Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

A Bayesian Shrinkage Approach for the Inverse Weibull Distribution under the Type-II Censoring Schemes

Mojtaba Delavari[1], Zahra Khodadadi[1*], Karim Zare[1], Einolah Deiri[2], Abdul Rasoul Ziaei [3]

[1] Department of Statistics, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran
[2] Department of Statistics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
[3] Department of Statistics, Dashtestan Branch, Islamic Azad University, Borazjan, Iran
E-mail: zkhodadadi@iau.ac.ir

Received: 31 August 2024; **Revised:** 18 November 2024; **Accepted:** 19 November 2024

Abstract: One main challenge in the application of the lifetime distribution models, such as inverse Weibull (IW) distribution is the need for an appropriate estimation method based on experimental conditions. When prior information and certain guessed values are available for model parameters the Bayesian shrinkage (BS) method becomes a valuable approach in this situation. This study considered the BS estimation method in the two-parameter IW distribution under the squared error loss function (SELF) and the type-II censored data. The maximum likelihood (ML), the least squares (LS), and Bayes estimation methods were also examined for a comparative study. Due to the complexity of calculations, the Lindley approach was utilized to approximate the Bayes estimates. The BS estimates were derived and a score test for the guessed value was presented. Additionally, a Monte Carlo simulation was conducted to evaluate the efficiency of all estimation methods. Furthermore, a real data set was implemented to illustrate and compare the BS estimates with the other estimates. The simulation study indicated the consistency of the estimators. The numerical studies also demonstrated that the BS estimators outperform the others.

***Keywords*:** inverse Weibull, bayes estimation, bayesian shrinkage estimation, lindley approximation, type-II censored data, least squares estimation, maximum likelihood estimation

MSC: 62E15, 62F10, 62F15

Abbreviation

APC	Average Percent of Changes
BS	Bayesian Shrinkage
CDF	Cumulative Distribution Function
HF	Hazard Function
IW	Inverse Weibull
LS	Least Squares
MCMC	Markov Chain Monte Carlo
ML	Maximum Likelihood
MSE	Mean Squared Error

DOI: https://doi.org/10.37256/cm.5420245609

PDF Probability Density Function
SELF Square Error Loss Function
TTT Total Time on Test

1. Introduction

The Weibull distribution has an extensive application in lifetime data analysis. The wide variety of the forms of the Weibull distribution that can be adjusted by changing the parameters makes this distribution popular. In literature, extensive research has been done on this distribution; for example, see Johnson et al. [1], Murthy [2] and Kundu [3]. Depending on the shape parameter value, the probability density function (PDF) of the Weibull distribution may be decreasing or unimodal, and its hazard function (HF) may be decreasing or increasing (See Figure 1(a)). Therefore, the Weibull distribution has been widely used in modeling survival and failure time data where the empirical estimates of the HF are monotonic. While, it may be inappropriate where the HF estimate of the data is non-monotonic, whatever the values of its parameters are. In many practical studies, it is usually established in advance that the hazard rate cannot be monotonic. For example, when the course of a disease may be depicted in a pattern where mortality rate peaks after a limited period and then slowly declines. A real example of this situation is in a study of the curability of breast cancer, where Langlands et al. [4] found that the death rate reaches its peak after about three years; as another real example, Bennett [5] analyzed data from the Veterans Administration lung cancer trial presented by Prentiss [6] and found the smoothed empirical HF estimates for both low and high-performance status groups were not clearly monotonic. It is reasonable to analyze such data sets with suitable models. If, after empirical studies, we conclude that the HF is non-monotonic, then the IW distribution is one of the appropriate choices in modeling (See Figure 1(b) which covers the range of the values of the IW parameters reported in the literature).

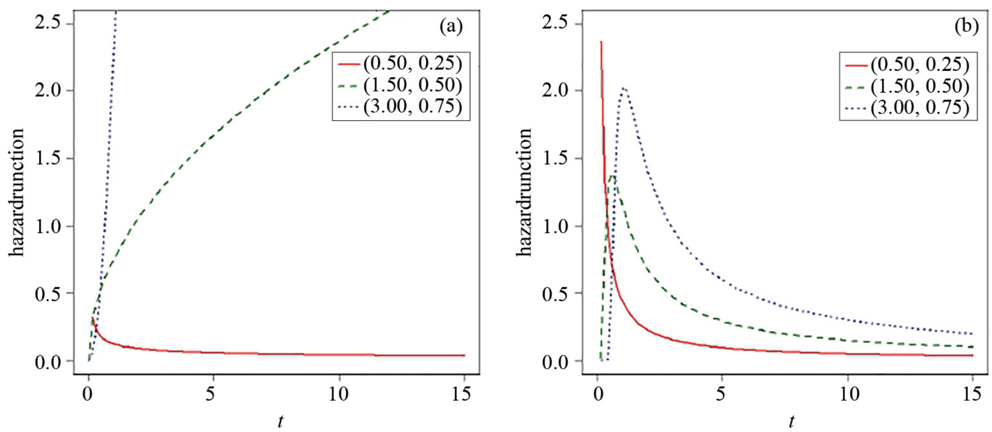

Figure 1. (a) Weibull distribution HF; (b) IW distribution HF for different amounts of (α, λ)

Based on theoretical considerations and also practical applications in many diverse fields, the IW distribution is a suitable distribution with high flexibility for modeling complete or censored lifetime data (See for example, Murthy [2]; Kundu and Howlader [7]). Consequently, numerous researchers have focused on defining the IW distribution in different forms under complete and censored lifetime data, and have introduced classical and Bayesian estimation methods for this distribution. Using a Bayesian framework is preferred to classical inferences including ML and LS estimation methods when prior information is available for the IW parameters. In many practical situations where there are also guessed values for the parameters, the combination of Bayesian and shrinkage approaches (called BS approach) can perform better than any other approaches. It is essential to establish the definition of the IW distribution before delving deeper into discussions.

Suppose that a random variable Y has the Weibull distribution with the scale parameter λ and the shape parameter α. Then, the random variable $X = 1/Y$ is called an IW random variable and we write, $X \sim \text{IW}\ (\alpha,\ \lambda)$. The PDF of X can be obtained as follows Keller et al. [8]:

$$f_X(x|\alpha,\ \lambda) = \alpha\lambda x^{-(\alpha+1)}e^{-\lambda x^{-\alpha}}, \quad x > 0, \quad \alpha > 0, \quad \lambda > 0. \tag{1}$$

Also, from (1), the cumulative distribution function (CDF) of X is derived as follows:

$$F_X(x|\alpha,\ \lambda) = e^{-\lambda x^{-\alpha}}. \tag{2}$$

The mean and variance are given by

$$E(X) = \lambda^{\frac{1}{\alpha}}\Gamma\left(1 - \frac{1}{\alpha}\right),$$

and

$$Var(X) = \lambda^{\frac{2}{\alpha}}\left(\Gamma\left(1 - \frac{2}{\alpha}\right) - \left(\Gamma\left(1 - \frac{1}{\alpha}\right)\right)^2\right).$$

According to the above equations, the mean and variance of the IW $(\alpha,\ \lambda)$ random variable exist when $\alpha > 2$.

One of the common problems in reliability and survival analysis is data censorship. So, the study of the lifetime distributions under the types of censored data is usually one of the interesting topics for statisticians. Some recent studies in this field are Dey et al. [9], Jia et al. [10], Dey et al. [11] and Asar and Arabi Belaghi [12].

Consider n items are to be tested in a lifetime experiment for which the observed failure times are the order statistics of a random sample, denoted by $X_{(1)} \leqslant \ldots \leqslant X_{(n)}$, from a random variable X. Since the waiting time for the final failure is unbounded, the experiment may be terminated before the last failure [13]. For this reason, in some cases, the experiment may end when the r-th failure, $X_{(r)}$, is observed, which is called a type-II censoring scheme. In this case, the value of the failure time r is usually fixed while the end point $X_{(r)}$ is a random variable. This form of censoring reduces time and cost, but information about essential parameters in the censored data is lost [14]. Therefore, type-II censored data will naturally be less efficient than complete data. In addition to the type-II censoring scheme, there are other schemes such as random censoring and combined (hybrid) censoring Epstein [15] and progressively Type-II censoring Balakrishnan and Aggarwala [16].

