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Research Article

Deriving Optimal Skew Polycyclic Codes Over $\mathbb{F}_q$ Using Skew Polycyclic Linear Codes Over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$

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Abstract: This paper investigates the theory and applications of linear and skew polycyclic codes over the ring $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, where $\mathscr{R}_i$ $(0 \leq i \leq 3)$ are finite commutative rings. We first explore the structure of linear codes over $\mathfrak{R}$, establishing foundational properties. Then, we introduce skew polycyclic codes over $\mathfrak{R}$, a generalization of polycyclic code over a finite field. We delve into the algebraic structure of these codes and demonstrate how they differ from their classical counterparts. Furthermore, we examine the dual codes of skew polycyclic codes over $\mathfrak{R}$, providing necessary and sufficient conditions for a code to be self-dual. Finally, we investigate the Gray images of skew polycyclic codes over $\mathfrak{R}$, focusing on codes with optimal parameters. We provide explicit construction of Gray maps that yield images with good properties, such as large minimum distances and favorable automorphism groups. These results have potential applications in constructing new classes of error-correcting codes. We demonstrate this through an example of skew polycyclic codes applied in secret sharing schemes.

***Keywords*:** linear codes, skew polycyclic codes, dual codes, gray images, additive rings

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1. Introduction

Coding theory plays a vital role in modern communication systems, ensuring reliable data transmission through noisy channels. Linear codes over finite rings, known for their algebraic structure, have emerged as a prominent class of codes with applications in error correction and information transmission [1–3]. Since the work of Hammons et al. [4], which linked non-linear codes over $\mathbb{Z}_2$ to linear codes over $\mathbb{Z}_4$, interest in codes over rings has significantly grown. Cyclic codes, due to their polynomial structure, have also garnered much attention. These codes are particularly valuable in various fields, including secret sharing schemes and DNA-based applications [5, 6]. Earlier studies focused on finite rings, often requiring the automorphism order to divide the code length, until Siap et al. [7] removed this condition. Recently, research has extended to skew polynomial rings with both automorphisms and derivations, as seen in the work of Boulagouaz and Leroy [8], and Ma et al. [9], further broadening the scope of coding theory applications. In this context, the phenomenon of triality in characteristic 0, a geometric concept with profound implications for principal bundles and fixed points, offers a valuable perspective on the algebraic structures underlying skew polycyclic codes. While triality is primarily

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associated with characteristic 0, it serves as a valuable counterpart to the objects we investigate in characteristic p. This relationship underscores the interconnectedness of concepts across different characteristics and enriches our understanding of fundamental algebraic structures. For further reading, it is recommended to [10–12]. The study of codes over the ring $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, where $\mathscr{R}_i$ $(0 \leq i \leq 3)$ are finite frameworks that provide a more efficient communication framework. This paper explores several classes of codes over $\mathfrak{R}$, focusing on their theoretical properties and potential applications.

This paper aims to contribute to the ongoing development of coding theory by introducing and analyzing new classes of codes over the product ring $\mathfrak{R}$. Through a combination of algebraic theory and practical constructions, we provide insights into the structure and potential applications of these codes. We begin with an exploration of linear codes over the ring $\mathfrak{R}$. These codes are an extension of classical linear codes over finite fields, and they inherit many good properties, such as straightforward encoding and decoding procedures. The product's ring structure of $\mathfrak{R}$ introduces additional complexity, resulting in new challenges and possibilities for the advancement and analysis of these codes. The next focus is on skew polycyclic codes over $\mathfrak{R}$. These skew polycyclic codes represent a broad class with potential applications in commutative algebra and cryptography. We investigate the algebraic properties of these codes, emphasizing their generator polynomials and the implications of the skew structure. The study of the dual codes of skew polycyclic codes over $\mathfrak{R}$ follows naturally from this exploration. Dual codes play a vital role in error detection and correction, and their properties are closely tied to the algebraic structure of the original codes. We derive conditions under which skew polycyclic codes are self-dual and analyze the resulting code structures. Finally, we consider the Gray images of skew polycyclic codes over $\mathfrak{R}$, particularly those with optimal parameters. Gray Maps is a powerful tool that transforms code over rings into codes over fields, often preserving or enhancing some code properties. By applying Gray maps to skew polycyclic codes, we construct new classes of codes with good properties, such as large minimum distances and high levels of symmetry. These results are not only theoretically interesting but also have practical significance in the design of efficient and robust coding schemes. Skew polycyclic codes offer a powerful framework for secret sharing, providing enhanced security features such as error detection and correction. A practical example is presented where an optimal code is used to securely distribute a password among participants.

The manuscript is organized as follows: Section 3 introduces linear codes over the ring $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, detailing their fundamental properties and construction methods within this composite ring structure. Section 4 delves into skew polycyclic codes over the same ring $\mathfrak{R}$, exploring their algebraic properties and how they extend classical polycyclic codes. In Section 5, we examine the duals of skew polycyclic codes, discussing their duality relationships and implications for code design. Finally, Section 6 examines the Gray images of skew polycyclic codes and explores their optimization. We also demonstrate an application of skew polycyclic codes in secret sharing schemes through an example, illustrating how secrets can be securely shared and reconstructed only when a threshold number of participants combine their shares.

2. Preliminaries

In this section, we recall some notions concerning the additive ring $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, defined as follows: The ring $\mathfrak{R}$ constructed as the Cartesian product of three rings, $\mathscr{R}_1 = \sum_{i=0}^{2} u^i \mathbb{F}_q$, with $u^3 = u$, $\mathscr{R}_2 = \sum_{i=0}^{3} v^i \mathbb{F}_q$, with $v^4 = v$, and $\mathscr{R}_3 = \sum_{i=0}^{4} w^i \mathbb{F}_q$, with $w^5 = w$, $\mathbb{F}_q$ represents a finite field with q elements where $q = p^n$ for some $n \in \mathbb{N}$. Each $\mathscr{R}_i$ is an individual ring with its operations of addition and multiplication. The additive structure of $\mathfrak{R}$ is defined as component-wise, meaning that for any two elements $(r_1,\ r_2,\ r_3)$ and $(s_1,\ s_2,\ s_3)$ in $\mathfrak{R}$, their sum is given by $(r_1 + s_1,\ r_2 + s_2,\ r_3 + s_3)$. This additive structure inherits properties from the individual rings, making $\mathfrak{R}$ an additive ring that plays a crucial role in our work.

Each element is represented by the ordered triple $c = (c_1 \mid c_2 \mid c_3)$, where $c_1 \in \mathscr{R}_1$, $c_2 \in \mathscr{R}_2$, and $c_3 \in \mathscr{R}_3$. As noted in reference [13], the components c_1, c_2, and c_3 are given by the following expressions:

$$c_1 = \sum_{i=1}^{3} \mu_i c_{1i}, \quad \mu_1 = \frac{u+u^2}{2}, \; \mu_2 = \frac{-u+u^2}{2} \quad \text{and} \quad \mu_3 = 1-u^2,$$

$$c_2 = \sum_{i=1}^{3} \rho_i c_{2i}, \quad \rho_1 = \frac{v+v^2+v^3}{3}, \; \rho_2 = \frac{-v-v^2+2v^3}{3} \quad \text{and} \quad \rho_3 = 1-v^3,$$

$$c_3 = \sum_{i=1}^{3} \eta_i c_{3i}, \quad \eta_1 = \frac{w+w^2+w^3+w^4}{4}, \; \eta_2 = \frac{-w-w^2-w^3+3w^4}{4} \quad \text{and} \quad \eta_3 = 1-w^4.$$

A linear code C of length n over $\mathscr{R}$ is defined as an $\mathscr{R}$-submodule of the module $\mathscr{R}^n$ and the elements of C are called codewords. If B and C are codes, we denote $B \oplus C$ to represent the code $\{b+c \mid b \in B,\ c \in C\}$ and $B \otimes C$ to denote the code $\{(b,\ c) \mid b \in B,\ c \in C\}$. The dual code of C is defined as $C^{\perp} = \{x \mid \forall y \in C,\ x \cdot y = 0\}$, which is also a linear code of length n over $\mathscr{R}$. A code C is self-orthogonal if $C \subseteq C^{\perp}$ and self-dual if $C = C^{\perp}$. A polycyclic code C induced by a vector $v = (v_0,\ v_1,\ \ldots,\ v_{n-1}) \in \mathscr{R}^n$ over $\mathscr{R}$, is a linear code with the property that for any $c = (c_0,\ c_1,\ \ldots,\ c_{n-1}) \in C$, we have $(0,\ c_0,\ c_1,\ \ldots,\ c_{n-2}) + c_{n-1}(v_0,\ v_1,\ \ldots,\ v_{n-1}) \in C$. If $v_0 = 1$ and $v_i = 0$ for all $1 \le i \le n-1$, then C is called a cyclic code. Moreover, if $v_0 = \theta \in \mathscr{R}^*$ and $v_i = 0$ for all $1 \le i \le n-1$, then C is called as a θ-constacyclic code.

The following lemma is fundamental to the subsequent analysis. It establishes a relationship between the elements of the ring R and their corresponding properties, providing the necessary results that follow. This lemma not only clarifies the structure of the problem but also serves as a critical tool for proving more theorems in the later sections.

lemma 2.1 [14] The element $\mu_i \rho_j \eta_k,\ 1 \le i,\ j,\ k \le 3$, form a fundamental set of idempotents for $\mathfrak{R}$, and we have

- $(\mu_i \rho_j \eta_k)(\mu_{i'} \rho_{j'} \eta_{k'}) = 0$, for $i \ne i',\ j \ne j'$, and $k \ne k'$, where $1 \le i,\ j,\ k,\ i',\ j',\ k' \le 3$.
- $(\mu_i \rho_j \eta_k)^2 = \mu_i \rho_j \eta_k$, for $1 \le i,\ j,\ k \le 3$.
- $\sum_{i=1}^{3} \left[\sum_{j=1}^{3} \left(\sum_{k=1}^{3} \mu_i \rho_j \eta_k \right) \right] = 1$.

