satisfied at $t = (q\pi)/n$, where $q = 0,\ 1,\ \ldots,\ 2n$.

Using the second derivative test, we conclude that we have n vertices at the maximum values which occur at $t = (2q\pi)/n$, where $q = 0,\ 1,\ \ldots,\ n-1$, and by Formula (7), at $\theta = (2q\pi)/n$, with the same values for q. The minimum values are corresponding to the opposite points and occur at $t = ((2q+1)\pi)/n$, where $q = 0,\ 1,\ \ldots,\ n-1$, i.e. at $\theta = ((2q+1)\pi)/n$, using Formula (7). □

To clarify the connection between the rectangular angle t and the polar angle θ, we present the following four theorems.

Theorem 5 Consider the oval $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$. If the polar angle θ and the rectangular angle t are related with Formula (7), then, θ and t are equal at the vertices points and their opposite points, namely, at

$$t = \frac{q\pi}{n},\ \ q = 0,\ 1,\ \ldots,\ 2n.$$

Proof. From Formula (7), θ and t are equal if

$$\tan^{-1}\frac{\alpha'(t)}{\alpha(t)} = 0,$$

which is satisfied if $\alpha'(t)/\alpha(t) = 0$, and since $\alpha(t) = a + \cos nt \neq 0$, then

$$\alpha'(t) = -n\sin nt = 0,$$

$$\Rightarrow nt = q\pi \Rightarrow t = \frac{q\pi}{n},\ \ q = 0,\ 1,\ \ldots,\ 2n.$$

This is where the vertices points, and their corresponding opposite points are located on the polar curve. □

Theorem 6 Consider the oval $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$. If the polar angle θ and the rectangular angle t are related with Formula (7), then, the difference between θ and t has relative extrema at

$$t = \frac{1}{n}\left(2q\pi + \cos^{-1}\left(\frac{-1}{a}\right)\right), \tag{8}$$

and at

$$t = \frac{1}{n}\left(2(q+1)\pi - \cos^{-1}\left(\frac{-1}{a}\right)\right), \tag{9}$$

for $q = 0,\ 1,\ \ldots,\ n-1$.

Proof. As $\tan(\theta(t) - t) = \alpha'(t)/\alpha(t)$ from Formula (7), the relative extrema of the function

$$\beta(t) = \tan(\theta(t) - t),$$

may be found among the ones of

$$\gamma(t) = \frac{\alpha'(t)}{\alpha(t)}.$$

$$\Rightarrow \frac{d}{dt}(\gamma(t)) = \frac{d}{dt}\left(\frac{\alpha'(t)}{\alpha(t)}\right) = \frac{\alpha\alpha'' - (\alpha')^2}{\alpha^2} = 0\,,$$

$$\Rightarrow \alpha\alpha'' - (\alpha')^2 = (a + \cos nt)(-n^2 \cos nt) - (-n \sin nt)^2$$

$$= -an^2 \cos nt - n^2 = 0\,,$$

$$\Longrightarrow\ a\cos nt + 1 = 0 \Longrightarrow\ \cos nt = (-1)/a\,,$$

$$\Longrightarrow t = \frac{1}{n}\left(2q\pi + \cos^{-1}\left(\frac{-1}{a}\right)\right),$$

and

$$t = \frac{1}{n}\left(2(q+1)\pi - \cos^{-1}\left(\frac{-1}{a}\right)\right),$$

for $q = 0,\ 1,\ \ldots,\ n-1$. □

Theorem 7 Consider the oval $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$. If the polar angle θ and the rectangular angle t are related with Formula (7), then, the difference between θ and t has an absolute maximum of

$$\tan^{-1} \frac{n}{\sqrt{a^2 - 1}}\,,$$

that occurs at

$$t = \frac{1}{n}\left(2(q+1)\pi - \cos^{-1}\left(\frac{-1}{a}\right)\right),$$

for $q = 0,\ 1,\ \ldots,\ n-1$.

Proof. To show that we have relative maxima at the determined values of the parameter

$$t = \frac{1}{n}\left(2(q+1)\pi - \cos^{-1}\left(\frac{-1}{a}\right)\right)$$

for $q = 0,\ 1,\ \ldots,\ n-1$, we use the second derivative test.

Since, from the proof of Theorem 6,

$$\frac{d}{dt}(\gamma(t)) = \frac{\alpha\alpha'' - (\alpha')^2}{\alpha^2}\,,$$

then,

$$\frac{d^2}{dt^2}(\gamma(t)) = \frac{d}{dt}\left(\frac{\alpha\alpha'' - (\alpha')^2}{\alpha^2}\right)$$

$$= \frac{\alpha^2(\alpha\alpha''' - \alpha'\alpha'') - \left(\alpha\alpha'' - (\alpha')^2\right)(2\alpha\alpha')}{(\alpha^2)^2}$$

$$= \frac{\alpha\alpha''' - \alpha'\alpha''}{\alpha^2}, \quad \text{since} \quad \alpha\alpha'' - (\alpha')^2 = 0.$$

