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Research Article

Decision-Making of Fredholm Operator on a New Variable Exponents Sequence Space of Supply Fuzzy Functions Defined by Leonardo Numbers

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Abstract: In this article, we will use a weighted regular matrix formed by Leonardo numbers and variable exponent sequence spaces to build a new stochastic space. We have proposed various geometric and topological structures for this new space, as well as the multiplication operator that operates on it.

Keywords: leonardo numbers, variable exponent, extended s-fuzzy numbers, multiplication operator, non-newtonian fluids

MSC: 46B15, 46C05, 46E05

Abbreviation

p-$q.N$	Pre-quasi norm
$\mathfrak{pssf}$	Private sequence space of fuzzy functions
p-m	Pre-modular
p-$q.B$	Pre-quasi Banach
Bs	Banach space
Cs	Cauchy sequence
CMs	Complete metric space
$\mathscr{M}.\mathscr{O}$	Multiplication operator
$Iy.O$	Isometry operator
$Iv.O$	Invertible operator
$\mathscr{C}.\mathscr{R}$	Closed range

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$Fr.O$ Fredholm operator.

Notations

$\mathscr{N} := \{0, 1, 2, ...\}$ and $\mathbf{R}$ is the set of real numbers.

$\mathbf{R}^{+\mathscr{N}}$: The space of all sequences of positive reals.

ℓ_m, ℓ_∞, and c_0: The spaces of m-absolutely summable, bounded, and convergent to zero sequences of reals, respectively.

b^ν: For $\nu \in (0, 1)$, the ν-level set of a fuzzy real b is defined by Matloka [1] as

$$b^\nu = \{y \in \mathbf{R} : b(y) \geq \nu\}.$$

$\mathbf{R}^{[0, 1]}$: The set of all convex fuzzy number, normal, upper semi-continuous, and b^ν is compact.

For $m \in \mathbf{R}^{[0, 1]}$, we have

$$\overline{m}(k) = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$$

μ^F: The space of all sequences of fuzzy reals.

$\mathfrak{G}$, $\mathfrak{V}$: Infinite dimensional Banach spaces.

Γ: Banach space of one dimension.

$\mathfrak{Bd} \Uparrow_{\mathfrak{G}}^{\mathfrak{V}}$, $\mathfrak{Ft} \Uparrow_{\mathfrak{G}}^{\mathfrak{V}}$,

$\mathfrak{Pt} \Uparrow_{\mathfrak{G}}^{\mathfrak{V}}$, and $\mathfrak{Ct} \Uparrow_{\mathfrak{G}}^{\mathfrak{V}}$: The space of all bounded, finite rank, approximable, and compact bounded linear mappings from $\mathfrak{G}$ into

$\mathfrak{V}$, respectively.

$\mathfrak{Bd} \Uparrow_{\mathfrak{G}}$, $\mathfrak{Ft} \Uparrow_{\mathfrak{G}}$, $\mathfrak{Pt} \Uparrow_{\mathfrak{G}}$, and $\mathfrak{Ct} \Uparrow_{\mathfrak{G}}$: The space of all bounded, finite rank, approximable, and compact bounded linear mappings from $\mathfrak{G}$ into itself, respectively.

$\mathfrak{Bd}$, $\mathfrak{Ft}$, $\mathfrak{Pt}$, and $\mathfrak{Ct}$: The ideal of bounded, finite rank, approximable and compact mappings between any arbitrary Banach spaces, respectively.

$\mathscr{E}^F$: The linear space of sequences of fuzzy functions.

$\overline{\mathfrak{e}_d} := (\overline{0}, \overline{0}, ..., \overline{1}, \overline{0}, \overline{0}, \cdots)$, while $\overline{1}$ displays at the d^{th} place.

$[d]$: The integral part of real number d.

$\mathscr{F}$: The space of finite sequences of fuzzy numbers.

$\mathfrak{N}_+$ and $\mathfrak{D}_-$: The space of all monotonic increasing and decreasing sequences of positive reals, respectively.

$\mathfrak{I}$: The space of all sets with finite number of elements.

ℓ_∞^F: The space of bounded sequences of fuzzy functions.