3. Linear codes over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$

Typically, additive codes over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$ are subgroups of $\mathscr{R}_1^{n_1} \times \mathscr{R}_2^{n_2} \times \mathscr{R}_3^{n_3}$, as shown in [15–17]. Consequently, we can express any code $C = C_1 \times C_2 \times C_3$ over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$ in the following manner:

$$C = (\mu_1 C_1^1 \oplus \mu_2 C_1^2 \oplus \mu_3 C_1^3) \times (\rho_1 C_2^1 \oplus \rho_2 C_2^2 \oplus \rho_3 C_2^3) \times (\eta_1 C_3^1 \oplus \eta_2 C_3^2 \oplus \eta_3 C_3^3), \tag{1}$$

Theorem 3.1 Consider C a linear code of length $n = 3n_1 + 3n_2 + 3n_3$ over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, then

$$C = \bigoplus_{i=1}^{3} \left[\bigoplus_{j=1}^{3} \left(\bigoplus_{k=1}^{3} \mu_i \rho_j \eta_k C_{ijk} \right) \right], \tag{2}$$

with $C_{ijk} = C_1^i \times C_2^j \times C_3^k$, for $1 \le i,\ j,\ k \le 3$.

Proof. The proof follows from Lemma 2.1 and (1).

3.1 *Gray map and gray images of linear codes over* $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$

We specify the Gray map, which is crucial in coding theory over rings. The Gray map is a bijective linear transformation that takes elements from a ring and maps them to a corresponding vector space, effectively allowing us to study codes over rings within the framework of classical linear codes. By using the Gray map, we can translate problems in ring theory into more familiar linear algebraic problems, facilitating the analysis and construction of codes with desirable properties. According to [18]:

$$\begin{aligned} \Phi : \mathfrak{R} =& \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3 \to \mathbb{F}_q^{9n} \\ & (s_1,\ s_2,\ s_3) \mapsto \Phi(s_1,\ s_2,\ s_3), \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Phi(s_1,\ s_2,\ s_3) = ((a_{11},\ a_{21},\ a_{31})M_1,\ (a_{12},\ a_{22},\ a_{32})M_1,\ \ldots,\ (a_{1n},\ a_{2n},\ a_{3n})M_1, \\ (b_{11},\ b_{21},\ b_{31})M_2,\ (b_{12},\ b_{22},\ b_{32})M_2,\ \ldots,\ (b_{1n},\ b_{2n},\ b_{3n})M_2, \\ (c_{11},\ c_{21},\ c_{31})M_3,\ (c_{12},\ c_{22},\ c_{32})M_3,\ \ldots,\ (c_{1n},\ c_{2n},\ c_{3n})M_3), \end{aligned}$$

with $s_1 = \sum_{i=1}^{3} \mu_i a_{ij}$, $s_2 = \sum_{i=1}^{3} \rho_i b_{ij}$ and $s_3 = \sum_{i=1}^{3} \eta_i c_{ij}$, for $1 \le j \le n$. Moreover, M_i, for $1 \le i \le 3$, are square matrices of order 3 satisfying $M_i M_i^t = \varepsilon I_3$, for some $\varepsilon \in \mathbb{F}_q^*$. We can confirm that Φ is a distance-preserving map from $(\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3,\ d_L)$ to $(\mathbb{F}_q,\ d_H)$. The parameters of the Gray image of a linear code C under Φ are provided in the following result.

Theorem 3.2 If $C = \bigoplus_{i=1}^{3} \left[\bigoplus_{j=1}^{3} \left(\bigoplus_{k=1}^{3} \mu_i \rho_j \eta_k C_{ijk}\right)\right]$ is an $[n,\ k,\ d_L]$ linear code over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, then $\Phi(C)$ is $[9n;\ k;\ d_H]_q$-linear codes over $\mathbb{F}_q$, where $d_L = d_H$.

Proof. Let $x = (x_1,\ x_2,\ x_3) = (\sum_{i=1}^{3} \mu_i a_{ij},\ \sum_{i=1}^{3} \rho_i b_{ij},\ \sum_{i=1}^{3} \eta_i c_{ij})$, $y = (y_1,\ y_2,\ y_3) = (\sum_{i=1}^{3} \mu_i a'_{ij},\ \sum_{i=1}^{3} \rho_i b'_{ij},\ \sum_{i=1}^{3} \eta_i c'_{ij})$ $\in C$, for $1 \le j \le n$, $\alpha \in \mathbb{F}_q$, then

$$\begin{aligned} \Phi(x+y) =& \Phi(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3) \\ =& \Big(\left(a_{11} + a'_{11},\ a_{21} + a'_{21},\ a_{31} + a'_{31}\right) M_1,\ \ldots,\ \left(a_{1n} + a'_{1n},\ a_{2n} + a'_{2n},\ a_{3n} + a'_{3n}\right) M_1, \\ & \left(b_{11} + b'_{11},\ b_{21} + b'_{21},\ b_{31} + b'_{31}\right) M_2,\ \ldots,\ \left(b_{1n} + b'_{1n},\ b_{2n} + b'_{2n},\ b_{3n} + b'_{3n}\right) M_2, \\ & \left(c_{11} + c'_{11},\ c_{21} + c'_{21},\ c_{31} + c'_{31}\right) M_3,\ \ldots,\ \left(c_{1n} + c'_{1n},\ c_{2n} + c'_{2n},\ c_{3n} + c'_{3n}\right) M_3 \Big) \\ =& \Big((a_{11},\ a_{21},\ a_{31})M_1,\ \ldots,\ (a_{1n},\ a_{2n},\ a_{3n})M_1,\ (b_{11},\ b_{21},\ b_{31})M_2,\ \ldots,\ (b_{1n},\ b_{2n},\ b_{3n})M_2, \end{aligned}$$

$$(c_{11},\ c_{21},\ c_{31})M_3,\ \ldots,\ (c_{1n},\ c_{2n},\ c_{3n})M_3\Big)+$$

$$\Big(\Big(a_{11}^{'},\ a_{21}^{'},\ a_{31}^{'}\Big)M_1,\ \ldots,\ \Big(a_{1n}^{'},\ a_{2n}^{'},\ a_{3n}^{'}\Big)M_1,\ \Big(b_{11}^{'},\ b_{21}^{'},\ b_{31}^{'}\Big)M_2,\ \ldots,\ \Big(b_{1n}^{'},\ b_{2n}^{'},\ b_{3n}^{'}\Big)M_2,$$

$$\Big(c_{11}^{'},\ c_{21}^{'},\ c_{31}^{'}\Big)M_3,\ \ldots,\ \Big(c_{1n}^{'},\ c_{2n}^{'},\ c_{3n}^{'}\Big)M_3\Big)$$

$$=\Phi(x)+\Phi(y).$$

$$\Phi(\alpha x)=\Phi(\alpha x_1,\alpha x_2,\ \alpha x_3)$$

$$=\Big((\alpha a_{11},\ \alpha a_{21},\alpha a_{31})M_1,\ \ldots,\ (\alpha a_{1n},\ \alpha a_{2n},\ \alpha a_{3n})M_1,\ (\alpha b_{11},\ \alpha b_{21},\alpha b_{31})M_2,\ \ldots,$$

$$(\alpha b_{1n},\alpha b_{2n},\ \alpha b_{3n})M_2,\ (\alpha c_{11},\ \alpha c_{21},\ \alpha c_{31})M_3,\ \ldots,\ (\alpha c_{1n},\ \alpha c_{2n},\ \alpha c_{3n})M_3\Big)$$

$$=\alpha\Big((a_{11},\ a_{21},a_{31})M_1,\ \ldots,\ (a_{1n},\ a_{2n},\ a_{3n})M_1,\ (b_{11},\ b_{21},\ b_{31})M_2,\ \ldots,\ (b_{1n},\ b_{2n},\ b_{3n})M_2,$$

$$(c_{11},\ c_{21},\ c_{31})M_3,\ \ldots,\ (c_{1n},\ c_{2n},\ c_{3n})M_3\Big)$$

$$=\alpha\Phi(x).$$

So Φ is linear. Since Φ is bijective, then $|C|=|\Phi(C)|$. As Φ is a distance preserving map, we have $d_L=d_H$.

Theorem 3.3 Consider C a linear code of length n over $\mathfrak{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3$, then we have

$$\Phi(C)=C_{111}\otimes C_{112}\otimes\cdots\otimes C_{333},\ \textit{and}\ |C|=|C_{111}|\times|C_{112}|\times\cdots\times|C_{333}|. \tag{4}$$

Proof. Similar to the approach utilized in proving Theorem 8 [19]. □

The generator matrix of the linear code $C=\bigoplus_{i=1}^{3}\left[\bigoplus_{j=1}^{3}\left(\bigoplus_{k=1}^{3}\mu_i\rho_j\eta_kC_{ijk}\right)\right]$ over $\mathfrak{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3$ can be expressed in terms of the generator matrices of C_{ijk}, for $1\le i,\ j,\ k\le 3$ as follows

$$G=\begin{pmatrix}\mu_1\rho_1\eta_1G_{111}\\ \mu_1\rho_1\eta_2G_{112}\\ \vdots\\ \mu_3\rho_3\eta_3G_{333}\end{pmatrix}, \tag{5}$$

where G_{ijk} are generator matrices of a linear code C_{ijk} over $\mathbb{F}_q$, for $1 \leq i,\ j,\ k \leq 3$. Additionally, by applying the Gray map Φ, the following result can be easily derived.

Theorem 3.4 If $C = \bigoplus_{i=1}^{3} \left[\bigoplus_{j=1}^{3} \left(\bigoplus_{k=1}^{3} \mu_i\rho_j\eta_k C_{ijk}\right)\right]$ is a linear code of length n over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$ with generator matrix G, then

$$\Phi(G) = \begin{bmatrix} G_{111} & 0 & \dots & 0 \\ 0 & G_{112} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G_{333} \end{bmatrix}. \tag{6}$$

Proof. The results can be obtained using the matrix provided below

$$\Phi(G) = \begin{pmatrix} \Phi(\mu_1\rho_1\eta_1 G_{111}) \\ \Phi(\mu_1\rho_1\eta_2 G_{112}) \\ \vdots \\ \Phi(\mu_3\rho_3\eta_3 G_{333}) \end{pmatrix}. \tag{7}$$

□

The corollary below examines the relationship between the Gray image of a linear code $C = \bigoplus_{i=1}^{3} \left[\bigoplus_{j=1}^{3} \left(\bigoplus_{k=1}^{3} \mu_i\rho_j \eta_k C_{ijk}\right)\right]$ over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$ and the Gray image of its dual.

Corollary 3.5 Let $C = \bigoplus_{i=1}^{3} \left[\bigoplus_{j=1}^{3} \left(\bigoplus_{k=1}^{3} \mu_i\rho_j\eta_k C_{ijk}\right)\right]$ be a linear code of length n over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, then $\Phi(C^{\perp}) = [\Phi(C)]^{\perp}$. Further, C is a self-dual code if and only if $\Phi(C)$ is a self-dual code.