To determine the sign of the second derivative, we have to evaluate at the critical values

$$t = \frac{1}{n}\left(2(q+1)\pi - \cos^{-1}\left(\frac{-1}{a}\right)\right),$$

so,

$$\alpha = a + \cos(nt) = a + \cos\left(2(q+1)\pi - \cos^{-1}\left(\frac{-1}{a}\right)\right) = \frac{a^2-1}{a},$$

$$\alpha' = -n\sin nt = \cdots = \frac{n\sqrt{a^2-1}}{a},$$

$$\alpha'' = -n^2\cos nt = \cdots = \frac{n^2}{a}, \text{ and}$$

$$\alpha''' = n^3\sin nt = \cdots = \frac{-n^3\sqrt{a^2-1}}{a}.$$

Therefore,

$$\frac{d^2}{dt^2}(\gamma(t)) = \frac{\alpha\alpha''' - \alpha'\alpha''}{\alpha^2} = \frac{-n^3a^2\sqrt{a^2-1}}{(a^2-1)^2} < 0.$$

Thus, there are relative maxima at $t = \left(2(q+1)\pi - \cos^{-1}\left(\frac{-1}{a}\right)\right)/n$, for $q = 0, 1, \ldots, n-1$. Since the relative maxima are equal at all values of t, we obtain the maximum value, and in order to determine this value, we substitute the value of t into the Formula (7), so

$$\text{Max}(\theta - t) = \tan^{-1}\frac{\alpha'(t)}{\alpha(t)} = \tan^{-1}\left(\frac{n\sqrt{a^2-1}}{a} \div \frac{a^2-1}{a}\right) = \tan^{-1}\frac{n}{\sqrt{a^2-1}}\ .$$

□

Theorem 8 Consider the oval $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$. If the polar angle θ and the rectangular angle t are related with Formula (7), then, the difference between θ and t has an absolute minimum of

$$\tan^{-1}\frac{-n}{\sqrt{a^2-1}}\ ,$$

that occurs at

$$t = \frac{1}{n}\left(2q\pi + \cos^{-1}\left(\frac{-1}{a}\right)\right),$$

for $q = 0,\ 1,\ \ldots,\ n-1$.

Proof. To show that we have relative minima at the determined values of the parameter $t = \frac{1}{n}\left(2q\pi + \cos^{-1}\left(\frac{-1}{a}\right)\right)$, we use the second derivative test.

From the proof of Theorem 6, we have

$$\frac{d}{dt}(\gamma(t)) = \frac{\alpha\alpha'' - (\alpha')^2}{\alpha^2}\ ,$$

and from the proof of Theorem 7, we have

$$\frac{d^2}{dt^2}(\gamma(t)) = \frac{\alpha\alpha''' - \alpha'\alpha''}{\alpha^2}\ .$$

To determine the sign of the second derivative, we have to evaluate at the critical values

$$t = \frac{1}{n}\left(2q\pi + \cos^{-1}\left(\frac{-1}{a}\right)\right),$$

so,

$$\alpha = a + \cos(nt) = a + \cos\left(2q\pi + \cos^{-1}\left(\frac{-1}{a}\right)\right) = \frac{a^2-1}{a}\ ,$$

$$\alpha' = -n\sin nt = \cdots = \frac{-n\sqrt{a^2-1}}{a}\ ,$$

$$\alpha'' = -n^2 \cos nt = \cdots = \frac{n^2}{a}\ , \text{ and}$$

$$\alpha''' = n^3 \sin nt = \cdots = \frac{n^3\sqrt{a^2-1}}{a}\ .$$

Therefore,

$$\frac{d^2}{dt^2}(\gamma(t)) = \frac{\alpha\alpha''' - \alpha'\alpha''}{\alpha^2} = \frac{n^3a^2\sqrt{a^2-1}}{(a^2-1)^2} > 0\ .$$

Thus, there are relative minima at $t = \left(2q\pi + \cos^{-1}\left(\frac{-1}{a}\right)\right)/n$, for $q = 0,\ 1,\ \ldots,\ n-1$. Since the relative minima are equal at all values of t, we obtain the minimum value, and in order to determine this value, we substitute the value of t into Formula (7), so

$$\text{Min}(\theta - t) = \tan^{-1}\frac{\alpha'(t)}{\alpha(t)} = \tan^{-1}\left(\frac{-n\sqrt{a^2-1}}{a} \div \frac{a^2-1}{a}\right) = \tan^{-1}\frac{-n}{\sqrt{a^2-1}}\ .$$

□

Theorem 9 In polar coordinates $(r,\ \theta)$, consider the curve $P(t) = (x(t),\ y(t))$, parameterized by (3). If the polar angle θ and the rectangular angle t are related with Formula (7), then, the arc length L of the polar curve, starting from any point on the polar curve at a polar angle θ, to its corresponding opposite point at the polar angle $\theta + \pi$ is

$$L = \pi a + 2\left(\frac{n^2-1}{n}\right)\cdot \sin\left[n\left(\theta - \tan^{-1}\frac{\alpha'}{\alpha}\right)\right]. \tag{10}$$