Reviewing the literature, a number of studies can be found on modeling censored lifetime data through the IW distribution. Calabria and Pulcini [17] studied the ML estimates of the parameters of the IW distribution for the complete and type-II censored data. Kundu and Howlader [7] used (Markov chain Monte Carlo) MCMC procedure to compute the Bayes estimates and prediction problems of the IW distribution under type-II censored data. Yaghmaei et al. [18] proposed the classical and Bayesian methods to estimate the scale parameter of the IW distribution. Sultan et al. [19] provided both the classical and Bayesian inference for a two-parameter IW distribution where type-II progressively censored data are available. Kazemi and Azizpoor [20] presented the classical and Bayesian inferences of the IW distribution under type-I hybrid censoring. Delavari et al. [21] presented the BS estimates for the scale parameter of the IW distribution based on the squared error and LINEX loss functions under type-II censored data (One can refer to Prakash and Singh [22], Naghizade Qomi [23], and Naghizadeh Qomi et al. [24] to see the application of the BS method in statistical models).

They showed the performance of the ML and the Bayesian estimates are quite satisfactory. Yaghoobzadeh Shahrestani et al. [25] obtained E-Bayesian estimation of parameters of IW distribution under the unified hybrid censoring scheme. Okasha et al. [26] investigated the Bayesian estimators of the rate parameter of the IW distribution under two error loss functions. However, they didn't present the estimate of the parameter under censoring due to the form of the survival function of the IW distribution. To see other studies in this field, one may refer to Singh et al. [27], Ateya [28], Alam and Nassar [29] and Ren and Hu [30].

The literature review of the two-parameter IW distribution defined in (1) under type-II censored data reveals a gap in the studying the Bayesian inference through the Lindley approximation technique and the BS approach. Additionally, comparing these suggested approaches with classical approaches can be useful in the application. To do so, we suppose the experiment is under a type-II censoring scheme, where observations end after the occurrence of r-th failure. The IW distribution is examined in order to model the censored data and derive the ML, LS, and Bayes estimates of the parameters. Also, we use Lindley approximation technique is considered to obtain the Bayes estimates due to the lack of explicit solutions. As a highlight of this paper, a BS method is suggested for the estimation of the parameters. As we know, when the experimenter has a guessed value about the parameter value, the shrinkage estimators are valuable in application. In this case, the guessed value can be used to infer the parameter. Here, a score test procedure is proposed for choosing the guessed value based on the sample data. Moreover, the comparison of the BS estimates with the other popular estimates through a Monte Carlo simulation and a real data analysis is another outstanding point of this work.

The continuation of the article is organized as follows. Section 2 focuses on estimating the parameters of IW distribution using ML and LS estimation methods under Type-II censored data. In Section 3 according to the method developed by Lindley [31], the Bayes and BS estimators are derived in a form that avoids integrals. In Section 4, the efficiency of the mentioned methods is evaluated based on a Monte Carlo simulation study. Section 5 provides a real data example to illustrate our results. Finally, Section 6 gives discussion and conclusions.

2. ML and LS estimation methods

2.1 *ML estimation method*

In this subsection, the ML estimates of the parameters are derived for a type-II censored IW random sample of size n. Let $x_{(1)}, \ldots, x_{(r)}$ be the r smallest observations of a random sample $X_1, \ldots, X_n$. Then, the likelihood function is presented as (Arnold et al. [32]) as

$$L(\alpha, \lambda | x_{(1)}, \ldots, x_{(r)}) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_{(i)})[1 - F(x_{(r)})]^{n-r}$$

$$\propto \alpha^r \lambda^r \prod_{i=1}^{r} x_{(i)}^{-(\alpha+1)} \exp\left(-\lambda \sum_{i=1}^{r} x_{(i)}^{-\alpha}\right) \left[1 - \exp(-\lambda x_{(r)}^{-\alpha})\right]^{n-r}. \tag{3}$$

Hence, the log-likelihood function from (3) becomes

$$\ell(\alpha, \lambda) = \ell(\alpha, \lambda | x_{(1)}, \ldots, x_{(r)}) = \ln(L(\alpha, \lambda | x_{(1)}, \ldots, x_{(r)}))$$

$$\propto r(ln(\alpha\lambda)) - (\alpha+1) \sum_{i=1}^{r} \ln\left(x_{(i)}\right) - \lambda \sum_{i=1}^{r} x_{(i)}^{-\alpha} + (n-r)\ln(1-B),$$

where $B = B\ (\alpha,\ \lambda) = \exp\{-\lambda x_{(r)}^{-\alpha}\}$. First, we take the derivatives of the log-likelihood function with respect to α and λ, and then set them equal to zero.

$$\frac{\partial \ell(\alpha,\ \lambda)}{\partial \alpha} = \frac{r}{\alpha} - \sum_{i=1}^{r} \ln\left(x_{(i)}\right) + \lambda \sum_{i=1}^{r} x_{(i)}^{-\alpha} \ln\left(x_{(i)}\right) + (n-r)\frac{B\ln(B)}{1-B} \ln\left(x_{(r)}\right) = 0, \tag{4}$$

and

$$\frac{\partial \ell(\alpha,\ \lambda)}{\partial \lambda} = \frac{r}{\lambda} - \sum_{i=1}^{r} x_{(i)}^{-\alpha} - \frac{n-r}{\lambda}\frac{B\ln(B)}{1-B} = 0. \tag{5}$$

The ML estimates of α and λ, i.e. $\hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$, are the solutions of the equations (4) and (5), respectively (Calabria and Pulcini [17]). These equations show that the solutions are not in closed forms. Therefore, an iterative algorithm is needed to solve the equations.

2.2 *LS estimation method*

Let $X_{(1)},\ \ldots,\ X_{(r)}$ be a type-II censored random sample from the IW distribution. If the CDF in (2) is changed to a linear function, then we have,

$$\ln(-\ln F(x)) = \ln(\lambda) - \alpha \ln(x). \tag{6}$$

Let $Y = \ln[-\ln F(x)]$, $X = \ln(x)$, $\beta_1 = -\alpha$ and $\beta_0 = \ln(\lambda)$. The equation (6) can be written as

$$Y = \beta_0 + \beta_1 X.$$

Now, if the rank average is used to estimate the values of the CDF, then the estimator of F is equal to

$$\hat{F}(X_{(i)}) = \frac{i}{r+1}.$$

In the following, the regression parameters β_0 and β_1 are chosen such that the sum of squared errors, that is

$$Q(\beta_0,\ \beta_1) = \sum_{i=1}^{r} \left(Y_i - \beta_0 - \beta_1 \ln\left(X_{(i)}\right)\right)^2,$$

is minimized. With differentiating Q with respect to β_0 and β_1 and setting equal to zero, the LS estimates of β_0 and β_1 respectively yield as follows:

$$\hat{\beta}_1 = \frac{r\sum_{i=1}^{r} \ln\left(X_{(i)}\right) \ln\left(-\ln \hat{F}(X_{(i)})\right) - \sum_{i=1}^{r} \ln\left(X_{(i)}\right) \sum_{i=1}^{r} \ln\left(-\ln\left(\hat{F}(X_{(i)})\right)\right)}{r\sum_{i=1}^{r} \left(\ln\left(X_{(i)}\right)\right)^2 - \left(\sum_{i=1}^{r} \ln\left(X_{(i)}\right)\right)^2}$$

and

$$\hat{\beta}_0 = \frac{1}{r}\sum_{i=1}^{r} \ln\left(-\ln \hat{F}(X_{(i)})\right) - \hat{\beta}_1 \frac{1}{r}\sum_{i=1}^{r} \ln\left(X_{(i)}\right).$$

Therefore, the LS estimates of the parameters α and λ, respectively, are given by

$$\hat{\alpha}_{LS} = -\frac{r\sum_{i=1}^{r} \ln\left(X_{(i)}\right) \ln\left(-\ln \hat{F}(X_{(i)})\right) - \sum_{i=1}^{r} \ln\left(X_{(i)}\right) sum_{i=1}^{r} \ln\left(-\ln \hat{F}(X_{(i)})\right)}{r\sum_{i=1}^{r}\left(\ln\left(X_{(i)}\right)\right)^2 - \left(\sum_{i=1}^{r} \ln\left(X_{(i)}\right)\right)^2}$$

and

$$\hat{\lambda}_{LS} = \exp\left\{\frac{1}{r}\sum_{i=1}^{r} \ln\left(-\ln\left(\hat{F}(X_{(i)})\right)\right) + \hat{\alpha}\frac{1}{r}\sum_{i=1}^{r} \ln\left(X_{(i)}\right)\right\}.$$

See also Calabria and Pulcini [17].