Proof. Consider $s = (s_1,\ s_2,\ s_3) \in C$ and $x = (x_1,\ x_2,\ x_3) \in C^{\perp}$, where $s_1 = \sum_{i=1}^{3} \mu_i a_{ij}$, $s_2 = \sum_{i=1}^{3} \rho_i b_{ij}$, and $s_3 = \sum_{i=1}^{3} \eta_i c_{ij}$, and $x_1 = \sum_{i=1}^{3} \mu_i a_{ij}^{'}$, $x_2 = \sum_{i=1}^{3} \rho_i b_{ij}^{'}$, and $x_3 = \sum_{i=1}^{3} \eta_i c_{ij}^{'}$, for $1 \leq j \leq n$. Then

$$\begin{aligned} < s,\ x >_{\mathfrak{R}} &= < (s_1,\ s_2,\ s_3),\ (x_1,\ x_2,\ x_3) >_{\mathfrak{R}} \\ &= < s_1,\ x_1 >_{\mathscr{R}_1} + < s_2,\ x_2 >_{\mathscr{R}_2} + < s_3,\ x_3 >_{\mathscr{R}_3} \\ &= \mu_1 \sum_{j=0}^{n-1} a_{1j}a_{1j}^{'} + \mu_2 \sum_{j=0}^{n-1} a_{2j}a_{2j}^{'} + \mu_3 \sum_{j=0}^{n-1} a_{3j}a_{3j}^{'} \\ &\quad + \rho_1 \sum_{j=0}^{n-1} b_{1j}b_{1j}^{'} + \rho_2 \sum_{j=0}^{n-1} b_{2j}b_{2j}^{'} + \rho_3 \sum_{j=0}^{n-1} b_{3j}b_{3j}^{'} \\ &\quad + \eta_1 \sum_{j=0}^{n-1} c_{1j}c_{1j}^{'} + \eta_2 \sum_{j=0}^{n-1} c_{2j}c_{2j}^{'} + \eta_3 \sum_{j=0}^{n-1} c_{3j}c_{3j}^{'} \\ &= 0, \end{aligned}$$

which suggests that $\sum_{j=0}^{n-1} a_{ij}a_{ij}^{'}=0$, $\sum_{j=0}^{n-1} b_{ij}b_{ij}^{'}=0$ and $\sum_{j=0}^{n-1} c_{ij}c_{ij}^{'}=0$, for $0\leq i\leq 3$.

Following the Gray map previously defined, we have

$$\Phi(s)=((a_{11},\, a_{21},\, a_{31})M_1,\, (a_{12},\, a_{22},\, a_{32})M_1,\, \ldots,\, (a_{1n},\, a_{2n},\, a_{3n})M_1,$$

$$(b_{11},\, b_{21},\, b_{31})M_2,\, (b_{12},\, b_{22},\, b_{32})M_2,\, \ldots,\, (b_{1n},\, b_{2n},\, b_{3n})M_2,$$

$$(c_{11},\, c_{21},\, c_{31})M_3,\, (c_{12},\, c_{22},\, c_{32})M_3,\, \ldots,\, (c_{1n},\, c_{2n},\, c_{3n})M_3),$$

and

$$\Phi(x)=\left(\left(a_{11}^{'},\, a_{21}^{'},\, a_{31}^{'}\right)M_1,\, \left(a_{12}^{'},\, a_{22}^{'},\, a_{32}^{'}\right)M_1,\, \ldots,\, \left(a_{1n}^{'},\, a_{2n}^{'},\, a_{3n}^{'}\right)M_1,\right.$$

$$\left(b_{11}^{'},\, b_{21}^{'},\, b_{31}^{'}\right)M_2,\, \left(b_{12}^{'},\, b_{22}^{'},\, b_{32}^{'}\right)M_2,\, \ldots,\, \left(b_{1n}^{'},\, b_{2n}^{'},\, b_{3n}^{'}\right)M_2,$$

$$\left.\left(c_{11}^{'},\, c_{21}^{'},\, c_{31}^{'}\right)M_3,\, \left(c_{12}^{'},\, c_{22}^{'},\, c_{32}^{'}\right)M_3,\, \ldots,\, \left(c_{1n}^{'},\, c_{2n}^{'},\, c_{3n}^{'}\right)M_3\right).$$

Building on these relationships, we obtain

$$<\Phi(s),\, \Phi(x)>_{\mathbb{F}_q}=\sum_{j=0}^{n-1}(a_{1j},\, a_{2j},\, a_{3j})M_1M_1^{\perp}\left[(a_{1j}^{'},\, a_{2j}^{'},\, a_{3j})^{'}\right]^t$$

$$+\sum_{j=0}^{n-1}(b_{1j},\, b_{2j},\, b_{3j})M_2M_2^{\perp}\left[(b_{1j}^{'},\, b_{2j}^{'},\, b_{3j}^{'})\right]^t$$

$$+\sum_{j=0}^{n-1}(c_{1j},\, c_{2j},\, c_{3j})M_3M_3^{\perp}\left[(c_{1j}^{'},\, c_{2j}^{'},\, c_{3j}^{'})\right]^t$$

$$=\varepsilon\sum_{j=0}^{n-1}(a_{1j},\, a_{2j},\, a_{3j})\left[(a_{1j}^{'},\, a_{2j}^{'},\, a_{3j}^{'})\right]^t$$

$$+\varepsilon\sum_{j=0}^{n-1}(b_{1j},\, b_{2j},\, b_{3j})\left[(b_{1j}^{'},\, b_{2j}^{'},\, b_{3j}^{'})\right]^t$$

$$+\varepsilon\sum_{j=0}^{n-1}(c_{1j},\, c_{2j},\, c_{3j}))\left[(c_{1j}^{'},\, c_{2j}^{'},\, c_{3j}^{'})\right]^t$$

$$=\varepsilon\left[\sum_{j=0}^{n-1} a_{1j}a_{1j}^{'}+\sum_{j=0}^{n-1} a_{2j}a_{2j}^{'}+\sum_{j=0}^{n-1} a_{3j}a_{3j}^{'}\right.$$

$$+\sum_{j=0}^{n-1} b_{1j}b_{1j}^{'}+\sum_{j=0}^{n-1} b_{2j}b_{2j}^{'}+\sum_{j=0}^{n-1} b_{3j}b_{3j}^{'}$$

$$\left.+\sum_{j=0}^{n-1} c_{1j}c_{1j}^{'}+\sum_{j=0}^{n-1} c_{2j}c_{2j}^{'}+\sum_{j=0}^{n-1} c_{3j}c_{3j}^{'}\right]$$

$$=0.$$

This leads us to the relation $\Phi(C^{\perp})\subseteq[\Phi(C)]^{\perp}$. Moreover, $\mid\Phi(C^{\perp})\mid=\mid[\Phi(C)]^{\perp}\mid$ we reach the conclusion $\Phi(C^{\perp})=[\Phi(C)]^{\perp}$. □

4. Skew polycyclic codes over $\mathfrak{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3$

In this section, we examine the structure of skew polycyclic codes over $\mathfrak{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3$. To investigate these codes, we start by defining an automorphism of $\mathfrak{R}$. Specifically, we consider the automorphisms $\varphi_i:\mathscr{R}_i\longrightarrow\mathscr{R}_i$ for $1\le i\le 3$, which are defined as follows:

$$\varphi_1(\mu_1d_1+\mu_2d_2+\mu_3d_3)=\mu_1\varphi_1(d_1)+\mu_2\varphi_1(d_2)+\mu_3\varphi_1(d_3)=\mu_1d_1^p+\mu_2d_2^p+\mu_3d_3^p,$$

$$\varphi_2(\rho_1e_1+\rho_2e_2+\rho_3e_3)=\rho_1\varphi_2(e_1)+\rho_2\varphi_2(e_2)+\rho_3\varphi_2(e_3)=\rho_1e_1^p+\rho_2e_2^p+\rho_3e_3^p,$$

$$\varphi_3(\eta_1f_1+\eta_2f_2+\eta_3f_3)=\eta_1\varphi_3(f_1)+\eta_2\varphi_3(f_2)+\eta_3\varphi_3(f_3)=\eta_1f_1^p+\eta_2f_2^p+\eta_3f_3^p.$$

We now define the automorphism on $\mathfrak{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3$ by:

$$\begin{aligned}\varphi:\ \mathfrak{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3&\longrightarrow\mathfrak{R}\\ c=(c_1,\ c_2,\ c_3)&\longmapsto\varphi(c),\end{aligned}\tag{8}$$

where $\varphi(c)=(\ \varphi_1(c_1),\ \ \varphi_2(c_2),\ \ \varphi_3(c_3)\ )=\varphi_1(c_1)+\varphi_2(c_2)+\varphi_3(c_3)$.

We consider the structure of skew polycyclic codes over $\mathfrak{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3$. By applying the automorphism defined previously, we can specify the properties and operations that characterize these codes.

Definition 4.1 [20] Consider C as a linear code of length n over the ring $\mathfrak{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3$. Let φ be an automorphism of $\mathfrak{R}$, and let $v=(e_0,\ e_1,\ \cdots,\ e_{n-1})\in\mathfrak{R}^n$. The code C is called a skew φ-polycyclic code induced by the vector v, if for every codeword $c=(c_0,\ c_1,\ \cdots,\ c_{n-1})\in C$, the following condition holds:

$$\sigma_{v,\ \varphi}(c) = (0,\ \varphi(c_0),\ \varphi(c_1),\ \ldots,\ \varphi(c_{n-2})) + \varphi(c_{n-1})(e_0,\ e_1,\ \ldots,\ e_{n-1}) \in C. \tag{9}$$

When $v = (1,\ 0,\ \ldots,\ 0)$ or $v = (-1,\ 0,\ \ldots,\ 0)$, the corresponding code is referred to as a skew φ-cyclic (or simply skew cyclic) code and a skew φ-negacyclic (or simply skew negacyclic) code, respectively. Let's introduce the concept of a skew-polynomial ring, denoted as $\mathfrak{R}[x,\ \varphi]$. This ring consists of polynomials, with coefficients from $\mathfrak{R}$, i.e.,

$$\mathfrak{R}[x,\ \varphi] = \left\{a_0 + a_1\, x + a_2\, x^2 + \ldots + a_n\, x^n \mid a_i \in \mathfrak{R},\ n \in \mathbb{N}\right\} \tag{10}$$

The addition of polynomials follows the standard procedure, but the multiplication defined by:

$$x \cdot a = \varphi(a) \cdot x,\ \text{for } a \in \mathfrak{R} \tag{11}$$

This multiplication rule generally makes $\mathfrak{R}[x,\ \varphi]$ non-commutative, unless φ happens to be the identity map. The elements of this ring are called skew-polynomials. When does a skew-polynomial $f(x)$ belong to the center of $\mathfrak{R}[x,\ \varphi]$? This is important because it aids in our understanding of skew φ-polycyclic codes, particularly when viewed as ring ideals.