Proof. The arc length of the polar curve from a polar angle θ to $\theta + \pi$ is

$$L = \int_{\theta}^{\theta+\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Using (2), and the proof of Theorem 4, we obtain

$$L = \int_{t}^{t+\pi} \sqrt{\left(\sqrt{\alpha^2+(\alpha')^2}\right)^2 + \left(\frac{\alpha'}{\alpha}\sqrt{\alpha^2+(\alpha')^2}\right)^2}\; \frac{\alpha^2+\alpha\alpha''}{\alpha^2+(\alpha')^2}\, dt$$

$$= \int_{t}^{t+\pi} \frac{\alpha^2+(\alpha')^2}{\alpha}\; \frac{\alpha^2+\alpha\alpha''}{\alpha^2+(\alpha')^2}\, dt$$

$$= \int_{t}^{t+\pi} \left(\alpha+\alpha''\right)\, dt = \int_{t}^{t+\pi} \left(a-(n^2-1)\cos nt\right) dt$$

$$= \left(at - \frac{n^2-1}{n}\sin n\theta\right)\Bigg|_{t}^{t+\pi}$$

$$= \pi a + 2\left(\frac{n^2-1}{n}\right)\sin nt,$$

where $t = \theta - \tan^{-1}\dfrac{\alpha'}{\alpha}$. □

Theorem 10 In polar coordinates $(r,\ \theta)$, consider the curve $P(t) = (x(t),\ y(t))$, parameterized by (3). If the polar angle θ and the rectangular angle t are related with Formula (7), then, the arc length L of the polar curve, starting from any vertex point on the polar curve at a polar angle θ, to its corresponding opposite point at the polar angle $\theta+\pi$ is $L = \pi a$.

Proof. We have proved that $\theta = t$ at the vertices and at their corresponding opposite points, and in this case, $\alpha' = 0$, thus,

$$\sin\ n\left(\theta - \tan^{-1}\frac{\alpha'}{\alpha}\right) = 0,$$

therefore, $L = \pi a$ using Equation (10). □

The following theorem, which is well-known as Barbier's theorem, was firstly proved by Mellish [1], and stated with a simple proof by Robertson in [15]. Al-rabtah and Al-Banawi [10] presented another proof using the parametrization in Equation (4). The new proof here uses the polar representation $(r,\ \theta)$ of the parametric curve, where r and θ are given by Formula (6), and Formula (7), respectively.

Theorem 11 In polar coordinates $(r,\ \theta)$, consider the curve $P(t) = (x(t),\ y(t))$, parameterized by (3). If the polar angle θ and the rectangular angle t are related with Formula (7), then, the perimeter of the polar curve is equal to $L = 2\pi a$.

Proof. The proof follows that of Theorem 9 over the whole domain. That is,

$$L = \int_{0}^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \left(\alpha + \alpha''\right)\, dt$$

$$= \left(at - \frac{n^2 - 1}{n} \sin n\theta\right)\Bigg|_{0}^{2\pi} = 2\pi a.$$

□

Theorem 12 In polar coordinates $(r,\ \theta)$, consider the curve $P(t) = (x(t),\ y(t))$, parameterized by (3). If the polar angle θ and the rectangular angle t are related with Formula (7). The arc length L of the polar curve, starting from any vertex point at a polar angle θ, to the next one is equal to $L = (2\pi a)/n$, which is equal to the perimeter of the polar curve divided by the total number of vertices.

Proof.

$$L = \int_{2q\pi/n}^{2(q+1)\pi/n} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{2q\pi/n}^{2(q+1)\pi/n} \alpha + \alpha''\, dt$$

$$= \left[at - \frac{n^2 - 1}{n} \sin n\theta\right]\Bigg|_{2q\pi/n}^{2(q+1)\pi/n} = \frac{2\pi a}{n}.$$

□

3. Intersection of an oval with particular circles in polar coordinates

It is interesting to compare our oval to particular circles related to the width of the oval and its vertices. In this section, we also state new results regarding the intersection points between the oval and such circles.

Theorem 13 Consider the oval $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$. The radial coordinate r has a maximum value of $r = a + 1$, which represents an origin-centered circle of radius $a + 1$, obtained at the vertices that are located by:

$$\theta = \frac{2q\pi}{n},\ \ q = 0,\ 1,\ \ldots,\ n - 1.$$

And, it has a minimum value of $r = a - 1$, which represents an origin-centered circle of radius $a - 1$, obtained at the opposite points, that are located by:

$$\theta = \frac{(2q+1)\pi}{n}, \quad q = 0,\ 1,\ \ldots,\ n-1.$$

Proof. In the proof of Theorem 4, we have shown that there was a maximum value for the radial coordinate r at the vertices. Since $\alpha = a + 1$ and $\alpha' = 0$ at those points, then the maximum value for the radial coordinate is

$$r = \sqrt{\alpha^2 + (\alpha')^2} = \sqrt{(a+1)^2 + 0} = a + 1.$$

Also, we have shown that there was a minimum value for the radial coordinate r at the corresponding opposite points. Since $\alpha = a - 1$ and $\alpha' = 0$ at those points, then the minimum value for the radial coordinate is

$$r = \sqrt{\alpha^2 + (\alpha')^2} = \sqrt{(a-1)^2 + 0} = a - 1.$$

□

Theorem 14 Consider the oval $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$, and the origin-centered circle $r = a + 1$. The points of intersections between the oval in polar coordinates, for any value of the parameter n that satisfies the condition of convexity, and the specified circle occur at the vertices on the polar oval given by

$$\theta = \frac{2q\pi}{n}, \quad q = 0,\ 1,\ \ldots,\ n-1.$$

Proof. Since the maximum values of the radial coordinate of the oval, which is equal to $a + 1$, occur only at the vertices, then the intersection points between the oval in polar coordinates, for any value of the parameter n that satisfies the condition of convexity, and the specified origin-centered circle $r = a + 1$ occur at the vertices points on the oval. □