3. Bayes and BS estimation methods

3.1 *Assumptions on priors*

When the shape parameter α is known, the scale parameter λ has a prior conjugate gamma distribution. When both parameters are unknown, they therefore have no prior conjugate. In this case, we consider the following assumptions on α and λ:

· α and λ are independent;

· α follows from the non-informative improper prior distribution $\pi_2(\alpha)$, where:

$$\pi_2(\alpha) \propto \frac{1}{\alpha}, \quad \alpha > 0;$$

· λ follows from the gamma prior distribution with the scale parameter a and the shape parameter b i.e. $\lambda \sim \Gamma(a, b)$;

Therefore, under the above assumptions, we can take the following joint prior distribution on α and λ as

$$\pi(\alpha, \lambda) = \frac{a^b}{\alpha\Gamma(b)}\lambda^{b-1}\exp(-\alpha\lambda), \quad \alpha > 0, \quad \lambda > 0, \quad a > 0, \quad b > 0.$$

3.2 *Posterior analysis and bayes estimators*

According to the observed type-II censored random sample as well as the above prior assumptions, the joint posterior density function of α and λ can be written as

$$\pi(\alpha, \lambda | x_{(1)}, \ldots, x_{(r)}) = \frac{\pi(\alpha, \lambda) L(\alpha, \lambda | x_{(1)}, \ldots, x_{(r)})}{K}$$

where $L\ (\alpha,\ \lambda|x_{(1)},\ \ldots,\ x_{(r)})$ is defined in (3) and

$$K=\int_0^\infty\int_0^\infty \pi(\alpha,\ \lambda)L(\alpha,\ \lambda|x_{(1)},\ \ldots,\ x_{(r)})\mathrm{d}\alpha\mathrm{d}\lambda.$$

is the marginal unconditional density. So, we have

$$\pi(\alpha,\ \lambda|x_{(1)},\ \ldots,\ x_{(r)})=\frac{\alpha^{r-1}\lambda^{r+b-1}}{K}\exp\left\{-\lambda\left(\alpha+\sum_{i=1}^{r}x_{(i)}^{-\alpha}\right)\right\}\prod_{i=1}^{r}x_{(i)}^{-(\alpha+1)}(1-B)^{n-r}.$$

Now, we obtain the Bayes estimator and the generalized Bayes estimator for the unknown parameters α and λ under the SELF. As we know, the Bayesian estimator under the SELF is the posterior mean of the parameter. Therefore, the Bayes estimate of λ and the generalized Bayes estimate of α under the SELF are respectively as below:

$$\hat{\lambda}_{Bayes}=E(\lambda|x_{(1)},\ \ldots,\ x_{(r)})=\int_0^\infty\lambda\int_0^\infty \pi(\alpha,\ \lambda|x_{(1)},\ \ldots,\ x_{(r)})\mathrm{d}\alpha\mathrm{d}\lambda$$

$$=\frac{1}{K}\int_0^\infty\int_0^\infty \alpha^{r-1}\lambda^{r+b}\exp\left\{-\lambda\left(a+\sum_{i=1}^{r}x_{(i)}^{-\alpha}\right)\right\}\prod_{i=1}^{r}x_{(i)}^{-(\alpha+1)}(1-B)^{n-r}\mathrm{d}\alpha\mathrm{d}\lambda, \tag{7}$$

and

$$\hat{\alpha}_{Bayes}=E(\alpha|x_{(1)},\ \ldots,\ x_{(r)})=\int_0^\infty\alpha\int_0^\infty \pi(\alpha,\ \lambda|x_{(1)},\ \ldots,\ x_{(r)})\mathrm{d}\lambda\mathrm{d}\alpha$$

$$=\frac{1}{K}\int_0^\infty\int_0^\infty \alpha^{r}\lambda^{r+b-1}\exp\left\{-\lambda\left(a+\sum_{i=1}^{r}x_{(i)}^{-\alpha}\right)\right\}\prod_{i=1}^{r}x_{(i)}^{-(\alpha+1)}(1-B)^{n-r}\mathrm{d}\lambda\mathrm{d}\alpha, \tag{8}$$

In the next section, we use the well-known approximated method to obtain all the considered estimators.

3.3 *Lindley approximation*

We obtained the Bayes estimators of λ and α under SELF in the previous subsection. It should be noted that these estimators are the ratio of two integrals that do not have a simple closed form. Here, with the approach developed by Lindley [31], an excellent approximation is provided for the Bayesian estimators that is really easy to use.

Consider the integral ratio $R=R\ (x_1,\ \ldots,\ x_n)$ as below

$$R(x_1,\ \ldots,\ x_n)=\frac{\int_{(\theta_1,\ \theta_2)}U(\theta_1,\ \theta_2)\exp\{\ell(\theta_1,\ \theta_2)+P(\theta_1,\ \theta_2)\}\mathrm{d}(\theta_1,\ \theta_2)}{\int_{(\theta_1,\ \theta_2)}\exp\{\ell(\theta_1,\ \theta_2)+P(\theta_1,\ \theta_2)\}\mathrm{d}(\theta_1,\ \theta_2)}, \tag{9}$$

where $x_1,\ \ldots,\ x_n$ is an observed random sample from a distribution with parameters θ_1 and θ_2, $U=U(\theta_1,\ \theta_2)$ is just a function of θ_1 and θ_2, $\ell=\ell\ (\theta_1,\ \theta_2)$ is the log-likelihood function, and $P=P\ (\theta_1,\ \theta_2)=\ln\ (\pi(\theta_1,\ \theta_2))$, in which π is the prior. For notational simplicity, we define

$$\hat{A} = A(\hat{\theta}_1,\ \hat{\theta}_2),$$

$$\hat{A}_i = A_i\Big|_{\theta_1=\hat{\theta}_1,\ \theta_2=\hat{\theta}_2}$$

where

$$A_i = \frac{\partial}{\partial \theta_i} A(\theta_1,\ \theta_2),$$

$$\hat{A}_{ij} = \frac{\partial}{\partial \theta_j} A_j\Big|_{\theta_1=\hat{\theta}_1,\ \theta_2=\hat{\theta}_2},$$

and

$$\hat{A}_{ijk} = \frac{\partial}{\partial \theta_j} A_{jk}\Big|_{\theta_1=\hat{\theta}_1,\ \theta_2=\hat{\theta}_2}, \quad i,\ j,\ k = 1,\ 2 \text{ where } A_{jk} = \tfrac{\partial}{\partial j} A_k,$$

in which $(\hat{\theta}_1,\ \hat{\theta}_2)^\top$ denotes the MLE of $(\theta_1,\ \theta_2)^\top$. Also, suppose that $\hat{\sigma}^{ij}$ is the $(i,\ j)$-th element of the inverse of the observed information matrix of the log-likelihood function ℓ evaluated at $(\hat{\theta}_1,\ \hat{\theta}_2)^\top$.