Lemma 4.2 Consider a skew-polynomial ring $\mathfrak{R}[x,\ \varphi]$ where φ is an automorphism of $\mathfrak{R}$. Let $f(x) = x^n - f_{n-1}x^{n-1} - \cdots - f_1 x - f_0$ be a polynomial in this ring, where $\varphi(f_i) = f_i$, for $0 \le i < n$, $f_n = 1$ and the order of φ divides i, for $0 \le i \le n$. Then f belongs to the center of $\mathfrak{R}[x,\ \varphi]$, i.e., $f \in Z(\mathfrak{R}[x,\ \varphi])$.

Proof. To show $f(x)$ is in the center, we need to prove that it commutes with all elements of $\mathfrak{R}[x,\ \varphi]$. Let's consider an arbitrary element $g(x) = a_0 + a_1 x + \ldots + a_m x^m$ in $\mathfrak{R}[x,\ \varphi]$, we first show that f_0 commutes with $g(x)$, we have

$$f_0 g(x) = f_0 a_0 + f_0 a_1 x + \cdots + f_0 a_m x^m$$

$$g(x) f_0 = a_0 f_0 + a_1 \varphi_h(f_0) x + \ldots + a_m \varphi_h^m(f_0) x^m$$

These are equal because $\varphi(f_0) = f_0$.

Next, we show that $f_i x^i$ commutes with $g(x)$ for $1 \leqslant i \leqslant n$, we obtain

$$f_i x^i g(x) = f_i \varphi_i(a_0) x^i + f_i \varphi_i(a_1) x^{i+1} + \cdots + f_i \varphi_i(a_n) x^{i+n}$$

$$g(x) f_i x^i = a_0 \varphi_i(f_i) x^i + a_1 \varphi_i(f_i) x^{i+1} + \cdots + a_n \varphi_i(f_i) x^{i+n}$$

These are equal because $\varphi(fi) = fi$ and $\varphi_i(a_k) = a_k$ (since the order of φ divides i). Therefore, $f(x)$ is in the center of $\mathfrak{R}[x,\ \varphi]$. □

Lemma 4.3 For a polynomial $f(x) = x^n - f_{n-1}x^{n-1} - \cdots - f_1 x - f_0$ in the center of $\mathfrak{R}[x,\ \varphi]$. If $f(x) = g(x)h(x)$ for some polynomials $g(x)$ and $h(x)$ in $\mathfrak{R}[x,\ \varphi]$, then $f(x) = h(x)g(x)$.

Proof. Consider $f(x) \in Z(\mathfrak{R}[x,\ \varphi])$ and $f(x) = g(x)h(x)$ for some polynomials $g(x)$ and $h(x)$ in $\mathfrak{R}[x,\ \varphi]$. Then, we have the sequence $g(x)(h(x)g(x)) = (g(x)h(x))g(x) = f(x)g(x) = g(x)f(x) = g(x)(g(x)h(x))$. This leads to $g(x)(h(x)g(x) - g(x)h(x)) = 0$.

Suppose $T(x)=g(x)\left(h(x)g(x)-g(x)h(x)\right)$, where $g(x)=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)$ and $h(x)=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)$ for some polynomials $g_{ijk}(x)$ and $h_{ijk}(x)$ in $\mathbb{F}_q[x;\ \varphi]$ where $1\leq i,\ j,\ k\leq 3$. Then, we obtain:

$$
\begin{aligned}
T(x)=&\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\Bigg[\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)\right)\\
&\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\right)-\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\right)\\
&\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)\right)\Bigg]\\
=&\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\Bigg[\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)g_{ijk}(x)\right)\\
&-\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)h_{ijk}(x)\right)\Bigg]\\
=&\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\right)\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)g_{ijk}(x)\right)\\
&-\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\right)\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)h_{ijk}(x)\right)\\
=&\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\left(g_{ijk}(x)h_{ijk}(x)g_{ijk}(x)-g_{ijk}(x)g_{ijk}(x)h_{ijk}(x)\right)\\
=&\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\left(h_{ijk}(x)g_{ijk}(x)-g_{ijk}(x)h_{ijk}(x)\right)\\
=&\ 0.
\end{aligned}
$$

This indicates that $g_{ijk}(x)\left(h_{ijk}(x)g_{ijk}(x)-g_{ijk}(x)h_{ijk}(x)\right)=0$, for $1\leq i,\ j,\ k\leq 3$. Consequently, this means that either $h_{ijk}(x)=0$ or $h_{ijk}(x)g_{ijk}(x)-g_{ijk}(x)h_{ijk}(x)=0$ since $\mathbb{F}_q[x;\ \varphi]$ has no non-trivial zero divisors. In other words, $h_{ijk}(x)g_{ijk}(x)=g_{ijk}(x)h_{ijk}(x)$, for $1\leq i,\ j,\ k\leq 3$. Therefore, we can conclude that

$$
g(x)h(x)=\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\right)\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)\right)
$$

$$=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)h_{ijk}(x)$$

$$=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)g_{ijk}(x)$$

$$=\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)\right)\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\right)$$

$$=h(x)g(x)$$

Thus, the lemma is proven. □

Let $f(x)\in Z(\mathfrak{R}[x,\ \varphi])$ as defined previously. Then the left ideal $\mathfrak{R}[x,\ \varphi]f(x)=\{r(x)f(x)\mid r(x)\in\mathfrak{R}[x,\ \varphi]\}$ generated by $f(x)$ is also a right ideal, forming a two-sided ideal. Consequently, $\frac{\mathfrak{R}[x,\ \varphi]}{\mathfrak{R}[x,\ \varphi]f(x)}$ is a ring. If the conditions of the preceding lemma are not satisfied, $\frac{\mathfrak{R}[x,\ \varphi]}{\mathfrak{R}[x,\ \varphi]f(x)}$ forms a left $\mathfrak{R}[x,\ \varphi]$-module under the left multiplication: $r(x)(h(x)+\mathfrak{R}[x,\ \varphi]f(x))=r(x)h(x)+\mathfrak{R}[x,\ \varphi]f(x)$ where $r(x),\ h(x)\in\mathfrak{R}[x,\ \varphi]$. We can identify an element $c=(c_0,\ c_1,\ \ldots,\ c_{n-1})\in\mathfrak{R}^n$ with the polynomial $p_n(c)=c_0+c_1x+\ldots+c_{n-1}x^{n-1}\in\frac{\mathfrak{R}[x,\ \varphi]}{\mathfrak{R}[x,\ \varphi]f(x)}$. Any code of length n over $\mathfrak{R}$ can be viewed as a subset of $\frac{\mathfrak{R}[x,\ \varphi]}{\mathfrak{R}[x,\ \varphi]f(x)}$, where $n=\deg(f(x))$.

Lemma 4.4 Let C be a linear code of length n over the ring $\mathfrak{R}$. The code C is a skew φ-polycyclic code induced by a vector v if and only if C can be represented as a left $\mathfrak{R}[x,\ \varphi]$-submodule of the module $\frac{\mathfrak{R}[x,\ \varphi]}{\mathfrak{R}[x,\ \varphi]f(x)}$, where the polynomial $f(x)$ is defined as $f(x)=x^n-p_n(v)$.

Proof. Consider C as a skew φ-polycyclic code of length n induced by vector v, and let the polynomial $f(x)=x^n-v(x)$ where $v(x)=p_n(v)=\sum_{i=0}^{n-1}e_ix^i$. For $c=(c_0,\ c_1,\ \ldots,\ c_{n-1})\in C$, we have $\sigma_{v,\ \varphi}(C)\subseteq C$. Let $c(x)=\sum_{i=0}^{n-1}c_ix^i$. To explore the behavior of $c(x)$ under multiplication by x, we first calculate the product $x\cdot c(x)$, which can be expressed as:

$$x\cdot c(x)=\sum_{i=0}^{n-2}\varphi(c_i)x^{i+1}+\varphi(c_{n-1})x^n. \tag{12}$$

Given that x^n is equivalent to $v(x)$ within the module $\frac{\mathfrak{R}[x,\ \varphi]}{\mathfrak{R}[x,\ \varphi]f(x)}$, we substitute x^n with $\sum_{i=0}^{n-1}e_ix^i$, resulting in

$$x\cdot c(x)=\sum_{i=0}^{n-2}\varphi(c_i)x^{i+1}+\varphi(c_{n-1})\sum_{i=0}^{n-1}e_ix^i. \tag{13}$$

This polynomial corresponds to $\sigma_{v,\ \varphi}(c)$. Therefore, $xc(x)$ belongs to C, and hence $r(x)c(x)\in C$ for any $r(x)\in\mathfrak{R}[x,\ \varphi]$. To establish the reverse implication, assume that C is indeed a left $\mathfrak{R}[x,\ \varphi]$-submodule of the module $\frac{\mathfrak{R}[x,\ \varphi]}{\mathfrak{R}[x,\ \varphi]f(x)}$ and $c(x)=\sum_{i=0}^{n-1}c_ix^i\in C$. Then $r(x)c(x)\in C$ for any $r(x)\in\mathfrak{R}[x,\ \varphi]$, implying $xc(x)\in C$. Note that $xc(x)$ is the polynomial corresponding to $\sigma_{v,\,\varphi}(c)$ for $c=(c_0,\,c_1,\,\ldots,\,c_{n-1})\in C$. Therefore, $\sigma_{v,\,\varphi}(C)\subseteq C$, and hence C is a skew φ-polycyclic code over $\mathfrak{R}$.

Lemma 4.5 Let C be a linear code of length n over $\mathfrak{R}$, and let $C=C_1\times C_2\times C_3$, where $C_1,\ C_2$ and C_3 are linear codes of length n over $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$. Given a codeword $c=(b_0,\,b_1,\,\ldots,\,b_{n-1})\in C$ and a vector $v=(e_0,\,e_1,\,\ldots,\,e_{n-1})\in\mathfrak{R}^n$. So we have:

$$\sigma_{v_1,\,\varphi_1}(c_1)\,\sigma_{v_2,\,\varphi_2}(c_2)\,\sigma_{v_3,\,\varphi_3}(c_3)\;=\;\sigma_{v,\,\varphi}(c), \tag{14}$$

where $v=(v_1|\,v_2|\,v_3)$ and $c=(c_1|\,c_2|\,c_3)$.