$(\mathscr{R}(U))^c$: The complement of $Range(U)$.

1. Introduction

The study of variable exponent Lebesgue spaces has gained more momentum due to its application in the mathematical modeling of non-Newtonian fluids in hydrodynamics, as discussed by Ružîĉka [2]. The utilization of electrorheological fluids, which are a type of non-Newtonian fluids, spans across diverse fields such as military science, civil engineering, and orthopedics. Diening et al. [3] discussed Lebesgue and Sobolev spaces with variable exponents.

We consider the following interesting references in which the norms and Lebesgue measures are considered for fluid applications [4–8]. The solution of discrete dynamical systems is contained in a specific sequence space. So there is a great interest in mathematics to construct new sequence spaces, see [9]. Mursaleen and Noman [10] examined some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞, and Mursaleen and Başar [11] constructed and investigated the domain of Cesàro mean of order one in some spaces of double sequences. Mustafa and Bakery [12] introduced the concept of $\mathfrak{pssf}$. They constructed the operators' ideal by a weighted binomial matrix in the Nakano sequence space of extended s-fuzzy functions. Komal et al. [13], investigated the multiplication operators acting on Cesàro sequence spaces under the Luxemburg norm. The multiplication operators acting on Cesàro second order function spaces examined by İlkhan et al. [14]. The aim of this paper is to construct a novel stochastic space using a weighted regular matrix defined by Leonardo numbers and variable exponent sequence spaces. We have provided certain geometric and topological structures to fuzzy functions, of the multiplication maps acting on it.

2. Definitions and preliminaries

Definition 2.1 [12] $\mathscr{E}^F$ is called a $\mathfrak{pssf}$, if it satisfies the next setups:

(1c) $\mathscr{E}^F$ is linear space and $\overline{\mathfrak{e}_r} \in \mathscr{E}^F$, for $r \in \mathscr{N}$,

(2c) $\mathscr{E}^F$ is solid i.e., for $\overline{m} = (\overline{m_r}) \in \mu^F$, $|\overline{k}| = (|\overline{k_r}|) \in \mathscr{E}^F$ and $|\overline{m_r}| \le |\overline{k_r}|$, where $r \in \mathscr{N}$, then $|\overline{m}| \in \mathscr{E}^F$,

(3c) $\left(\left|\overline{k_{[\frac{r}{2}]}}\right|\right)_{r\in\mathscr{N}} \in \mathscr{E}^F$, whenever $\left(|\overline{k_x}|\right)_{r\in\mathscr{N}} \in \mathscr{E}^F$.

Definition 2.2 [15] A subspace $\mathfrak{pssf}$ $\mathscr{E}^F_{\|\cdot\|_{p-qN}}$ is called a ***p-m*** $\mathfrak{pssf}$, if $\|.\|_{p-qN} : \mathscr{E}^F \to [0, \infty)$ holds the next setups for every $\overline{m}$, $\overline{k} \in \mathscr{E}^F$, and $\delta \in \mathbf{R}$:

(a1) $\overline{k} = \overline{\vartheta} \iff \|(|\overline{k}|)\|_{p-qN} = 0$, and $\|\overline{k}\|_{p-qN} \ge 0$,

(a2) one gets $C_1 \ge 1$ under $\|\delta\overline{m}\|_{p-qN} \le |\delta| C_1 \|\overline{m}\|_{p-qN}$,

(a3) $\|\overline{m}+\overline{k}\|_{p-qN} \le C_2(\|\overline{m}\|_{p-qN} + \|\overline{k}\|_{p-qN})$ holds with $C_2 \ge 1$,

(a4) for $|\overline{m_r}| \le |\overline{k_r}|$, we have $\|(|\overline{m_r}|)\|_{p-qN} \le \|(|\overline{k_r}|)\|_{p-qN}$,

(a5) the inequality, $\|(|\overline{k_r}|)\|_{p-qN} \le \|(|\overline{k_{[\frac{r}{2}]}}|)\|_{p-qN} \le C_3\|(|\overline{k_r}|)\|_{p-qN}$ verifies, for $C_3 \ge 1$,

(a6) the closure of $\mathscr{F} = \mathscr{E}^F_{\|\cdot\|_{p-qN}}$,

(a7) the inequality, $\|(\overline{m}, \overline{0}, \overline{0}, \overline{0}, \ldots)\|_{p-qN} \ge \alpha|m|\|\overline{\mathfrak{e}_1}\|_{p-qN}$ holds for $\alpha > 0$.