Theorem 15 Consider the oval $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$, and the specified origin-centered circle $r = a - 1$. The points of intersections between the oval in polar coordinates, for any value of the parameter n that satisfies the condition of convexity, and the specified circle occur at the corresponding opposite points of the vertices on the polar oval given by

$$\theta = \frac{(2q+1)\pi}{n}, \quad q = 0,\ 1,\ \ldots,\ n-1.$$

Proof. Since the minimum values for the radial coordinate of the oval, which is equal to $a - 1$, occur only at the opposite points of vertices, then the intersection points between the oval in polar coordinates, for any value of the parameter n that satisfies the condition of convexity, and the specified origin-centered circle $r = a - 1$ occur at the corresponding opposite points of the vertices on the polar oval. □

Corollary 1 All ovals $P(t) = (x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$, for any value of the parameter n that satisfies the condition of convexity, are enclosed between two specific origin-centered circles: the bigger circle is $r = a + 1$, and the smaller one is $r = a - 1$.

Proof. We have shown in Theorem 13 that the maximum radial coordinate of the polar oval is $a + 1$ which occurs at the vertices points, and in Theorem 14 we have shown that the polar ovals, for any value of the parameter n that satisfies the condition of convexity, intersect the circle at those vertices, which means that all other radial coordinates are less than $a+1$.

On the other hand, we have shown in Theorem 13 that the minimum radial coordinate of the polar oval is $a-1$ which occurs at the corresponding opposite points of the vertices, and in Theorem 15 we have shown that the polar ovals, for any value of the parameter n that satisfies the condition of convexity, intersect the circle at those points, which means that all other radial coordinates are greater than $a-1$.

Thus, all radial coordinates of the oval in polar coordinates lie between $a-1$ and $a+1$. Therefore, all ovals in polar coordinates, discussed in this study, are enclosed between the two particular circles. □

Corollary 2 Consider the oval $P(t)=(x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$, and the two particular circles $r=a+1$, and $r=a-1$ centered at the origin. The distance between any of the two circles and the resulting polar oval, for any value of the parameter n that satisfies the condition of convexity, lies in the interval $[0,\ 2]$.

Proof. Since the maximum value of the radial coordinate of the considered polar oval, which is $a+1$, coincides with the circle $r=a+1$, and the minimum value of the radial coordinate of that oval, which is $a-1$, coincides with the circle $r=a-1$, then the distance between any of the two circles and the polar oval lies between zero and two. □

Theorem 16 Consider the oval $P(t)=(x(t),\ y(t))$, parameterized by (3), in polar coordinates $(r,\ \theta)$, and any origin-centered circle $r=a$. The points of intersections between the oval in polar coordinates, for any value of the parameter n that satisfies the condition of convexity, and the circle occur for

$$\theta=\frac{1}{n}\left[2q\pi+\cos^{-1}\left(\frac{a-\sqrt{a^2+n^2(n^2-1)}}{n^2-1}\right)\right]$$

$$+\tan^{-1}\left[\frac{\left(-n\right)\sqrt{-2a\left(a-\sqrt{a^2+n^2(n^2-1)}\right)-n^2+1}}{an^2-\sqrt{a^2+n^2(n^2-1)}}\right],$$

and for

$$\theta=\frac{1}{n}\left[2(q+1)\pi-\cos^{-1}\left(\frac{a-\sqrt{a^2+n^2(n^2-1)}}{n^2-1}\right)\right]$$

$$+\tan^{-1}\left[\frac{\left(n\right)\sqrt{-2a\left(a-\sqrt{a^2+n^2(n^2-1)}\right)-n^2+1}}{an^2-\sqrt{a^2+n^2(n^2-1)}}\right].$$

Proof. Since $r=\sqrt{\alpha^2+(\alpha')^2}$, then the intersections between the polar curve $(r,\ \theta)$ and the circle $r=a$ occur when $a=\sqrt{\alpha^2+(\alpha')^2}$, this gives

$$a^2 = (a + \cos nt)^2 + n^2 \sin^2 nt \ ,$$

$$\Longrightarrow (n^2 - 1)\cos^2 nt - 2a\cos nt - n^2 = 0 \ ,$$

$$\Longrightarrow \cos nt = \frac{a - \sqrt{a^2 + n^2(n^2 - 1)}}{n^2 - 1} \ ,$$

$$\Longrightarrow t = \frac{1}{n}\left[2q\pi + \cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2 - 1)}}{n^2 - 1}\right)\right], \tag{11}$$

and

$$t = \frac{1}{n}\left[2(q+1)\pi - \cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2 - 1)}}{n^2 - 1}\right)\right]. \tag{12}$$

To find the value of θ that corresponds to the value of t obtained in Equation (11), we use the Formula (7), where

$$\alpha = a + \cos\left[2q\pi + \cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2 - 1)}}{n^2 - 1}\right)\right]$$

$$= a + \frac{a - \sqrt{a^2 + n^2(n^2 - 1)}}{n^2 - 1} = \frac{an^2 - \sqrt{a^2 + n^2(n^2 - 1)}}{n^2 - 1} \ , \text{ and}$$

$$\alpha' = -n\sin\left[2q\pi + \cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2 - 1)}}{n^2 - 1}\right)\right]$$

$$= -n\sin\left[\cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2 - 1)}}{n^2 - 1}\right)\right]$$

$$= \frac{-n}{n^2 - 1}\sqrt{-2a\left(a - \sqrt{a^2 + n^2(n^2 - 1)}\right) - n^2 + 1} \ .$$