Proof. Let $c_s=(b_{0,\,s},\,b_{1,\,s},\,\ldots,\,b_{n-1,\,s})\in\mathscr{R}_s$ and $v_s=(e_{0,\,s},\,e_{1,\,s},\,\ldots,\,e_{n-1,\,s})\in\mathscr{R}_s^n$, for $1\leq s\leq 3$. Consider $c=(c_1|\,c_2|\,c_3)$ and $v=(v_1|\,v_2|\,v_3)$, we have

$$\begin{aligned}
\sigma_{v,\,\varphi(c)} =&(0,\ \varphi(b_0),\ \varphi(b_1),\ \ldots,\ \varphi(b_{n-2}))+\varphi(b_{n-1})(e_0,\,e_1,\,\ldots,\,e_{n-1})\\
=&\Big(0,\ \varphi_1(b_{0,\,1})+\varphi_2(b_{0,\,2})+\varphi_3(b_{0,\,3}),\varphi_1(b_{1,\,1})+\varphi_2(b_{1,\,2})+\varphi_3(b_{1,\,3}),\ldots,\varphi_1(b_{n-2,\,1})\\
&+\varphi_2(b_{n-2,\,2})+\varphi_3(b_{n-2,\,3})\Big)+\Big(\varphi_1(b_{n-1,\,1}\,e_{0,\,1})+\varphi_2(b_{n-1,\,2}\,e_{0,\,2})+\varphi_3(b_{n-1,\,3}\,e_{0,\,3})\\
&,\ldots,\varphi_1(b_{n-1,\,1}\,e_{n-1,\,1})+\varphi_2(b_{n-1,\,2}\,e_{n-1,\,2})+\varphi_3(b_{n-1,\,3}\,e_{n-1,\,3})\Big)\\
=&\Big[(0,\ \varphi_1(b_{0,\,1}),\varphi_1(b_{1,\,1}),\ldots,\varphi_1(b_{n-2,\,1}))+\varphi_1(b_{n-1,1})(e_{0,\,1},e_{1,\,1},\ldots,\,e_{n-1,\,1})\Big]\\
&+\Big[(0,\ \varphi_2(b_{0,\,2}),\varphi_2(b_{1,\,2}),\ \ldots,\ \varphi_2(b_{n-2,\,2}))+\varphi_2(b_{n-1,\,2})(e_{0,\,2},\ e_{1,\,2},\ \ldots,\ e_{n-1,\,2})\Big]\\
&+\Big[(0,\ \varphi_3(b_{0,\,3}),\ \varphi_3(b_{1,\,3}),\ \ldots,\ \varphi_3(b_{n-2,\,3}))+\varphi_3(b_{n-1,\,3})(e_{0,\,3},\ e_{1,\,3},\ \ldots,\ e_{n-1,\,3})\Big]\\
=&\Big[(0,\ \varphi_1(b_{0,\,1}),\ \varphi_1(b_{1,\,1}),\ \ldots,\ \varphi_1(b_{n-2,\,1}))+\varphi_1(b_{n-1,\,1})(e_{0,\,1},\ e_{1,\,1},\ \ldots,\ e_{n-1,1}),\\
&(0,\ \varphi_2(b_{0,\,2}),\ \varphi_2(b_{1,\,2}),\ \ldots,\ \varphi_2(b_{n-2,\,2}))+\varphi_2(b_{n-1,\,2})(e_{0,\,2},e_{1,\,2},\ \ldots,\ e_{n-1,\,2}),\\
&(0,\ \varphi_3(b_{0,\,3}),\varphi_3(b_{1,\,3}),\ \ldots,\ \varphi_3(b_{n-2,\,3}))+\varphi_3(b_{n-1,\,3})(e_{0,\,3},e_{1,\,3},\ \ldots,\ e_{n-1,\,3})\Big].
\end{aligned}$$

Therefore,

$$\sigma_{v,\ \varphi}(c) = \sigma_{v_1,\ \varphi_1}(c_1)\,\sigma_{v_2,\ \varphi_2}(c_2)\,\sigma_{v_3,\ \varphi_3}(c_3).$$

□

Theorem 4.6 Let $C = \bigoplus_{i=1}^{3}\left[\bigoplus_{j=1}^{3}\left(\bigoplus_{k=1}^{3}\mu_i\rho_j\eta_k C_{ijk}\right)\right]$ be a linear code of length n over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, and let $v = (e_0,\ e_1,\ \ldots,\ e_{n-1}) \in \mathfrak{R}^n$ where

$$v_r = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k e_{r,\ ijk} \in \mathfrak{R},\ 0 \le r \le n-1. \tag{15}$$

Then C is a skew φ-polycyclic code induced by the vector v if and only if each C_{ijk} is a skew φ-polycyclic code induced by the vector v_{ijk}, for $1 \le i,\ j,\ k \le 3$, where $v_{ijk} = (e_{0,\ ijk},\ e_{1,\ ijk},\ \ldots,\ e_{(n-1),\ ijk}) \in \mathbb{F}_q^{3n}$, for $1 \le i,\ j,\ k \le 3$. Specifically, C is a skew cyclic (or skew negacyclic) code over $\mathfrak{R}$ if and only if each C_{ijk} is a skew cyclic (or skew negacyclic) code over $\mathbb{F}_q^{3n}$, for $1 \le i,\ j,\ k \le 3$.

Proof. Let C be a skew φ-polycyclic code over $\mathfrak{R}$ induced by the vector

$$v = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\ \mu_i\rho_j\eta_k v_{ijk} \tag{16}$$

and consider $b_{ijk} = (b_{0,\ ijk},\ b_{1,\ ijk},\ \ldots,\ b_{(n-1),\ ijk}) \in C_{ijk}$, for $1 \le i,\ j,\ k \le 3$. Then

$$c = (b_0,\ b_1,\ \ldots,\ b_{n-1}) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k b_{ijk} \in C$$

where $c_r = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k b_{r,\ ijk}$, for $0 \le r \le n-1$. Since C is a skew φ-polycyclic code over $\mathfrak{R}$, it follows that

$$\sigma_{v,\ \varphi}(c) = (0,\ \varphi(b_0),\ \varphi(b_1),\ \ldots,\ \varphi(b_{n-2})) + \varphi(b_{n-1})(e_0,\ e_1,\ \ldots,\ e_{n-1}) \in C. \tag{17}$$

Note that

$$\begin{aligned}\sigma_{v,\ \varphi}(c) =& (0,\ \varphi(b_0),\ \varphi(b_1),\ \ldots,\ \varphi(b_{n-2})) + \varphi(b_{n-1})(e_0,\ e_1,\ \ldots,\ e_{n-1})\\ =& \left(0,\ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\varphi(b_{0,\ ijk}),\ \ldots,\ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\varphi(b_{(n-2),\ ijk})\right)\\ &+ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\varphi(b_{(n-1),\ ijk})\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k e_{0,\ ijk}, \ldots,\ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k e_{(n-1),\ ijk}\right)\end{aligned}$$

$$=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\Big(0,\ \varphi(b_{0,\ ijk}),\ \ldots,\ \varphi(b_{(n-2),\ ijk})\Big)$$

$$+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\varphi(b_{(n-1),\ ijk})\Big(e_{0,\ ijk},\ldots,\ e_{(n-1),\ ijk}\Big)$$

$$=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\Big[\big(0,\ \varphi(b_{0,\ ijk}),\ \ldots,\ \varphi(b_{(n-2),\ ijk})\big)$$

$$+\varphi(b_{(n-1),\ ijk})\big(e_{0,\ ijk},\ \ldots,\ e_{(n-1),\ ijk}\big)\Big]$$

$$=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\sigma_{v_{ijk},\ \varphi}(b_{ijk}),$$

where $v_{ijk}=(e_{0,\ ijk},\ e_{1,\ ijk},\ \ldots,\ e_{n-1,\ ijk})\in\mathbb{F}_q^{3n}$, for $1\leq i,\ j,\ k\leq 3$ consequently

$$\sigma_{v_{ijk},\ \varphi}(b_{ijk})\in C_{ijk},\ \text{ for }\ 1\leq i,\ j,\ k\leq 3,$$

thus, C_{ijk} is a skew φ-polycyclic code over $\mathbb{F}_q$ induced by $v_{ijk}=(e_{0,\ ijk}, e_{1,\ ijk},\ \ldots,\ e_{n-1,\ ijk})\in\mathbb{F}_q^{3n}$ for each $1\leq i,\ j,\ k\leq 3$.

Conversely, suppose C_{ijk} is a skew φ-polycyclic code over $\mathbb{F}_q$ induced by

$$v_{ijk}=(e_{0,\ ijk},\ e_{1,\ ijk},\ \ldots,\ e_{n-1,\ ijk})\in\mathbb{F}_q^{3n}\ \textit{and}\ c=(b_0,\ b_1,\ \ldots,\ b_{n-1})\in C,$$

where

$$c_r=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k b_{r,\ ijk},\ \textit{for}\ 0\leq r\leq n-1.$$

Then

$$b_{ijk}=(b_{0,\ ijk},\ b_{1,\ ijk},\ \ldots,\ b_{n-1,\ ijk})\in C_{ijk},\ \ \textit{for}\ 1\leq i,\ j,\ k\leq 3$$

since

$$c=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k b_{ijk}.$$

As C_{ijk} is a skew φ-polycyclic code induced by v_{ijk}, we have $\sigma_{v_{ijk},\ \varphi}(b_{ijk}) \in C_{ijk}$ for, $1 \le i,\ j,\ k \le 3$, which implies

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\sigma_{v_{ijk},\ \varphi}(b_{ijk}) \in C.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\sigma_{v_{ijk},\ \varphi}(b_{ijk}) =& \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k \\
& \left[\left(0,\ \varphi(b_{0,\ ijk}),\ \dots,\ \varphi(b_{(n-2),\ ijk})\right) + \varphi(b_{(n-1),\ ijk})\left(e_{0,\ ijk},\ \dots,\ e_{(n-1),\ ijk}\right)\right] \\
=& \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\left(0,\ \varphi(b_{0,\ ijk}),\ \dots,\ \varphi(b_{(n-2),\ ijk})\right) + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\varphi(b_{(n-1),\ ijk}) \\
& \left(e_{0,\ ijk},\ \dots,\ e_{(n-1),\ ijk}\right) \\
=& \left(0,\ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\varphi(b_{0,\ ijk}),\ \dots,\ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\varphi(b_{(n-2),\ ijk})\right) \\
& + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k\varphi(b_{(n-1),\ ijk}) \\
& \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k e_{0,\ ijk},\ \dots,\ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k e_{(n-1),\ ijk}\right) \\
=& (0,\ \varphi(b_0),\ \varphi(b_1),\ \dots,\ \varphi(b_{n-2})) + \varphi(b_{n-1})(e_0,\ e_1,\ \dots,\ e_{n-1}) \\
=& \sigma_{v,\ \varphi}(c).
\end{aligned}$$

Thus, $\sigma_{v,\ \varphi}(c) \in C$, establishing that C is indeed a skew φ-polycyclic code over $\mathfrak{R}$. □

By applying the above arguments and Theorem 4.6, we can now derive the generator polynomial for any skew φ-polycyclic code over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$ induced by a polynomial $f \in \mathfrak{R}[x,\ \varphi]$.