Definition 2.3 [15] The $\mathfrak{pssf}$ $\mathscr{E}^F_{\|\cdot\|_{p-qN}}$ is called a ***p-q.N*** $\mathfrak{pssf}$, if $\|.\|_{p-qN}$ confirms the setups (a1)-(a3) of Definition 2. If $\mathfrak{pssf}$ $\mathscr{E}^F_{\|\cdot\|_{p-qN}}$ is complete ***p-q.N*** $\mathfrak{pssf}$, then $\mathscr{E}^F_{\|\cdot\|_{p-qN}}$ is said to be a ***p-q.B*** $\mathfrak{pssf}$.

Theorem 2.4 [12] Each ***p-m*** $\mathfrak{pssf}$ $\mathscr{E}^F_{\|\cdot\|_{p-qN}}$ is a ***p-q.N*** $\mathfrak{pssf}$.

Definition 2.5 [15] Suppose $\lambda = (\lambda_k) \in \mathbf{R}^{\mathscr{N}}$ and $\mathscr{E}^F_{\|\cdot\|_{p-qN}}$ is a ***p-q.N*** $\mathfrak{pssf}$. The operator $H_\lambda : \mathscr{E}^F_{\|\cdot\|_{p-qN}} \to \mathscr{E}^F_{\|\cdot\|_{p-qN}}$ is named a $\mathscr{M}.\mathscr{O}$ on $\mathscr{E}^F_{\|\cdot\|_{p-qN}}$, if $H_\lambda\overline{f} = \left(\lambda_b\overline{f_b}\right) \in \mathscr{E}^F_{\|\cdot\|_{p-qN}}$, with $f \in \mathscr{E}^F_{\|\cdot\|_{p-qN}}$. The $\mathscr{M}.\mathscr{O}$ is named created by λ, if $H_\lambda \in \mathfrak{B}\mathfrak{d}(\mathscr{E}^F_{\|\cdot\|_{p-qN}})$.

Definition 2.6 [16] For $X \in \mathfrak{B}\mathfrak{d} \Uparrow_{\mathscr{E}}$ is called ***Fr.O*** if $\mathscr{R}(U)$ is closed, $\dim(\ker(U)) < \infty$, and $\dim(\mathscr{R}(U))^c < \infty$.

Theorem 2.7 [17] For a ***Bs*** $\mathscr{E}^F$ under $\dim(\mathscr{E}^F) = \infty$, one has

$$\mathfrak{Ft} \Uparrow_{\mathscr{E}^F} \subsetneqq \mathfrak{Pt} \Uparrow_{\mathscr{E}^F} \subsetneqq \mathfrak{Ct} \Uparrow_{\mathscr{E}^F} \subsetneqq \mathfrak{B}\mathfrak{d} \Uparrow_{\mathscr{E}^F} .$$

Lemma 2.8 [18] Assume $r_m > 1$ and $\alpha_m,\ \delta_m \in \mathbf{R}$, for every $m \in \mathscr{N}$, and $\beth = \sup_m r_m$, then

$$|\alpha_m + \delta_m|^{r_m} \le 2^{\beth-1}\left(|\alpha_m|^{r_m} + |\delta_m|^{r_m}\right). \tag{1}$$

3. Configuration and properties of $\left(\gamma_{\mathfrak{r}}^{F}(q,t)\right)_{\|.\|_{p-qN}}$

In this section, we introduce the definition and some inclusion relations of the sequence space $\left(\gamma_{\mathfrak{r}}^{F}(q,\, t)\right)_{\|.\|_{p-qN}}$ equipped with the function $\|.\|_{p-qN}$.
Assume $\mathfrak{r}_l$, $l \in \mathscr{N}$ mark the l^{th} Leonardo number. Where, the Leonardo numbers are defined as:

$$\mathfrak{r}_0 = \mathfrak{r}_1 = 1, \mathfrak{r}_l = \mathfrak{r}_{l-1} + \mathfrak{r}_{l-2} + 1,\ l \geq 2.$$