Then, based on Lindley [31], an approximation for $R\ (x_1,\ \ldots,\ x_n)$, when the sample size n is sufficiently large, would be as

$$\begin{aligned} R(x_1,\ \ldots,\ x_n) =& \hat{U} + \frac{1}{2}[(\hat{U}_{11} + 2\hat{U}_1\hat{P}_1)\hat{\sigma}^{11} + (\hat{U}_{12} + 2\hat{U}_1\hat{P}_2)\hat{\sigma}^{12} + (\hat{U}_{21} + 2\hat{U}_2\hat{P}_1)\hat{\sigma}^{21} + (\hat{U}_{22} + 2\hat{U}_2\hat{P}_2)\hat{\sigma}^{22}] \\ &+ \frac{1}{2}[(\hat{U}_1\hat{\sigma}^{11} + \hat{U}_2\hat{\sigma}^{12})(\hat{\ell}_{111}\hat{\sigma}^{11} + \hat{\ell}_{121}\hat{\sigma}^{12} + \hat{\ell}_{211}\hat{\sigma}^{21} + \hat{\ell}_{221}\hat{\sigma}^{22}) \\ &+ (\hat{U}_1\hat{\sigma}^{21} + \hat{U}_2\hat{\sigma}^{22})(\hat{\ell}_{112}\hat{\sigma}^{11} + \hat{\ell}_{122}\hat{\sigma}^{12} + \hat{\ell}_{212}\hat{\sigma}^{21} + \hat{\ell}_{222}\hat{\sigma}^{22})]. \end{aligned} \tag{10}$$

Now, we can obtain the Lindley approximation for the Bayes estimators by matching the equations (7) and (8) with the integral ratio in (9) and then using the equation (10). The common items that we need in (10) to approximate the equations (7) and (8) are as follows

$$\hat{\ell}_{\alpha\alpha}=-\left\{\frac{r}{\hat{\alpha}^2}+\hat{\lambda}\sum_{i=1}^{r}\left(\ln\left(x_{(i)}\right)\right)^2 x_{(i)}^{-\hat{\alpha}}+(n-r)\left(\ln\left(x_{(r)}\right)\right)^2 C\right\},$$

$$\hat{\ell}_{\alpha\lambda}=\hat{\ell}_{\lambda\alpha}=\sum_{i=1}^{r}\left(\ln\left(x_{(i)}\right)\right)x_{(i)}^{-\hat{\alpha}}+\frac{(n-r)\ln\left(x_{(r)}\right)}{\hat{\lambda}}C,$$

$$\hat{\ell}_{\lambda\lambda}=-\frac{1}{\hat{\lambda}^2}(r+(n-r)D),$$

$$\hat{\ell}_{\alpha\alpha\alpha}=\frac{2r}{\hat{\alpha}^3}+\hat{\lambda}\sum_{i=1}^{r}\left(\ln\left(x_{(i)}\right)\right)^3 x_{(i)}^{-\hat{\alpha}}+(n-r)\left(\ln\left(x_{(r)}\right)\right)^3 E,$$

$$\hat{\ell}_{\alpha\alpha\lambda}=\hat{\ell}_{\alpha\lambda\alpha}=\hat{\ell}_{\lambda\alpha\alpha}=-\sum_{i=1}^{r}\left(\ln\left(x_{(i)}\right)\right)^2 x_{(i)}^{-\hat{\alpha}}-\frac{(n-r)\left(\ln\left(x_{(r)}\right)\right)^2}{\hat{\lambda}}E,$$

$$\hat{\ell}_{\lambda\lambda\alpha}=\hat{\ell}_{\lambda\alpha\lambda}=\hat{\ell}_{\alpha\lambda\lambda}=\frac{(n-r)\ln\left(x_{(r)}\right)}{\hat{\lambda}^2}F,$$

$$\hat{\ell}_{\lambda\lambda\lambda}=\frac{1}{\hat{\lambda}^3}(2r+(n-r)(2D-F)),$$

$$\hat{P}_{\alpha}=-\frac{1}{\hat{\alpha}},$$

$$\hat{P}_{\lambda}=\frac{b-1}{\hat{\lambda}}-a,$$

where

$$\hat{C}=C(\hat{\alpha},\ \hat{\lambda})=\frac{\hat{B}\ln(\hat{B})(\ln(\hat{B}e)-\hat{B})}{(1-\hat{B})^2},$$

$$\hat{D}=D(\hat{\alpha},\ \hat{\lambda})=\frac{\hat{B}(\ln(\hat{B}))^2}{(1-\hat{B})^2},$$

$$\hat{E}=E(\hat{\alpha},\ \hat{\lambda})=\frac{\hat{B}\ln(\hat{B})\left\{(\ln(\hat{B}e)-\hat{B})(\ln(\hat{B}e)-\hat{B}+\hat{B}\ln(\hat{B}))+(1-\hat{B})^2\ln(\hat{B})\right\}}{(1-\hat{B})^3},$$

$$\hat{F}=F(\hat{\alpha},\ \hat{\lambda})=\frac{\hat{B}(\ln(\hat{B}))^2\left\{2(1-\hat{B})+\ln(\hat{B})(1+\hat{B})\right\}}{(1-\hat{B})^3},$$

$$\hat{B} = B(\hat{\alpha},\, \hat{\lambda}) = \exp\{-\hat{\lambda} x_{(r)}^{-\hat{\alpha}}\}.$$

To approximate the equation (7), we have to implement the following equations in (10):

$$\hat{U} = \hat{\lambda}, \quad \hat{U}_{\lambda} = 1, \quad \hat{U}_{\alpha} = \hat{U}_{\alpha\alpha} = \hat{U}_{\alpha\lambda} = \hat{U}_{\lambda\alpha} = \hat{U}_{\lambda\lambda} = 0.$$

Also, to approximate the equation (8), we have to implement the following equations in (10):

$$\hat{U} = \hat{\alpha}, \quad \hat{U}_{\alpha} = 1, \quad \hat{U}_{\lambda} = \hat{U}_{\alpha\alpha} = \hat{U}_{\alpha\lambda} = \hat{U}_{\lambda\alpha} = \hat{U}_{\lambda\lambda} = 0.$$

Therefore, the Bayes estimation of λ and the generalized Bayes estimation of α under SELF are approximated, respectively, as follows

$$\begin{aligned}\hat{\lambda}_{Bayes} =& \hat{\lambda} + \left[\hat{\sigma}^{\lambda\lambda}\left(\frac{b-1}{\hat{\lambda}} - a\right) - \hat{\sigma}^{\alpha\lambda}\frac{1}{\hat{\alpha}}\right] \\ &+ \frac{1}{2}\left[\hat{\sigma}^{\alpha\lambda}(\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}^{\alpha\alpha} + 2\hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}^{\alpha\lambda} + \hat{\ell}_{\lambda\lambda\alpha}\hat{\sigma}^{\lambda\lambda}) + \hat{\sigma}^{\lambda\lambda}(\hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}^{\alpha\alpha} + 2\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}^{\alpha\lambda} + \hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}^{\lambda\lambda})\right].\end{aligned}$$

and

$$\begin{aligned}\hat{\alpha}_{Bayes} =& \hat{\alpha} + \left[\hat{\sigma}^{\alpha\lambda}\left(\frac{b-1}{\hat{\lambda}} - a\right) - \hat{\sigma}^{\alpha\alpha}\frac{1}{\hat{\alpha}}\right] \\ &+ \frac{1}{2}\left[\hat{\sigma}^{\alpha\alpha}(\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}^{\alpha\alpha} + 2\hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}^{\alpha\lambda} + \hat{\ell}_{\lambda\lambda\alpha}\hat{\sigma}^{\lambda\lambda}) + \hat{\sigma}^{\lambda\lambda}(\hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}^{\alpha\alpha} + 2\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}^{\alpha\lambda} + \hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}^{\lambda\lambda})\right].\end{aligned}$$

3.4 *BS estimation method*

Shrinkage estimation approach is to find an estimator through optimization of any usual estimator with a desirable criterion measure like mean squared error (MSE). Let $\hat{\alpha}_{Bayes}$ and $\hat{\lambda}_{Bayes}$ be the usual Bayes estimators, respectively, for α and λ. Then, following Thompson [33] and some related works Dey [9]; Singh [34]; Sing et al. [35]; Vishwakarma and Gupta [36], the Bayes shrinkage estimators of α and λ, respectively, are proposed as follows:

$$\hat{\alpha}_{BS} = w_1\hat{\alpha}_{Bayes} + (1-w_1)\alpha_0,$$

and

$$\hat{\lambda}_{BS} = w_2\hat{\lambda}_{Bayes} + (1-w_2)\lambda_0,$$

where $0 \leqslant w_i \leqslant 1,\ i = 1,\ 2$ are called the shrinkage coefficients and, α_0 and λ_0 are our guess for the parameters α and λ.

Note: When there is no guarantee that the guessed values α_0 and λ_0 are close to the true values of α and λ, we may conduct the following hypothesis test:

$H_0 : \alpha = \alpha_0$ and $\lambda = \lambda_0$ versus $H_1 : \alpha \neq \alpha_0$ or $\lambda \neq \lambda_0$.