Lemma 4.7 For a polynomial $f = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k f_{ijk} \in \mathfrak{R}[x,\ \varphi]$, where each f_{ijk} belongs to $\mathbb{F}_q[x;\ \varphi]$ for $0 \le i,\ j,\ k \le 3$, we can construct a skew φ-polycyclic code $C = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k C_{ijk}$ over $\mathfrak{R}$, where C_{ijk} is a skew φ-polycyclic code over $\mathbb{F}_q$ induced by g_{ijk}, is a right divisor of f_{ijk}, for $0 \le i,\ j,\ k \le 3$. Then the code C can be described as

$$C = \langle \mu_1\rho_1\eta_1 g_{111}(x),\ \mu_1\rho_1\eta_2 g_{112}(x),\ \cdots,\ \mu_3\rho_3\eta_3 g_{333}(x)\rangle. \tag{18}$$

Proof. First, we observe that C consists of all elements of the form:

$$c(x) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \mu_i\rho_j\eta_k h_{ijk}(x) g_{ijk}(x),$$

where $h_{ijk}(x) \in \mathbb{F}_q[x;\ \varphi]$, clearly any such $c(x)$ can be expressed as a combination of $\langle \mu_1\rho_1\eta_1 g_{111}(x),\ \mu_1\rho_1\eta_2 g_{112}(x),\ \cdots,$ $\mu_3\rho_3\eta_3 g_{333}(x)\rangle$, so

$$C \subseteq \langle \mu_1\rho_1\eta_1 g_{111}(x),\ \mu_1\rho_1\eta_2 g_{112}(x),\ \cdots,\ \mu_3\rho_3\eta_3 g_{333}(x)\rangle. \tag{19}$$

Conversely, consider an arbitrary element of $\langle \mu_1\rho_1\eta_1 g_{111}(x),\ \mu_1\rho_1\eta_2 g_{112}(x),\ \cdots,\ \mu_3\rho_3\eta_3 g_{333}(x)\rangle$, it is written as follows

$$r(x) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \mu_i\rho_j\eta_k s_{ijk}(x) g_{ijk}(x),\ \text{where } s_{ijk} \in \mathfrak{R}[x,\ \varphi]. \tag{20}$$

Therefore, the element $\mu_i\rho_j\eta_k s_{ijk}(x)$, for $0 \le i,\ j,\ k \le 3$ can be simplified to $\mu_i\rho_j\eta_k h_{ijk}(x)$ for some $h_{ijk}(x) \in \mathbb{F}_q[x;\ \varphi]$. Thus, we have

$$C \supseteq \langle \mu_1\rho_1\eta_1 g_{111}(x),\ \mu_1\rho_1\eta_2 g_{112}(x),\ \cdots,\ \mu_3\rho_3\eta_3 g_{333}(x)\rangle. \tag{21}$$

□

Theorem 4.8 Suppose $f = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \mu_i\rho_j\eta_k f_{ijk}$ in $\mathfrak{R}[x,\ \varphi]$, where $f_{ijk} \in \mathbb{F}_q[x;\ \varphi]$.

Let $C = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \mu_i\rho_j\eta_k C_{ijk}$ represent a skew φ-polycyclic code over $\mathfrak{R}$, induced by a monic polynomial f, with $C_{ijk} = \langle g_{ijk}(x)\rangle$, where g_{ijk} is a right divisor of f_{ijk}, for $1 \le i,\ j,\ k \le 3$. Then the code $C = \langle g(x)\rangle$, where $g(x) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \mu_i\rho_j\eta_k g_{ijk}(x)$ serves as a right divisor of $f(x)$.

Proof. According to Theorem 4.6, the code C is a skew φ-polycyclic code over $\mathfrak{R}$, induced by a monic polynomial f if and only if each C_{ijk} is a skew φ-polycyclic code generated by a right divisor g_{ijk} of f_{ijk}, for $1 \le i,\ j,\ k \le 3$. We know that

$$C = \langle \mu_1\rho_1\eta_1 g_{111}(x),\ \mu_1\rho_1\eta_2 g_{112}(x),\ \cdots,\ \mu_3\rho_3\eta_3 g_{333}(x)\rangle,$$

define $g(x) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \mu_i\rho_j\eta_k g_{ijk}(x)$, and consequently $\langle g(x)\rangle$ is a subset of C. Simultaneously, for $1 \le i,\ j,\ k \le 3,\ \mu_i\rho_j\eta_k g_{ijk}(x) = \mu_i\rho_j\eta_k g(x)$, implying $C \subseteq \langle g(x)\rangle$. Therefore, $C = \langle g(x)\rangle$. Now, assume that f_{ijk} can be decomposed as $f_{ijk}(x) = h_{ijk}(x)g_{ijk}(x)$ for each $1 \le i,\ j,\ k \le 3$, and consider $h(x) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \mu_i\rho_j\eta_k h_{ijk}(x)$. Then

$$h(x)g(x) = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)\right)\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k g_{ijk}(x)\right)$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k h_{ijk}(x)g_{ijk}(x)$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k f_{ijk}(x)$$

$$= f(x).$$

This shows that g is the right divisor of f, thus completing the proof. □

Theorem 4.9 Let $C = \bigoplus_{i=1}^{3}\left[\bigoplus_{j=1}^{3}\left(\bigoplus_{k=1}^{3}\mu_i\rho_j\eta_k C_{ijk}\right)\right]$ be a skew polycyclic codes over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, then $\Phi(C)$ is a skew polycyclic code over $\mathbb{F}_q$.

Proof. The result is derived by applying (3) from Theorem 3.4. □

5. The dual of skew polycyclic codes over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$

This section explores the intricate structure of dual-skew polycyclic codes, focusing on their unique properties and classification. Additionally, we have established sufficient conditions for a skew polycyclic code to be self-dual.

Theorem 5.1 Let $C = \bigoplus_{i=1}^{3}\left[\bigoplus_{j=1}^{3}\left(\bigoplus_{k=1}^{3}\mu_i\rho_j\eta_k C_1^i \times C_2^j \times C_3^k\right)\right]$ be a skew polycyclic code over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, then $C^{\perp} = \bigoplus_{i=1}^{3}\left[\bigoplus_{j=1}^{3}\left(\bigoplus_{k=1}^{3}\mu_i\rho_j\eta_k (C_1^i)^{\perp} \times (C_2^j)^{\perp} \times (C_3^k)^{\perp}\right)\right]$, is a skew polycyclic code over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$.

Proof. Suppose $c = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k c_{ijk}$, where $c_{ijk} \in C_{ijk}$, for $1 \le i,\ j,\ k \le 3$. If $r_{ijk} \in C_{ijk}^{\perp}$, for $1 \le i,\ j,\ k \le 3$, then it follows that $c_{ijk} \cdot r_{ijk} = 0$, for $1 \le i,\ j,\ k \le 3$. Now, consider the vector $r = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k r_{ijk}$, then we have

$$c \cdot r = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k (c_{ijk} \cdot r_{ijk}) = 0,$$

which implies that $r \in C^{\perp}$.

Therefore, $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k (C_{ijk})^{\perp} \subseteq C^{\perp}$. Now, take any vector $r = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k r_{ijk} \in C^{\perp}$. Let c_{ijk} be any element from C_{ijk} for $1 \le i,\ j,\ k \le 3$, and consider $c = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k c_{ijk} \in C$. Then,

$$0 = c \cdot r = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k (c_{ijk} \cdot r_{ijk}),$$

which implies that $c_{ijk} \cdot r_{ijk} = 0$ for $1 \le i,\ j,\ k \le 3$. Consequently, $r_{ijk} \in (C_{ijk})^{\perp}$, so $r \in \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k (C_{ijk})^{\perp}$, and thus $C^{\perp} \subseteq \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k (C_{ijk})^{\perp}$. Note that this decomposition is unique, so this sum is direct. Thus, the desired result is obtained. □

Corollary 5.2 The code C is a self-dual skew polycyclic code over $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$ if and only if C_{ijk} are self-dual skew polycyclic codes over $\mathbb{F}_q$, for $0 \leq i,\ j,\ k \leq 3$.