Thus,

$$\theta = t + \tan^{-1}\frac{\alpha'}{\alpha}$$

$$= \frac{1}{n}\left[2q\pi + \cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2-1)}}{n^2-1}\right)\right]$$

$$+ \tan^{-1}\left[\frac{\left(-n\right)\ \sqrt{-2a\left(a - \sqrt{a^2+n^2(n^2-1)}\right) - n^2 + 1}}{an^2 - \sqrt{a^2+n^2(n^2-1)}}\right].$$

To find the value of θ that corresponds to the value of t obtained in Equation (12), we use the Formula (7), where

$$\alpha = a + \cos\left[2(q+1)\pi - \cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2-1)}}{n^2-1}\right)\right]$$

$$= a + \frac{a - \sqrt{a^2 + n^2(n^2-1)}}{n^2-1} = \frac{an^2 - \sqrt{a^2 + n^2(n^2-1)}}{n^2-1}, \text{ and}$$

$$\alpha' = -n\sin\left[2(q+1)\pi - \cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2-1)}}{n^2-1}\right)\right]$$

$$= n\sin\left[\cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2-1)}}{n^2-1}\right)\right]$$

$$= \frac{n}{n^2-1}\sqrt{-2a\left(a - \sqrt{a^2+n^2(n^2-1)}\right) - n^2 + 1}\ .$$

Thus,

$$\theta = t + \tan^{-1}\frac{\alpha'}{\alpha}$$

$$= \frac{1}{n}\left[2(q+1)\pi - \cos^{-1}\left(\frac{a - \sqrt{a^2 + n^2(n^2-1)}}{n^2-1}\right)\right]$$

$$+ \tan^{-1}\left[\frac{\left(n\right)\ \sqrt{-2a\left(a - \sqrt{a^2+n^2(n^2-1)}\right) - n^2 + 1}}{an^2 - \sqrt{a^2+n^2(n^2-1)}}\right].$$

□

Next, we are going to simulate our theoretical results, so that all examples will be shown in a matter of sketching in polar coordinates, and all results will be more clarified.

4. Simulation results

Here we give some simulation findings to demonstrate and validate the theoretical analysis and conclusions in this study.

Example 1 Consider the relation between the polar angle θ and the rectangular angle t, given by the Formula (7). We show numerically that the polar angles and the rectangular angles are equal at the vertices and opposite points, and locate and evaluate the minimum and maximum difference between them, and where these extrema occur.

We consider two convex curves of CW with $a = 30$. Figure 2 shows the two curves of constant width $2a = 60$, the first curve when $n = 3$, while the second one when $n = 5$. Figure 2 shows the square with side length that is equal to $2a = 60$, and that each of the two curves touches all four sides of that square.

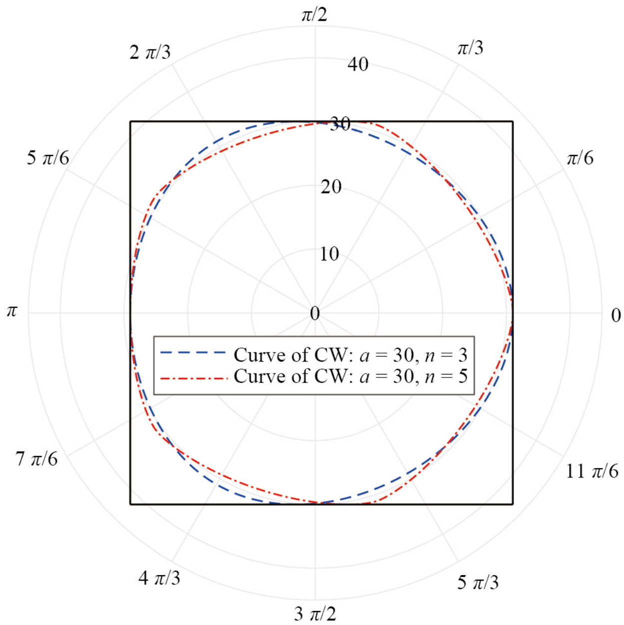


Figure 2. Convex curves of constant width $2a$, touching all four sides of the square with side length $2a$, where $a = 30$, $n = 3$, and $n = 5$

In Figure 3, we show the difference between the polar angles θ, and the corresponding values of the rectangular angles t, in the case where $a = 30$, and the parametric value $n = 3$ that satisfies the condition of convexity. It is clear that the difference between θ and t is equal to zero at 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$, and 2π, which correspond to where the vertices and their opposite points of the polar oval are located, as proved in Theorem 5. Figure 3 shows that we have an absolute maximum value of $\tan^{-1}(3/\sqrt{30^2-1}) \approx 0.0997$ occurs at $t = (1/3)(2\pi - \cos^{-1}(-1/30)) \approx 1.5597$, $t = (1/3)(4\pi - \cos^{-1}(-1/30)) \approx 3.6541$, and at $t = (1/3)(6\pi - \cos^{-1}(-1/30)) \approx 5.7485$, which concurs with the results in Theorem 7. The figure also shows that we have an absolute minimum value of $\tan^{-1}(-3/\sqrt{30^2-1}) \approx -0.0997$ occurs at $t = (1/3)\cos^{-1}(-1/30) \approx 0.5347$, $t = (1/3)(2\pi + \cos^{-1}(-1/30)) \approx 2.6291$, and at $t = (1/3)(4\pi + \cos^{-1}(-1/30)) \approx 4.7235$, which also concurs with the results in Theorem 8.