Here, the score test is proposed for testing the null hypothesis (Rao [37]), H_0. In this case, the score test statistic is given by

$$SC_0 = (\ell_{\alpha_0},\ \ell_{\lambda_0})^T \mathscr{J}^{-1}(\alpha_0,\ \lambda_0)(\ell_{\alpha_0},\ \ell_{\lambda_0}) \sim \chi_2^2,$$

where $\mathscr{J}(\alpha_0,\ \lambda_0)$ is the observed information matrix of the log-likelihood function ℓ evaluated at $(\alpha_0,\ \lambda_0)$,

$$\ell_{\alpha_0} = \frac{\partial \ell(\alpha,\ \lambda)}{\alpha}\Big|_{\alpha=\alpha_0,\ \lambda=\lambda_0},$$

and

$$\ell_{\lambda_0} = \frac{\partial \ell(\alpha,\ \lambda)}{\lambda}\Big|_{\alpha=\alpha_0,\ \lambda=\lambda_0}.$$

We reject H_0, at the significance level η, if:

$$SC_0 > \chi^2_{2,\ 1-\eta/2} \text{ or } SC_0 < \chi^2_{2,\ \eta/2}.$$

In application, the guessed values are chosen randomly from a uniform distribution around the ML estimates, and the score test is done.

By using MSE, the shrinkage coefficients are equal to

$$w_1 = \frac{(\alpha - \alpha_0)(\kappa_{\alpha,\ 1} - \alpha_0)}{\kappa_{\alpha,\ 2} - 2\alpha_0\kappa_{\alpha,\ 1} + \alpha_0^2},$$

and

$$w_2 = \frac{(\lambda - \lambda_0)(\kappa_{\lambda,\ 1} - \lambda_0)}{\kappa_{\lambda,\ 2} - 2\lambda_0\kappa_{\lambda,\ 1} + \lambda_0^2},$$

where $\kappa_{\theta,\ r} = E(\hat{\theta}_B)^r$. Since w_1 and w_2 depend on the unknown parameters α and λ, so we replace them by $\hat{\alpha}_B$ and $\hat{\lambda}_B$, respectively, and we get

$$\hat{w}_1 = \frac{(\hat{\alpha}_B - \alpha_0)(\hat{\kappa}_{\alpha,\ 1} - \alpha_0)}{\hat{\kappa}_{\alpha,\ 2} - 2\alpha_0\hat{\kappa}_{\alpha,\ 1} + \alpha_0^2},$$

and

$$\hat{w}_2 = \frac{(\hat{\lambda} - \lambda_0)(\hat{\kappa}_{\lambda,\ 1} - \lambda_0)}{\hat{\kappa}_{\lambda,\ 2} - 2\lambda_0\hat{\kappa}_{\lambda,\ 1} + \lambda_0^2}.$$

$\hat{\kappa}_{\theta,\ r}$ can be calculated numerically using, e.g., the MCMC procedure.

4. Monte carlo simulation study

Here, a Monte Carlo simulation was presented to evaluate the empirical performance of all estimates in finite sample sizes in terms of the bias and MSE criteria. Moreover, the R 4.2.3 software (R Core Team, [38]) was used to conduct all programs.

We took the true parameter values for the IW distribution as $(\alpha,\ \lambda) = (0.5,\ 0.25),\ (1.5,\ 0.5)$ and $(3,\ 0.75)$ based on the values used in Figure 1. For any combination of parameters, 1,000 samples with sample sizes $n = 100,\ 300$ and 500 were first generated and then we produced type-II censored data with different amounts of r. Also, the following algorithm was used to produce a random sample data with size n from IW distribution:

Step 1: Determine the values of $n,\ \alpha$ and λ.

Step 2: Generate a random sample $u_1,\ \ldots,\ u_n$ from $U \sim Uniform\ (0,\ 1)$.

Step 3: Compute the i-th IW random observation using the below formula:

$$x_i = \left(-\frac{1}{\lambda}\ln(u_i)\right)^{-\frac{1}{\alpha}}.$$

In each iteration, the ML, LS, Bayes and BS estimates of the parameters were first derived and then, the bias and MSE of all iterations were calculated, respectively, through the following formulae:

$$Bias = \frac{1}{1{,}000}\sum_{j=1}^{1{,}000}(\hat{\theta}_j - \theta),$$

and

$$x_i = \left(-\frac{1}{\lambda}\ln(u_i)\right)^{-\frac{1}{\alpha}}.$$

$$MSE = \frac{1}{1{,}000}\sum_{j=1}^{1{,}000}(\hat{\theta}_j - \theta)^2,$$

where θ is the true value of parameter and $\hat{\theta}_j$ is the estimate of θ in j-th iteration. It should be noted that, in each iteration, for the BS estimate, the guessed values are chosen randomly from a uniform distribution around the true values, and the score test is conducted. If H_0 is rejected, the process is repeated.

Tables 1-3 present the simulation results. From these results, it can be clearly seen that the biases and MSEs for all the estimators decrease in almost all cases when the sample size n and also effective sample size r increase. It can also be seen that for sufficiently large effective sample sizes (i.e., $r \geqslant 100$ in any sample size), all estimation procedures have almost the same performance. However, when the effective sample size r is too small with respect to the sample size n, the BS estimation method has the best performance.

Compared to other estimation methods, the LS estimation method has the worst performance due to having large biases and MSEs in almost all settings. The performance of Bayes estimates in different settings in terms of the bias and MSE criteria is slightly better than ML estimates. Furthermore, as expected, the bias and MSE of all estimation methods increase naturally when the true values of parameters increase. However, in this case, the bias and MSE of BS estimators are much better than those of the other estimators. This shows that knowing a suitable guessed point for the parameters can effectively improve the performance of the BS estimator in any situation.

Based on simulation results, it is proposed that the use of Bayes and the BS estimation methods in consideration of bias and MSE in general, especially when the effective sample size, r, in type II censoring schemes, is too small with respect to the original sample size, n.

Table 1. The bias and MSE's of the estimators for $\alpha = 0.5$, $\lambda = 0.25$, $a = 1.1$, $b = 0.2$ and different values of n and r

Parameter			$\alpha = 0.5$				$\lambda = 0.25$			
n	r	Criterion	ML	LS	Bayes	BS	ML	LS	Bayes	BS
100	20	Bias	0.04513	-0.02748	0.03696	0.02363	-0.01759	0.06354	0.00569	-0.00529
		MSE	0.01340	0.01025	0.01152	0.00571	0.00830	0.01801	0.00790	0.00241
	30	Bias	0.01712	-0.03029	0.01280	0.00402	-0.01554	0.02797	-0.00202	-0.00278
		MSE	0.00626	0.00709	0.00578	0.00141	0.00606	0.00863	0.00569	0.00120
	50	Bias	0.01515	-0.01358	0.01300	0.00150	-0.00478	0.01924	0.00154	0.00576
		MSE	0.00344	0.00325	0.00329	0.00131	0.00307	0.00394	0.00299	0.00122
	100	Bias	0.00524	-0.02677	0.00485	0.00280	-0.00300	0.01713	0.00208	0.00437
		MSE	0.00151	0.00327	0.00149	0.00128	0.00209	0.00276	0.00205	0.00183
300	30	Bias	0.02695	-0.03038	0.02211	0.01784	-0.01622	0.05228	0.00294	-0.00805
		MSE	0.00700	0.00753	0.00626	0.00343	0.00825	0.01644	0.00793	0.00297
	60	Bias	0.01357	-0.01479	0.01143	0.00284	-0.00521	0.02929	0.00280	0.00079
		MSE	0.00288	0.00311	0.00274	0.00084	0.00294	0.00500	0.00289	0.00067
	150	Bias	0.00493	-0.00650	0.00426	0.00315	-0.00312	0.00559	-0.00098	-0.00010
		MSE	0.00094	0.00109	0.00092	0.00083	0.00111	0.00130	0.00109	0.00099
	300	Bias	0.00141	-0.01362	0.00128	0.00129	0.00040	0.00751	0.00111	0.00106
		MSE	0.00058	0.00116	0.00058	0.00058	0.00063	0.00083	0.00063	0.00063
500	25	Bias	0.03719	-0.0302	0.02893	0.02801	-0.01876	0.077683	0.00100	0.00017
		MSE	0.01011	0.00950	0.00845	0.00575	0.01033	0.02798	0.01010	0.00549
	50	Bias	0.01807	-0.01840	0.01459	0.00838	-0.00965	0.04269	0.00305	-0.00274
		MSE	0.00401	0.00430	0.00369	0.00141	0.00443	0.00897	0.00432	0.00113
	100	Bias	0.00757	-0.01220	0.00600	0.00024	-0.00719	0.01665	-0.00195	0.00205
		MSE	0.00169	0.00185	0.00163	0.00054	0.00185	0.00288	0.00179	0.00065
	250	Bias	0.00216	-0.00520	0.00166	0.00169	-0.00181	0.00358	-0.00040	-0.00055
		MSE	0.00053	0.00059	0.00053	0.00052	0.00063	0.00071	0.00062	0.00062
	500	Bias	0.00323	-0.00634	0.00309	0.00309	0.00019	0.00652	0.00070	0.00068
		MSE	0.00034	0.00062	0.00034	0.00034	0.00037	0.00048	0.00037	0.00037