Proof. Assume $C = \bigoplus_{i=1}^{3}\left[\bigoplus_{j=1}^{3}\left(\bigoplus_{k=1}^{3}\mu_i\rho_j\eta_k C_1^i \times C_2^j \times C_3^k\right)\right]$ is a linear code over $\mathfrak{R}$. Follow Theorem 5.1, the dual code of C is given by

$$C^{\perp} = \bigoplus_{i=1}^{3}\left[\bigoplus_{j=1}^{3}\left(\bigoplus_{k=1}^{3}\mu_i\rho_j\eta_k (C_1^i)^{\perp} \times (C_2^j)^{\perp} \times (C_3^k)^{\perp}\right)\right]. \tag{22}$$

If each C_{ijk} is self-dual, meaning $C_{ijk} = C_{ijk}^{\perp}$, for $1 \leq i,\ j,\ k \leq 3$, then

$$C^{\perp} = \bigoplus_{i=1}^{3}\left[\bigoplus_{j=1}^{3}\left(\bigoplus_{k=1}^{3}\mu_i\rho_j\eta_k C_1^i \times C_2^j \times C_3^k\right)\right] = C. \tag{23}$$

Thus, C is self-dual. Conversely, suppose C is self-dual, so $C = C^{\perp}$. Let $c_{ijk} \in C_{ijk}$, for $1 \leq i,\ j,\ k \leq 3$. Then, the element $c = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k c_{ijk} \in C$ involves that $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k c_{ijk} \in C^{\perp}$. This means $c_{ijk} \in C_{ijk}^{\perp}$, for $1 \leq i,\ j,\ k \leq 3$, so $C_{ijk} \subseteq C_{ijk}^{\perp}$, for $1 \leq i,\ j,\ k \leq 3$. To show the reverse inclusion, take any $c_{ijk} \in C_{ijk}^{\perp}$. Then, $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\mu_i\rho_j\eta_k c_{ijk} \in C^{\perp} = C$, implying $c_{ijk} \in C_{ijk}$, for $1 \leq i,\ j,\ k \leq 3$. Thus, $C_{ijk}^{\perp} \subseteq C_{ijk}$, confirming that C_{ijk} is self-dual, for $1 \leq i,\ j,\ k \leq 3$. □

6. Gray images of skew polycyclic codes with optimal parameters

Based on the reference [21], a linear code over a finite field is deemed to have optimal parameters if it meets specific bounds such as those given by Singleton, Griesmer, or Gilbert-Varshamov. These bounds are specified by the following formulas:

$$d \leq n - k + 1, \tag{24}$$

$$n \geq \sum_{i=0}^{k-1}\frac{d_H}{q^i} \tag{25}$$

and

$$A_q(n,\ d) \geq \frac{q^n}{\sum\limits_{i=0}^{d-1} C_n^i (q-1)^i}. \tag{26}$$

In this context, $A_q(n,\ d)$ denotes the maximum size of a q-ary code with length n and minimum distance d. In our research, we aim to construct codes with optimal parameters over the finite field $\mathbb{F}_q$, drawing from the rich structure of skew polycyclic codes over the ring $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$. This goal is driven by the practical need for error-correcting codes in finite field settings, often encountered in digital communication systems, cryptography, and various information processing applications. By utilizing tools such as Magma, Sage, and the database (http://www.codetables.de), we have discovered several codes with optimal parameters.

Example 6.1 Consider $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, where $q = 3$ the factorization of $x^{20} - 1 = (x+1)(x+2)(x^2+1)(x^4+x^3+2x+1)(x^4+x^3+x^2+x+1)(x^4+2x^3+x+1)(x^4+2x^3+x^2+2x+1)$. Note that for $0 \leq i,\ j,\ k \leq 3$, C_{ijk} denotes a skew polycyclic code over $\mathbb{F}_3$ defined by the generator polynomial $\langle x^8+x^7+2x^6+2x^5+2x^4+x^3+1 \rangle$, and C is a code characterized by the parameters $[20,\ 8,\ 9]$. In accordance with Theorems 3.2, 4.9 and Lemma 4.7, it can be concluded that $\Phi(C)$ $[180,\ 72,\ 81]$ forms a skew polycyclic code over $\mathbb{F}_3$ with good parameters.

Example 6.2 Consider $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, where $q = 4$. The factorization of $x^{18} - 1$ is given by: $x^{18} - 1 = (x+1)^2(x+\alpha)^2(x+\alpha^2)^2(x^3+\alpha)^2(x+\alpha^2)^2$. For $0 \leq i,\ j,\ k \leq 3$, let C_{ijk} denote a skew polycyclic code over $\mathbb{F}_4$ defined by the generator polynomial $\langle x^4+\alpha^2x^2+1 \rangle$. The code C is characterized by the parameters $[18,\ 4,\ 12]$. According to Theorems 3.2, 4.9 and Lemma 4.7, it can be concluded that $\Phi(C)$ with parameters $[162,\ 36,\ 108]$ forms a skew polycyclic code over $\mathbb{F}_4$ with good parameters.

Example 6.3 Consider $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, where $q = 8$. The factorization of $x^{38} - 1$ is given by: $x^{38} - 1 = (x+1)^2(x+\alpha)^2(x^6+\alpha^3x^5+\alpha^6x^4+\alpha^6x^3+\alpha^6x^2+\alpha^3x+1)^2(x+\alpha)^2(x^6+\alpha^5x^5+\alpha^3x^4+\alpha^3x^3+\alpha^3x^2+\alpha^5x+1)^2(x^6+\alpha^6x^5+\alpha^5x^4+\alpha^5x^3+\alpha^5x^2+\alpha^6x+1)^2$. For $0 \leq i,\ j,\ k \leq 3$, let C_{ijk} denote a skew polycyclic code over $\mathbb{F}_8$ defined by the generator polynomial $\langle \alpha^2x^{17}+\alpha^4x^{16}+\alpha^4x^{13}+\alpha^3x^{11}+\alpha^4x^{10}+\alpha^6x^9+\alpha^3x^8+\alpha^3x^7+x^6+\alpha^2x^5+\alpha^2x^4+\alpha x^3+\alpha^2x+\alpha^3 \rangle$. The code C is characterized by the parameters $[38,\ 12,\ 20]$. According to Theorems 3.2, 4.9 and Lemma 4.7, it can be concluded that $\Phi(C)$ with parameters $[342,\ 108,\ 180]$ forms a skew polycyclic code over $\mathbb{F}_8$ with good parameters.

Example 6.4 Consider $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, where $q = 9$. The factorization of $x^{58} - 1$ is given by: $x^{58} - 1 = (x+1)(x+2)(x^{14}+wx^{13}+x^{12}+w^2x^{11}+w^3x^{10}+2x^9+w^2x^8+w^2x^7+w^2x^6+2x^5+w^3x^4+w^2x^3+x^2+wx+1)(x^{14}+w^3x^{13}+x^{12}+w^6x^{11}+wx^{10}+2x^9+w^6x^8+w^6x^7+w^6x^6+2x^5+wx^4+w^6x^3+x^2+w^3x+1)(x^{14}+w^5x^{13}+x^{12}+w^6x^{11}+w^3x^{10}+x^9+w^2x^8+w^6x^7+w^2x^6+x^5+w^3x^4+w^6x^3+x^2+w^5x+1)(x^{14}+w^7x^{13}+x^{12}+w^2x^{11}+wx^{10}+x^9+w^6x^8+w^2x^7+w^6x^6+x^5+wx^4+w^2x^3+x^2+w^7x+1)$. For $0 \leq i,\ j,\ k \leq 3$, let C_{ijk} denote a skew polycyclic code over $\mathbb{F}_9$ defined by the generator polynomial $\langle x^{18}+w^6x^{17}+w^7x^{16}+x^{15}+w^7x^{14}+x^{13}+w^6x^{12}+w^5x^{11}+w^5x^{10}+w^2x^8+w^3x^7+w^5x^6+x^5+x^4+w^2x^3+w^2x^2+w^2x+w^5 \rangle$. The code C is characterized by the parameters $[58,\ 40,\ 12]$. According to Theorems 3.2, 4.9 and Lemma 4.7, it can be concluded that $\Phi(C)$ with parameters $[522,\ 360,\ 108]$ forms a skew polycyclic code over $\mathbb{F}_8$ with good parameters.

The tables below provides details of some optimal linear skew polycyclic codes derived using the Gray map Φ.

Table 1. Linear skew polycyclic codes $\Phi(C)$ with good parameters over $\mathbb{F}_3$

$C_{ijk}[n,\ k,\ d]$	$C_{ijk} = \langle g_{ijk}(x) \rangle,\ 0 \leq i,\ j,\ k \leq 3$	$\Phi(C)$	O
[32, 24, 5]	$\begin{pmatrix} x^{24}+x^{22}+x^{20}+x^{18}+x^{16}+x^{14}+x^{12} \\ +x^8+x^6+x^4+x^2+1 \end{pmatrix}$	[288, 216, 45]	yes
[48, 40, 4]	$\begin{pmatrix} x^{42}+2x^{41}+x^{40}+x^{38}+x^{36}+2x^{35} \\ +x^{34}+2x^{33}+x^{32}+x^{30}+2x^{29} \\ +x^{28}+2x^{27}+2x^{26}+x^{24}+2x^{23} \\ +x^{22}+2x^{21}+2x^{20}+2x^{18} \\ +x^{17}+2x^{16}+2x^{14}+x^{13}+2x^{12} \\ +2x^{11}+2x^9+x^8+x^6 \\ +2x^5+x^4+2x^3+x^2+1 \end{pmatrix}$	[432, 360, 36]	yes
[52, 24, 16]	$\left(x^{24}+x^{22}+x^{14}+2x^{12}+2x^8+2x^6+1\right)$	[468, 216, 144]	yes
[60, 30, 18]	$\left(x^{30}+x^{27}+x^{24}+x^{21}+x^{12}+x^9+x^6+2\right)$	[144, 64, 16]	yes

Table 2. Linear skew polycyclic codes $\Phi(C)$ with good parameters over $\mathbb{F}_4$

$C_{ijk}[n, k, d]$	$C_{ijk} = \langle g_{ijk}(x) \rangle, 0 \le i, j, k \le 3$	$\Phi(C)$	O
[22, 8, 11]	$\begin{pmatrix} x^{12}+\alpha x^{11}+\alpha^2 x^{10}+\alpha x^9+\alpha^2 x^8 \\ +\alpha x^7+\alpha^2 x^6+x^5+x^4+x^3+x^2+1 \end{pmatrix}$	[198, 72, 99]	yes
[35, 23, 8]	$\begin{pmatrix} x^{12}+x^{10}+\alpha^2 x^9+\alpha^2 x^8+\alpha^2 x^7 \\ +\alpha^2 x^6+\alpha^2 x^5+\alpha x^3+\alpha x^2 \\ +\alpha x+1 \end{pmatrix}$	[315, 207, 72]	yes
[65, 36, 15]	$\begin{pmatrix} x^{29}+\alpha^2 x^{28}+\alpha x^{27}+x^{26}+x^{25} \\ +\alpha x^{23}+\alpha^2 x^{20}+\alpha x^{19}+\alpha x^{18} \\ +\alpha^2 x^{17}+\alpha x^{16}+x^{15}+x^{14}+\alpha x^{13} \\ +\alpha^2 x^{12}+\alpha x^{11}+\alpha x^{10}+\alpha^2 x^9 \\ +\alpha x^6+x^4+x^3+\alpha x^2+\alpha^2 x+1 \end{pmatrix}$	[585, 324, 135]	yes
[85, 44, 20]	$\begin{pmatrix} x^{41}+\alpha^2 x^{40}+\alpha^2 x^{39}+\alpha^2 x^{38}+x^{37} \\ +\alpha^2 x^{36}+\alpha^2 x^{35}+x^{34}+\alpha^2 x^{33}+x^{32} \\ +\alpha x^{31}+\alpha x^{29}+x^{27}+x^{22}+\alpha x^{21} \\ +\alpha^2 x^{19}+\alpha x^{18}+x^{17}+\alpha x^{16}+x^{14} \\ +\alpha x^{13}+x^{12}+\alpha^2 x^{11}+\alpha x^{10}+x^9 \\ +\alpha x^8+\alpha^2 x^7+\alpha x^6+\alpha^2 x^5+\alpha^2 x^4 \\ +\alpha x^3+\alpha^2 x^2+x+1 \end{pmatrix}$	[765, 396, 180]	yes