In Figure 4, we show the difference between the polar angles θ, and the corresponding values of the rectangular angles t, in the case where $a = 30$, and the parametric value $n = 5$ that satisfies the condition of convexity. It is clear

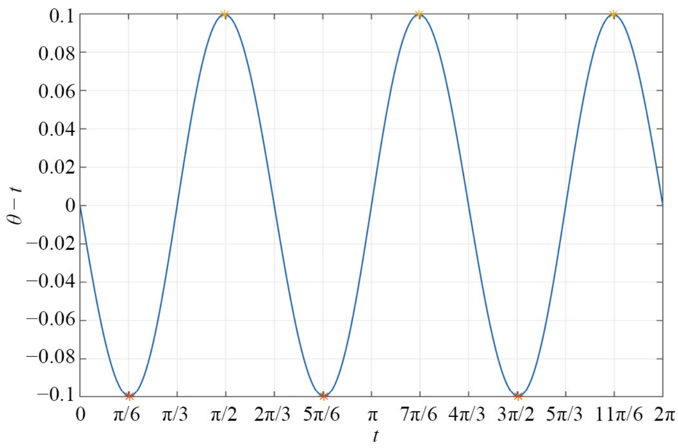


Figure 3. Difference between the polar angles θ and the corresponding values of the rectangular angles t, where $a = 30$, and $n = 3$

that the difference between θ and t is equal to zero at 0, $\pi/5$, $2\pi/5$, π, $6\pi/5$,$7\pi/5$, $8\pi/5$, $9\pi/5$, and 2π, which correspond to where the vertices and their corresponding opposite points of the polar oval are located, as proved in Theorem 5. The figure shows that we have an absolute maximum value of $\tan^{-1}(5/\sqrt{30^2-1}) \approx 0.1652$ occurs at $t = (1/5)(2\pi - \cos^{-1}(-1/30)) \approx 0.9358$, $t = (1/5)(4\pi - \cos^{-1}(-1/30)) \approx 2.1924$, $t = (1/5)(6\pi - \cos^{-1}(-1/30)) \approx 3.4491$, $t = (1/5)(8\pi - \cos^{-1}(-1/30)) \approx 4.7057$, and at $t = (1/5)(10\pi - \cos^{-1}(-1/30)) \approx 5.9624$, which is consistent with the results in Theorem 7. The figure also shows that we have an absolute minimum value of $\tan^{-1}(-5/\sqrt{30^2-1}) \approx -0.1652$ occurs at $t = (1/5)(\cos^{-1}(-1/30)) \approx 0.3208$, $t = (1/5)(2\pi + \cos^{-1}(-1/30)) \approx 1.5775$, $t = (1/5)(4\pi + \cos^{-1}(-1/30)) \approx 2.8341$, $t = (1/5)(6\pi + \cos^{-1}(-1/30)) \approx 4.0907$, and at $t = (1/5)(8\pi + \cos^{-1}(-1/30)) \approx 5.3474$, which concurs with the results in Theorem 8.

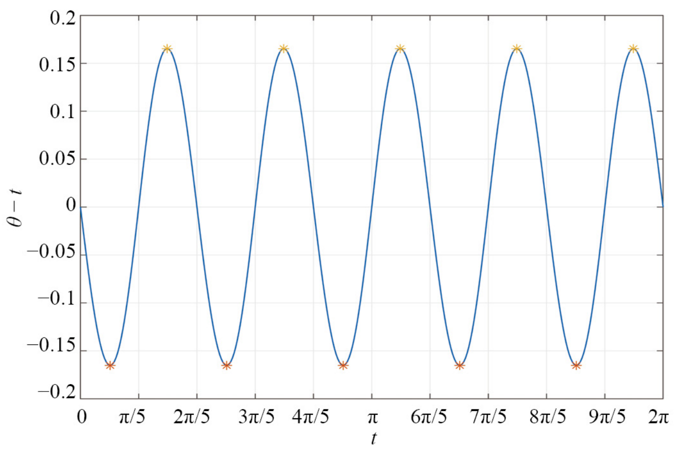


Figure 4. Difference between the polar angles θ and the corresponding values of the rectangular angles t, where $a = 30$, and $n = 5$

Example 2 Consider the oval parameterized with $P(t) = (x(t),\ y(t))$, given by (3) with $\alpha(t)$ defined by Formula (5), in polar coordinates $(r,\ \theta)$, and the particular circle $r = a+1$ that is centered at the origin, which represents the maximum radial coordinate of the polar curve, which can also be obtained by substituting $n = 0$ into the parametric function given by Formula (3). We compare the obtained ovals at two values of the parameter n with the particular predefined circle, and discuss the distances between the ovals and the selected circle.