Table 2. The bias and MSE's of the estimators for $\alpha = 0.15,\ \lambda = 0.5,\ a = 1.3,\ b = 0.4$ and different values of n and r

Parameter			$\alpha = 0.5$				$\lambda = 0.25$			
n	r	Criterion	ML	LS	Bayes	BS	ML	LS	Bayes	BS
100	20	Bias	0.14274	-0.07481	0.12788	0.02786	-0.03256	0.07059	-0.01198	-0.00041
		MSE	0.11782	0.09585	0.10437	0.02715	0.01961	0.03123	0.01696	0.00469
	30	Bias	0.09551	-0.06019	0.08716	0.00819	-0.01904	0.03301	-0.00866	0.00939
		MSE	0.06781	0.05539	0.06320	0.01886	0.01273	0.01543	0.01163	0.00537
	50	Bias	0.05097	-0.03747	0.04731	0.03034	-0.00710	0.02013	-0.00371	0.00255
		MSE	0.03355	0.03117	0.03250	0.02622	0.00745	0.00824	0.00707	0.00594
	100	Bias	0.01783	-0.07405	0.01790	0.01702	-0.00264	0.00927	-0.00302	-0.00312
		MSE	0.01464	0.02624	0.01454	0.01438	0.00387	0.00384	0.00374	0.003751
300	30	Bias	0.08229	-0.09045	0.07553	0.03114	-0.03339	0.06710	-0.01523	-0.00711
		MSE	0.06104	0.06249	0.05588	0.02316	0.01706	0.02888	0.01509	0.00573
	60	Bias	0.04101	-0.04991	0.03778	0.00665	-0.01121	0.03217	-0.00456	0.00684
		MSE	0.02641	0.02956	0.02547	0.01217	0.00640	0.00951	0.00608	0.00323
	150	Bias	0.02166	-0.01088	0.02054	0.02058	-0.00264	0.00900	-0.00152	-0.00163
		MSE	0.00950	0.00940	0.00939	0.00940	0.00245	0.00279	0.00241	0.00241
	300	Bias	0.00517	-0.03570	0.00520	0.00518	-0.00206	0.00391	-0.00218	-0.00219
		MSE	0.00434	0.00887	0.00433	0.00433	0.00147	0.00151	0.00145	0.00145
500	25	Bias	0.10237	-0.10589	0.08565	0.05907	-0.03381	0.10070	-0.00309	-0.01057
		MSE	0.08386	0.08768	0.07043	0.03519	0.02432	0.04897	0.02110	0.00866
	50	Bias	0.03835	-0.07013	0.03199	0.00040	-0.02118	0.05212	-0.00841	-0.00127
		MSE	0.03136	0.03883	0.02935	0.00778	0.01029	0.01719	0.00948	0.00321
	100	Bias	0.02751	-0.03133	0.02428	0.01343	-0.00534	0.02417	-0.00060	0.00454
		MSE	0.01654	0.01827	0.01605	0.01264	0.00421	0.00582	0.00409	0.00328
	250	Bias	0.00604	-0.01524	0.00499	0.00501	-0.00099	0.00532	-0.00005	-0.00009
		MSE	0.00553	0.00599	0.00549	0.00549	0.00129	0.00143	0.00128	0.00128
	500	Bias	0.00569	-0.02349	0.00551	0.00550	0.00028	0.00331	0.00038	0.00037
		MSE	0.00268	0.00548	0.00267	0.00267	0.00087	0.00088	0.00087	0.00087

Table 3. The bias and MSE's of the estimators for $\alpha = 3,\ \lambda = 0.75,\ a = 1.4,\ b = 0.6$ and different values of n and r

Parameter			$\alpha = 0.5$				$\lambda = 0.25$			
n	r	Criterion	ML	LS	Bayes	BS	ML	LS	Bayes	BS
100	20	Bias	0.26671	-0.16850	0.25825	0.06921	-0.04580	0.06735	-0.03476	0.00038
		MSE	0.51586	0.42292	0.47899	0.17543	0.02778	0.03540	0.02401	0.00912
	30	Bias	0.16159	-0.12967	0.15683	0.07601	-0.02960	0.03133	-0.02619	-0.00319
		MSE	0.24458	0.22040	0.23621	0.15553	0.01758	0.01937	0.01609	0.01051
	50	Bias	0.09637	-0.08295	0.09381	0.08931	-0.00593	0.01495	-0.00799	-0.00804
		MSE	0.11777	0.11255	0.11616	0.11323	0.01150	0.01149	0.01089	0.01092
	100	Bias	0.05008	-0.13413	0.05208	0.05172	0.00119	0.00086	-0.00353	-0.00352
		MSE	0.06096	0.11415	0.06111	0.06106	0.00805	0.00729	0.00770	0.00771
300	30	Bias	0.19792	-0.13918	0.20133	0.09899	-0.03742	0.08820	-0.02644	0.00193
		MSE	0.28224	0.27701	0.26826	0.12359	0.02520	0.04209	0.02246	0.00918
	60	Bias	0.10371	-0.07694	0.10320	0.07628	-0.01289	0.03282	-0.00986	-0.00468
		MSE	0.11125	0.11166	0.10913	0.09042	0.00905	0.01188	0.00864	0.00769
	150	Bias	0.04238	-0.02646	0.04162	0.04150	-0.00542	0.00288	-0.00609	-0.00611
		MSE	0.03819	0.03955	0.03802	0.03801	0.00322	0.00319	0.00317	0.00318
	300	Bias	0.02121	-0.06606	0.02184	0.02180	0.00490	0.00420	0.00332	0.00332
		MSE	0.01974	0.03822	0.01976	0.01976	0.00220	0.00211	0.00216	0.00216
500	25	Bias	0.23145	-0.18722	0.21921	0.10440	-0.03611	0.12667	-0.01302	-0.00790
		MSE	0.38736	0.35968	0.33489	0.15454	0.03423	0.06635	0.02891	0.01208
	50	Bias	0.08753	-0.12816	0.08402	0.01649	-0.01296	0.06355	-0.00426	0.01137
		MSE	0.13187	0.15034	0.12590	0.07117	0.01426	0.02480	0.01334	0.00683
	100	Bias	0.05293	-0.07132	0.04996	0.04937	-0.00738	0.02273	-0.00454	-0.00478
		MSE	0.05790	0.06554	0.05681	0.05643	0.00571	0.00770	0.00554	0.00554
	250	Bias	0.01800	-0.02651	0.01672	0.01668	-0.00366	0.00172	-0.00367	-0.00368
		MSE	0.02064	0.02228	0.02053	0.02053	0.00190	0.00198	0.00188	0.00188
	500	Bias	0.01902	-0.04022	0.01900	0.01899	0.00099	0.00018	0.00036	0.00036
		MSE	0.01144	0.02131	0.01143	0.01143	0.00159	0.00152	0.00158	0.00158

5. Guinea pigs data application

In the following, the methods presented in the previous sections are examined through a real data analysis. The data set is a particular set with 72 observations involving the survival times (in days) of guinea pigs after injection with tuberculosis bacillus, studied by Bjerkedal [39]. Indeed, guinea pigs are used as a model to study human tuberculosis, since they are strongly susceptible to this type of bacillus. To see the data, one can refer to Kundu and Howlader [7].