Table 3. Linear skew polycyclic codes $\Phi(C)$ with good parameters over $\mathbb{F}_8$

$C_{ijk}[n, k, d]$	$C_{ijk} = \langle g_{ijk}(x) \rangle, 0 \le i, j, k \le 3$	$\Phi(C)$	O
[37, 25, 9]	$\begin{pmatrix} x^{12}+\alpha^5 x^{11}+\alpha^5 x^{10}+\alpha^2 x^9+\alpha^6 x^8 \\ +\alpha^2 x^7+\alpha^6 x^6+\alpha^2 x^5+\alpha^6 x^4+\alpha^2 x^3 \\ +\alpha^5 x^2+\alpha^5 x+1 \end{pmatrix}$	[333, 225, 81]	yes
[57, 50, 6]	$\begin{pmatrix} x^7+\alpha^4 x^6+\alpha x^5+\alpha^4 x^4+\alpha^4 x^3+\alpha x^2 \\ +\alpha^4 x+1 \end{pmatrix}$	[513, 450, 54]	yes
[73, 61, 8]	$\begin{pmatrix} x^{12}+x^{11}+\alpha^5 x^{10}+\alpha^5 x^9+\alpha^3 x^8 \\ +\alpha^2 x^6+\alpha^3 x^4+\alpha^5 x^3+\alpha^5 x^2+x+1 \end{pmatrix}$	[657, 549, 72]	yes
[95, 65, 16]	$\begin{pmatrix} x^{30}+\alpha^5 x^{29}+\alpha^6 x^{28}+\alpha^4 x^{27}+\alpha^3 x^{26} \\ +\alpha^2 x^{25}+\alpha^5 x^{24}+\alpha^3 x^{23}+\alpha^6 x^{22}+\alpha^6 x^{21} \\ +\alpha^3 x^{20}+\alpha^6 x^{19}+\alpha^3 x^{18}+\alpha^5 x^{17}+\alpha^6 x^{16} \\ +\alpha^4 x^{15}+\alpha^5 x^{14}+\alpha^3 x^{13}+\alpha^6 x^{12}+\alpha^5 x^{11} \\ +\alpha^3 x^{10}+\alpha^4 x^9+\alpha^3 x^8+\alpha^6 x^7+\alpha^4 x^6 \\ +\alpha^5 x^5+\alpha^4 x^4+\alpha^3 x^3+\alpha^5 x^2 \\ +\alpha^4 x+1 \end{pmatrix}$	[144, 64, 16]	yes

6.1 *Practical application: optimal skew polycyclic code in secret sharing schemes*

Skew polycyclic codes offer a robust framework for secret sharing, leveraging their unique structural properties to enhance security and robustness. The generation of secret shares enables effective integration of these codes into secret-sharing mechanisms, providing built-in error detection and correction and offering enhanced security features that make the system more resilient to attacks. Moreover, polycyclic codes are well-suited for implementing multi-secret sharing schemes, where multiple secrets are shared simultaneously. Their applicability extends to several cryptographic contexts, such as threshold cryptography, complex access structures, distributed key management, and the emerging field of DNA-based multi-secret sharing schemes. In cases where standard secret-sharing schemes might prove inadequate, polycyclic codes provide a more efficient and secure alternative, ensuring that shared secrets remain protected even in distributed or high-risk environments.

6.1.1 *Secret sharing context*

According to [3, 5], secret sharing schemes are employed to split a secret s into multiple shares for distribution among participants. The objective is to guarantee that the secret can only be reconstituted when a minimum number of participants gather and combine their shares. This requirement is known as the reconstruction threshold. Additionally, the properties of secret-sharing schemes include confidentiality and robustness. Confidentiality ensures that several shares below the threshold do not reveal any information about the secret. Robustness means that the secret should be able to be reconstructed even if some shares are lost or corrupted. Optimal codes of length n and dimension k possess properties that make them excellent candidates for secret-sharing schemes because they enable efficient information encoding while ensuring error detection and correction capabilities. Their algebraic properties guarantee that any group of k participants can reconstruct the secret, while any subset of fewer than k participants does not provide enough information to deduce the secret. Suppose we want to securely share a password among n members of an organization, such that the password (the secret) can only be recovered if at least k members come together. If fewer than k members gather, they should not be able to retrieve the password. In the case of corruption of some shares (up to e shares), it should be possible to identify and fix errors before reconstructing the password. To achieve this, we will use an optimal polynomial code of length 32 and dimension 24 over the finite field $\mathbb{F}_3$.

6.1.2 *Construction and application steps*

The construction and application steps involve defining the mathematical structure, generating the necessary codewords, and distributing these codewords as secret shares.

• The secret s is a vector of 24 symbols, each belonging to the field $\mathbb{F}_3$. For example, let's consider the codeword $s = 201210221002101221120121$.

• The vector represents the password we wish to share with the 32 members.

• We use a generator matrix G of the optimal polycyclic code of length $n = 32$ and dimension $k = 24$, where

$$G = \begin{pmatrix} 20121022100211022112012110110210 \\ 01210221002110221120121101102102 \\ 12102210021102211201211011021020 \\ 21022100211022112012110110210201 \\ 10221002110221120121101102102012 \\ 02210021102211201211011021020121 \\ 22100211022112012110110210201210 \\ 21002110221120121101102102012102 \\ 10021102211201211011021020121022 \\ 00211022112012110110210201210221 \\ 02110221120121101102102012102210 \\ 21102211201211011021020121022100 \\ 11022112012110110210201210221002 \\ 10221120121101102102012102210021 \\ 02211201211011021020121022100211 \\ 22112012110110210201210221002110 \\ 21120121101102102012102210021102 \\ 11201211011021020121022100211022 \\ 12012110110210201210221002110221 \\ 20121101102102012102210021102211 \\ 01211011021020121022100211022112 \\ 12110110210201210221002110221120 \\ 21101102102012102210021102211201 \\ 11011021020121022100211022112012 \end{pmatrix}.$$

The matrix G is selected so that its rows are vectors of the polycyclic code and allow encoding messages of 24 symbols into codewords of 32 symbols.

• The codeword c is obtained by multiplying the vector s by the matrix G:

$$c = 20121022100211022112012110110210.$$

• Each share of the secret corresponds to a component of the codeword c.

• We have 32 participants, each receiving a share corresponding to one of the 32 symbols of c.

• This scheme is designed so that any 24 participants (or more) can reconstruct the secret.

• The 24 combined shares can be used to solve the system of linear equations given by the generator matrix G, allowing the recovery of the vector s.

• If fewer than 24 shares are combined, it is impossible to reconstruct s, as there are too many possible solutions to the linear equation.

6.1.3 *Error detection and correction*

The use of skew polycyclic code in secret sharing not only ensures secure distribution of shares but also enables effective error detection and correction. If a share is corrupted or altered, the code's structure can identify and correct the error, ensuring that the original secret can still be accurately reconstructed.

• If some shares are corrupted or incorrect (e.g., due to transmission errors), the properties of the polycyclic code allow for calculating a syndrome and locating the errors.

• For example, if the codeword received by 24 participants is:

$$\overline{c} = 20121022100211022112010110110220$$

• The parity check matrix H, where

$$H = \begin{pmatrix} 10212011200122011221021210000000 \\ 02120112001220112210212201000000 \\ 21201120012201122102122000100000 \\ 12011200122011221021220200010000 \\ 20112001220112210212202200001000 \\ 01120012201122102122022000000100 \\ 11200122011221021220220100000010 \\ 12001220112210212202201200000001 \end{pmatrix}$$

allows for calculating the syndrome corresponding to $\overline{c}$, which indicates the position of the error in the 23^{rd} share, where parity-check matrix H must be orthogonal to the generator matrix G, meaning that $G \cdot H^T = 0$, which guarantees that valid codewords lie in the null space of H. Additionally, the control matrix should have a rank, ensuring the number of independent rows equals the number of parity checks, thereby maximizing error-detection capabilities. Furthermore, the rows of H must be linearly independent to avoid redundant parity checks and ensure that distinct error patterns produce unique error syndromes. These conditions are essential for the code's functionality in detecting and correcting errors.

• The code is corrected, and s is reconstructed with the correct shares. Fewer than 24 shares provide no information about s. This protects the secret against any attempts at reconstruction by unauthorized subsets.

• The secret can be reconstructed even if up to 8 shares are lost or corrupted.

• Error correction ensures the integrity of the received information before reconstruction.

6.1.4 *Real-world use case: secret sharing in cryptography*

This scheme can be used to share a cryptographic key or other sensitive information among multiple parties within an organization. Here's an example:

• The financial system uses a secret sharing scheme to distribute a digital signature key among 32 members of the board of directors.

• If at least 24 members come together, they can reconstruct the key and validate the transaction.

• To ensure that any transaction is approved by a qualified majority, only 24 members can reconstruct the key.

7. Conclusion

In this article, we explored the structure and properties of linear codes over the ring $\mathfrak{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3$, focusing specifically on skew polycyclic codes. The study began by investigating the fundamental properties of linear codes within this algebraic framework. We then delved into skew polycyclic codes over $\mathfrak{R}$, demonstrating their significance and utility in coding theory. These codes, which generalize classical polycyclic codes by incorporating polynomial rings, possess rich algebraic structures that make them suitable for various applications. The analysis was further extended to the duals of skew polycyclic codes over $\mathfrak{R}$, revealing significant duality relationships that can be used to construct new codes with good properties. Additionally, a practical example demonstrated the effectiveness of skew polycyclic codes in secure secret sharing schemes, emphasizing their utility in cryptographic contexts like distributed key management.

Conflict of interest

The authors declare no competing financial interest.

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