In Figure 5, we show the circle $r = a + 1 = 32 + 1 = 33$ centered at the origin with comparisons to two ovals in polar coordinates, with parametric values $n = 3$ and $n = 5$, respectively, obtained by substituting these values of n, that satisfy the condition of convexity, and $a = 32$, into the same considered parametric function (3). The figure shows that each oval has n different vertices points, and n different opposite points, and these ovals intersect the circle at the vertices points, which satisfy the result of Theorem 14.

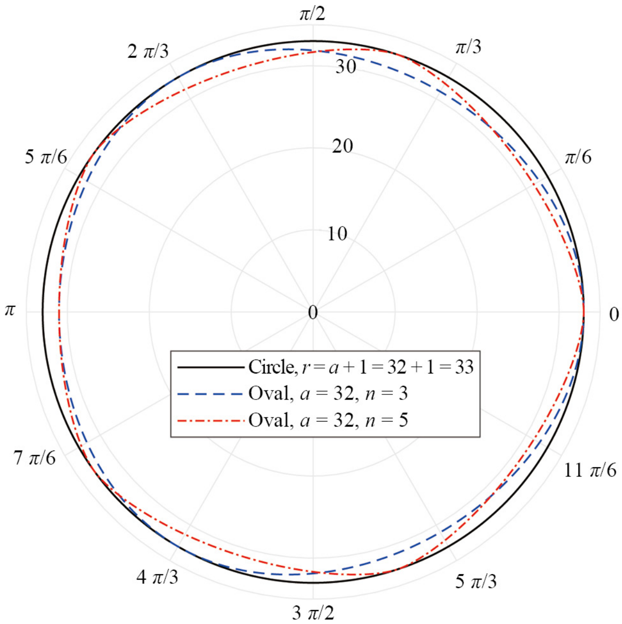
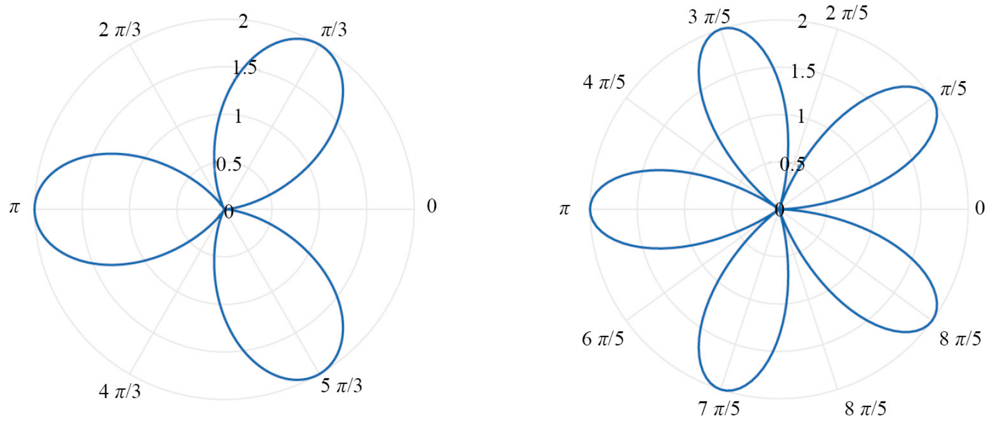


Figure 5. Ovals with different values of n, in polar coordinates, compared to the circle $r = a + 1$ centered at the origin, $a = 32$

In Figure 6, we show the difference between the radius of the particular circle $r = a + 1 = 32 + 1 = 33$ and the radial coordinates of the discussed ovals in polar coordinates. Figure 6a shows the differences in the first case when $n = 3$, we see that the minimum difference value of 0 occurs at the three vertices points, located at $\theta = (q\pi/3),\ q = 0,\ 2,\ 4$, and we see that the maximum difference value of 2 occurs at the corresponding three opposite points of the vertices, located at $\theta = (q\pi/3),\ q = 3,\ 5,\ 1$, and all other differences lie in the open interval $(0,\ 2)$. Figure 6b shows the differences in the second case when $n = 5$, you can see that the minimum difference value of 0 occurs at the five vertices points, located at $\theta = (q\pi/5),\ q = 0,\ 2,\ 4,\ 6,\ 8$, and that the maximum difference value of 2 occurs at the corresponding five opposite points of the vertices, located at $\theta = (q\pi/5),\ q = 5,\ 7,\ 9,\ 1,\ 3$, and all other differences lie in the open interval $(0,\ 2)$. Therefore, the distance between any point on the circle and the corresponding point on any of the two polar ovals lies in the closed interval $[0,\ 2]$, this confirms with the result of Corollary 2.



(a) $n = 3$ (b) $n = 5$

Figure 6. Differences between the radius of the concentric circle $r = a+1$, and the radial coordinates of the polar ovals with different values of n, where $a = 32$. (a) First case $n = 3$. (b) Second case $n = 5$

Example 3 Consider the oval parameterized with $P(t) = (x(t),\ y(t))$, given by (3) with $\alpha(t)$ defined by Formula (5), in polar coordinates $(r,\ \theta)$. In this example we present comparisons between three ovals, in polar coordinates, to two particular circles, we show graphically that the three ovals are bounded by the two circles, and the radial coordinates for any of these ovals are enclosed between the radii of the two circles.