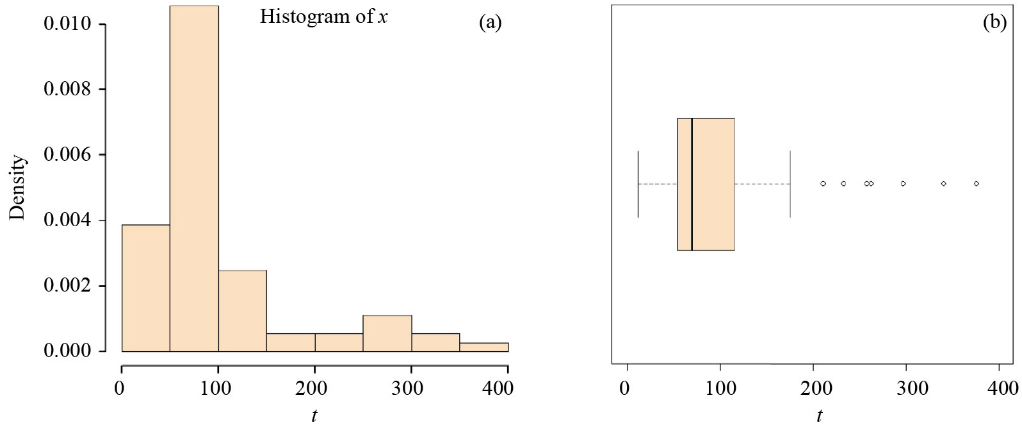

Figure 2. (a) Weibull distribution HF; (b) IW distribution HF for different amounts of (α, λ)

Table 4. Summary statistics for the survival times of guinea pigs injected with tuberculosis bacillus

Min	Max	Mean	Median	SD	Skewness	Kurtosis
12	376	99.82	70	81.12	1.83	2.89

Figure 2 shows the histogram and the boxplot for the survival times of the guinea pigs data that indicate an asymmetric and leptokurtic distribution for the data. Also, Table 4 provides descriptive statistics for the survival times (in days) of guinea pigs injected with tuberculosis bacillus. As seen, the data has strictly positive sample skewness and kurtosis, and also there is a very long distance between the sample mean and median. The results of Figure 2 and Table 4 verify using a distribution such as IW to fit the data.

Additionally, we plotted $\ln(t)$ against $\ln\left(-\ln\left(\hat{F}(t)\right)\right)$ in Figure 3, where t is the survival time and $\hat{F}(t)$ is its empirical cumulative distribution function. This plot shows a strictly negative relationship between $\ln(t)$ and $\ln\left(-\ln\left(\hat{F}(t)\right)\right)$ and this is another reason to use IW distribution for the data. Another way to propose a suitable distribution for a data is the shape of its HF. Based on Kundu and Howlader [7], the empirical HF of the guinea pigs data by applying TTT plot indicates a bathtub (unimodal) hazard rate. Hence, using the IW distribution could be a reasonable suggestion to analyze the data (See Figure 4).

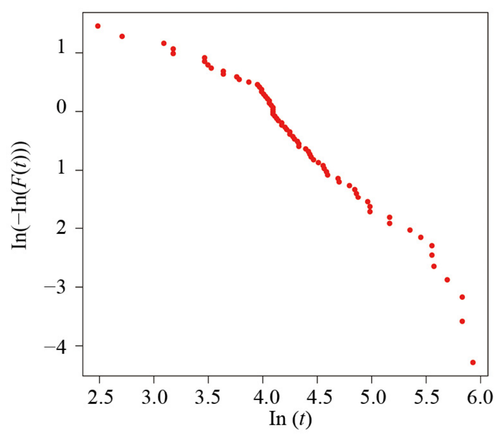

Figure 3. The scatter plot of $\ln(t)$ against $\ln(-\ln(\hat{F}(t)))$

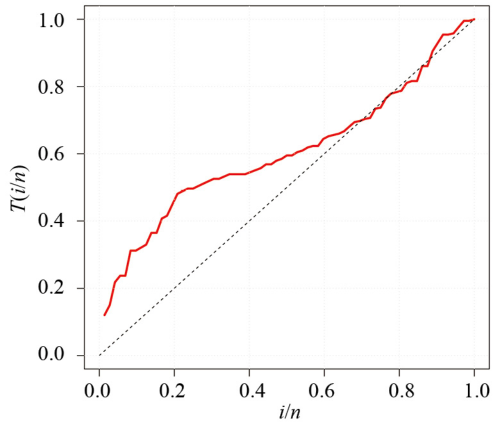

Figure 4. Scaled TTT plot for the guinea pig data

According to the above results, IW distribution was first fitted to the data and then, based on type-II censored schemes on the data, involving different values of effective sample sizes, and also the complete data, i.e. $r = n = 72$, the values of ML, LS, Bayes and BS estimators were derived. Table 5 shows the results along with the average percent of changes (APC) of the estimated parameters, where

$$APC = \left[\frac{1}{2}\left(\left|\frac{\hat{\alpha}_r - \hat{\alpha}_n}{\hat{\alpha}_n}\right| + \left|\frac{\hat{\lambda}_r - \hat{\lambda}_n}{\hat{\lambda}_n}\right|\right)\right] \times 100,$$

in which r and n indices denote the sample sizes for any estimator.

As expected, when the value of r increases, the estimated parameters are closed to their corresponding for complete data. The APC of BS method is lower than the other method in all of schemes. The ML and Bayes estimates are approximately the same and the LS estimate is the worst. Figure 5 presents the histogram of the complete data and the estimated density curves created by four methods with $r = 10$. As seen, the BS estimated density curve is superior to the other methods in terms of model fitting.

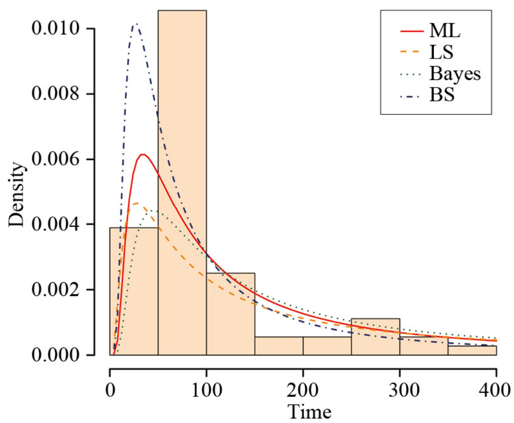

Figure 5. The histogram of the complete data and the estimated density curves created by four methods with $r = 10$

Table 5. Estimation methods of α and λ based on type-II censored schemes for the guinea pig data

r	method	α	λ	APC
10	ML	0.850	43.655	62.27
	LS	0.642	21.372	78.70
	Bayes	0.821	50.361	61.02
	BS	0.989	50.655	54.10
20	ML	0.918	56.992	57.51
	LS	0.760	30.727	74.37
	Bayes	0.888	52.107	58.26
	BS	1.038	56.490	51.21
30	ML	1.171	123.467	36.88
	LS	0.933	53.482	67.39
	Bayes	1.125	120.149	37.22
	BS	1.130	120.964	36.17
40	ML	1.275	174.503	24.21
	LS	1.119	100.136	58.22
	Bayes	1.234	169.336	24.25
	BS	1.225	171.980	23.41
50	ML	1.342	218.777	14.05
	LS	1.300	190.695	45.94
	Bayes	1.308	214.608	13.28
	BS	1.292	216.270	12.94
60	ML	1.369	240.803	13.28
	LS	1.442	323.869	31.71
	Bayes	1.334	231.471	9.25
	BS	1.319	238.199	7.96
72	ML	1.415	283.853	0.00
	LS	1.628	675.391	0.00
	Bayes	1.379	273.029	0.00
	BS	1.345	277.012	0.00

6. Discussion and conclusions

In many practical studies in lifetime analysis, the empirical HF may not be monotonic. Consequently, using a flexible distribution with a non-monotonic HF, such as the IW distribution, has been addressed by many researchers in lifetime data analysis, particularly in the context of various censoring schemes. Since there are different forms of this distributions, several methods have been proposed for estimating parameters. Finding an appropriate estimation method in a special situation with high efficiency, especially when the data are subject to types of censoring, is an outstanding work for statisticians. The ML, LS methods are the most popular procedures which can be found in the literature for estimating the IW parameters in complete and censored data. In situations where there are priors and also some guessed points for the IW parameters, using the Bayesian and BS approaches can be a better choice compared with the classical methods.