Figure 7 shows two origin-centered circles, the first circle is $r = a+1 = 49+1 = 50$, and the second one is $r = a-1 = 49-1 = 48$. The figure also shows three ovals, with $a = 49$, obtained by substituting each of the parametric values: $n = 3,\ 5$, and 7, in the considered parametric curve (3), the chosen values of the parameter n must satisfy the condition of convexity, $n < \sqrt{a+1}$. The figure also confirms that the three polar ovals are enclosed between the two circles, which satisfies Corollary 1.

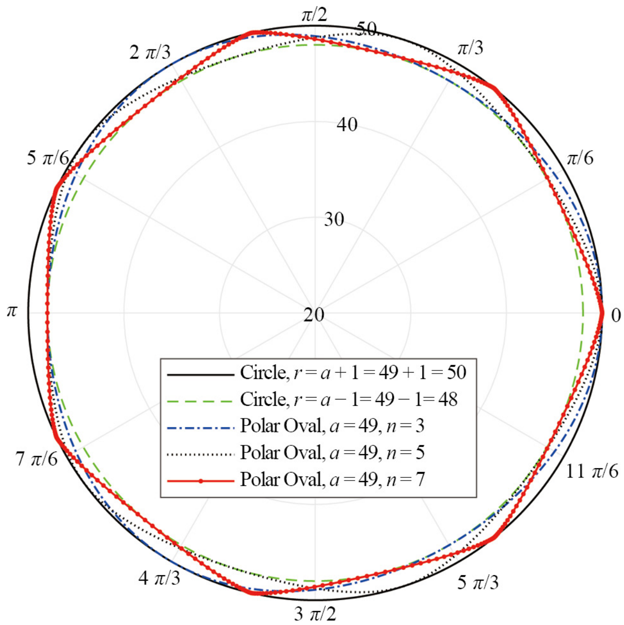


Figure 7. Ovals with different values of n bounded by two concentric circles $r = a+1$, and $r = a-1$, where $a = 49$

To clarify the results in this example, we compare the radii of the two circles with the radial coordinates of the three ovals with respect to the polar angles. Figure 8 shows comparisons between the radii of the two circles, $r = 50$ and $r = 48$, and the radial coordinates of the three ovals with $a = 49$, and the parameter values $n = 3$, $n = 5$, and $n = 7$. Notice that, for each of the three ovals, the radial coordinates lie in the closed interval $[48, 50]$, where the end points of the interval represent the radii of the two particular circles, also, the distances between any of the two circles and the three polar ovals lie in the interval $[0, 2]$, these conclusions are consistent with the results in Corollary 1, and Corollary 2.

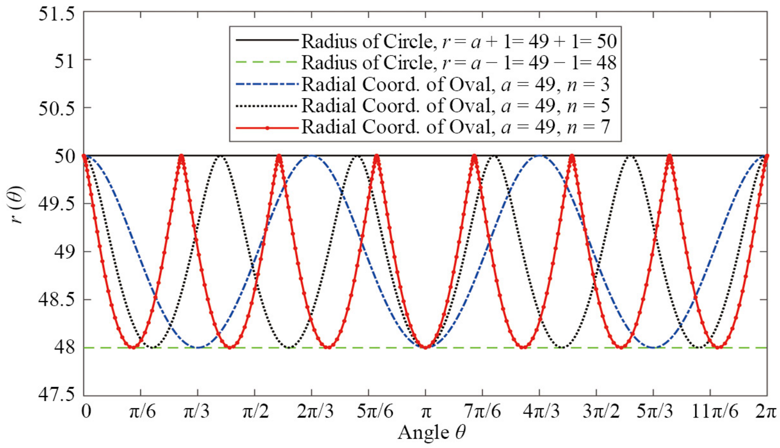


Figure 8. Radial coordinates of ovals with different values of n bounded by radii of two circles $r = a+1$, and $r = a-1$, where $a = 49$

5. Conclusions

In this research, we formulated ovals of CW in polar coordinates. The interesting relationship properties between the rectangular and polar angles of these ovals were discussed analytically and geometrically. Interesting properties of ovals in polar coordinates were examined, the vertices and lengths of such polar curves, in addition to many lengths formulas for these polar curves between opposite points, and from any vertex to the next one, we introduced a new verification of Barbier's theorem. The radii of the considered polar ovals attain their extreme values at their vertices and the corresponding opposite points. The extreme values of a polar oval coincide with two circles related to it, we found that all these types of polar ovals are enclosed between these two specific concentric circles. Finally, we analytically determined the intersection points between the general form of a polar oval and any circle related to it.

Through our work, we arrived at the fact that the radial coordinate and the angular coordinate both can be written in terms of the support function. Moreover, for future work, the support function itself, in both cases, is a solution of a first-order ODE. Furthermore, we used polar coordinates to show that not only vertices on ovals of CW govern the behavior of such geometrical structures, but differences in Cartesian and polar coordinates play a similar role in describing them.

In comparison with Cartesian coordinates, polar coordinates raised the importance of the study of ovals of CW in the sense that new formulas of length (as well as arc length) of such curves have more mathematically attractive evidence for conducting future work, especially if this is combined with the process of enclosing polar ovals of CW between particular circles, as seen in Section 3.

To conclude the list of findings, it is worthwhile to mention that this work is rich in simulation results and can serve as a future reference for scholars who have an interest in this topic.

Conflict of interest

The author declares no competing financial interest.

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