The primary objective of our study was to introduce Bayesian estimation method for the IW parameters in the presence of type-II censored data and then present a method derived from Bayesian estimates, named by BS estimation method which could potentially yield superior performance. First of all, ML and LS methods were introduced for comparative purposes. Then Bayesian method was presented and the parameter estimates were derived. Due to the complexity of Bayesian estimators' computations, a simpler but approximated approach called the Lindley approach was suggested. Furthermore, the BS method was proposed to estimate the IW parameters in the presence of any guessed point based on the Bayesian estimators. Also, a score test for testing our guessed values for the parameters as well as a way to calculate the shrinkage coefficients was suggested.

Our investigations on ML, LS and Bayesian methods verified the previous results in this field. The simulation results demonstrated the consistency, and hence the efficiency, of all estimates for the type-II censored data. Notably, The BS estimators outperformed other estimators under type-II censored data, especially with a small effective sample size. Additionally, through a real data set, it was noticed that the average percent of changes of the estimates in BS method is lower than the others in all schemes. Moreover, the estimated density curves by all methods were drawn under a small effective sample size and it was observed that BS estimated density curve is more consistent with the complete data than the other curves. These results in real data analysis also verify the simulation results.

Extending this study under types of censoring data will be our main goal in near future. Moreover, applying the BS estimation method to other statistical models is proposed for future research.

Data availability statement

The codes are available from the authors upon request.

Conflicts of interest

The authors declare no conflict of interest.

References

[1] Johnson NL, Kotz S, Balakrishnan N. *Continuous Univariate Distributions, vol. 2*. USA: John Wiley and Sons; 1995.

[2] Murthy D. *Weibull Models*. USA: John Wiley and Sons; 2004.

[3] Kundu D. Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. *Technometrics*. 2008; 50(2): 144-154.

[4] Langlands AO, Pocock SJ, Kerr GR, Gore SM. Long-term survival of patients with breast cancer: A study of the curability of the disease. *The British Medical Journal*. 1979; 2(6200): 1247-1251.

[5] Bennett S. Log-logistic regression models for survival data. *Journal of the Royal Statistical Society Series C: Applied Statistics*. 1983; 32(2): 165-171.

[6] Prentice RL. Exponential survivals with censoring and explanatory variables. *Biometrika*. 1973; 60(2): 279-288.

[7] Kundu D, Howlader H. Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data. *Computational Statistics and Data Analysis*. 2010; 54(6): 1547-1558.

[8] Keller A, Goblin M, Farnworth N. Reliability analysis of commercial vehicle engines. *Reliability Engineering*. 1985; 10(1): 15-25.

[9] Dey S, Dey T, Maiti SS. Bayes shrinkage estimation of the parameter of Rayleigh distribution for progressive type-II censored data. *Austrian Journal of Statistics*. 2015; 44(4): 3-15.

[10] Jia X, Nadarajah S, Guo B. Exact inference on Weibull parameters with multiply type-I censored data. *IEEE Transactions on Reliability*. 2018; 67(2): 432-445.

[11] Dey S, Singh S, Tripathi YM, Asgharzadeh A. Estimation and prediction for a progressively censored generalized inverted exponential distribution. *Statistical Methodology*. 2016; 32(4): 185-202.

[12] Asar Y, Arabi Belaghi R. Estimation in Weibull distribution under progressively type-I hybrid censored data. *REVSTAT-Statistical Journal*. 2023; 20(5): 563-586.

[13] Muenz LR, Green SB. Time savings in censored life testing. *Journal of the Royal Statistical Society Series B: Statistical Methodology*. 1977; 39(2): 269-275.

[14] Zheng G, Park S. A note on time savings in censored life testing. *Journal of Statistical Planning and Inference*. 2004; 124(2): 289-300.

[15] Epstein B. Truncated life tests in the exponential case. *The Annals of Mathematical Statistics*. 1954; 25(3): 555-564.

[16] Balakrishnan N, Aggarwala R. *Progressive Censoring: Theory, Methods, and Applications*. Basel, Switzerland: Birkhäuser; 2012.

[17] Calabria R, Pulcini G. On the maximum likelihood and least-squares estimation in the inverse Weibull distribution. *Statistica Applicata*. 1990; 2(1): 53-66.

[18] Yahgmaei F, Babanezhad M, Moghadam OS. Bayesian estimation of the scale parameter of inverse Weibull distribution under the asymmetric loss functions. *Journal of Probability and Statistics*. 2013; 2013(1): 890914.

[19] Sultan K, Alsadat N, Kundu D. Bayesian and maximum likelihood estimations of the inverse Weibull parameters under progressive type-II censoring. *Journal of Statistical Computation and Simulation*. 2014; 84(10): 2248-2265.

[20] Kazemi M, Azizpoor M. Estimation of the inverse Weibull distribution parameters under type-I hybrid censoring. *Austrian Journal of Statistics*. 2021; 50(5): 38-51.

[21] Delavari M, Deiri E, Khodadadi Z, Zare K. A Bayesian shrinkage estimation in inverse Weibull distribution type-II censored data using prior point information. *Life Cycle Reliability and Safety Engineering*. 2022; 11(1): 1-10.

[22] Prakash G, Singh D. A Bayesian shrinkage approach in Weibull type-II censored data using prior point information. *REVSTAT-Statistical Journal*. 2009; 7(2): 171-187.

[23] Naghizadeh Qomi M. Bayesian shrinkage estimation based on Rayleigh type-II censored data. *Communications in Statistics-Theory and Methods*. 2017; 46(19): 9859-9868.

[24] Naghizadeh Qomi M, Dey S, Fathollahi M. Performance of a class of bayes shrinkage estimators based on rayleigh record data under reflected gamma loss function. *Journal of The Iranian Statistical Society*. 2022; 18(2): 155-172.

[25] Yaghoubzadeh Shahrestani S, Zarei R, Malekzadeh P. E-Bayesian estimation of parameters of inverse Weibull distribution based on a unified hybrid censoring scheme. *International Journal of Nonlinear Analysis and Applications*. 2021; 12(1): 461-471.

[26] Okasha HM, Basheer AM, Lio Y. The e-bayesian methods for the inverse Weibull distribution rate parameter based on two types of error loss functions. *Mathematics*. 2022; 10(24): 4826.

[27] Singh SK, Singh U, Sharma VK. Bayesian analysis for type-II hybrid censored sample from inverse Weibull distribution. *International Journal of System Assurance Engineering and Management*. 2013; 4: 241-248. Available from: https://doi.org/10.1007/s13198-013-0172-7.

[28] Ateya SF. Estimation under inverse Weibull distribution based on Balakrishnan's unified hybrid censored scheme. *Communications in Statistics-Simulation and Computation*. 2017; 46(5): 3645-3666.

[29] Alam FMA, Nassar M. On entropy estimation of inverse Weibull distribution under improved adaptive progressively type-II censoring with applications. *Axioms*. 2023; 12(8): 751.

[30] Ren H, Hu X. Estimation for inverse Weibull distribution under progressive type-II censoring scheme. *AIMS Mathematics*. 2023; 8(10): 22808-22829.

[31] Lindley DV. Approximate bayesian methods. *Statistics and Operations Research Transactions*. 1980; 31: 223-245. Available from: https://doi.org/10.1007/BF02888353.

[32] Arnold BC, Balakrishnan N, Nagaraja HN. *A First Course in Order Statistics*. USA: Society for Industrial and Applied Mathematics; 2008.

[33] Thompson JR. Some shrinkage techniques for estimating the mean. *Journal of the American Statistical Association*. 1968; 63(321): 113-122.

[34] Singh J, Bhatkulikar SG. Shrunken estimation in Weibull distribution. *Sankhyā: The Indian Journal of Statistics, Series B*. 1978; 39(4): 382-393.

[35] Singh HP, Saxena S, Joshi H. A family of shrinkage estimators for Weibull shape parameter in censored sampling. *Statistical Papers*. 2008; 49(3): 513-529.

[36] Vishwakarma GK, Gupta S. Shrinkage estimator for scale parameter of gamma distribution. *Communications in Statistics-Simulation and Computation*. 2022; 51(6): 3073-3080.

[37] Rao C. *Score Test: Historical Review and Recent Developments*. Heidelberg: Springer; 2005.

[38] R Core Team. *R: A Language and Environment for Statistical Computing*. Vienna, Austria: Foundation for Statistical Computing; 2013.

[39] Bjerkedal T. Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Epidemiology*. 1960; 72(1): 130-148.