Catarino and Borges [19] proved that: $\sum_{k=0}^{v} \mathfrak{r}_v = \mathfrak{r}_{v+2} - (v+2),\ v \in \mathscr{N}$.
We have presented a novel stochastic space $\left(\gamma_{\mathfrak{r}}^{F}(q,\, t)\right)_{\|.\|_{p-qN}}$ of fuzzy functions.
Definition 3.1 If $(t_l),\ (q_l) \in \mathbf{R}^{+\mathscr{N}}$.

$$\left(\gamma_{\mathfrak{r}}^{F}(q,\, t)\right)_{\|.\|_{p-qN}} := \left\{\bar{d} = (\overline{d_b}) \in \mu^F : \|\delta\bar{d}\|_{p-qN} < \infty,\ \text{ for some } \delta > 0\right\},$$

where

$$\|\bar{d}\|_{p-qN} = \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \overline{d_z},\ \bar{0}\right)}{\mathfrak{r}_{l+2} - (l+2)}\right)^{t_l},$$

$$\bar{\hbar}(\bar{k},\, \bar{m}) = \sup_{0\leq\beta\leq1} \max\left\{\left|\bar{k}_1^{\beta} - \bar{m}_1^{\beta}\right|,\ \left|\bar{k}_2^{\beta} - \bar{m}_2^{\beta}\right|\right\}$$

and

$$\bar{k}^{\beta} = [\bar{k}_1^{\beta},\, \bar{k}_2^{\beta}],\ \ \overline{m}^{\beta} = [\overline{m}_1^{\beta},\, \overline{m}_2^{\beta}] \in \mathbf{R}^{[0,\ 1]}.$$

Clearly, when $(t_l) \in \mathbf{R}^{+\mathscr{N}} \cap \ell_\infty$, one has

$$\left(\gamma_{\mathfrak{r}}^{F}(q,\, t)\right)_{\|.\|_{p-qN}} = \left\{\bar{d} = (\overline{d_b}) \in \mu^F : \|\delta\bar{d}\|_{p-qN} < \infty,\ \text{for any } \delta > 0\right\}.$$

In [20], Yaying et al., studied new Banach sequence spaces involving Leonardo numbers and its associated mappings ideal.
Theorem 3.2 The space $\left(\gamma_{\mathfrak{r}}^{F}(q,\, t)\right)_{\|.\|_{p-qN}}$ is a ***NAT***, whenever $(t_l) \in [1,\, \infty)^{\mathscr{N}} \cap \ell_\infty$.
Proof. Evidently, since

$$\|\overline{\mathfrak{e}_0} - \overline{\mathfrak{e}_1}\|_{p-qN} = (q_0)^{t_0} + \left(\frac{|q_0 - q_1|}{2}\right)^{t_1} + \left(\frac{|q_0 - q_1|}{5}\right)^{t_2} + \cdots$$

$$\neq (q_0)^{t_0} + \left(\frac{|q_0+q_1|}{2}\right)^{t_1} + \left(\frac{|q_0+q_1|}{5}\right)^{t_2} + \cdots = \|(|\overline{\mathfrak{e}_0} - \overline{\mathfrak{e}_1}|)\|_{p-qN}.$$

Theorem 3.3 Assume $t_l \geq 1$ and $(t_l) \in \mathbf{R}^{+\mathscr{N}}$, for any $l \in \mathscr{N}$.

$$\left(|\gamma_{\mathfrak{r}}^{F}|(q,\, t)\right)_{\varphi} := \left\{\overline{f} = (\overline{f_k}) \in \mu^F : \varphi(\delta f) < \infty, \text{ for some } \delta > 0\right\},$$

where

$$\varphi(\overline{f}) = \sum_{l=0}^{\infty}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{f_z}|,\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}.$$

Theorem 3.4 Suppose $(t_l) \in (1,\, \infty)^{\mathscr{N}} \cap \ell_\infty$ with $\left(\frac{l+1}{\mathfrak{r}_{l+2}-(l+2)}\right) \notin \ell_{(t_l)}$, hence $\left(|\gamma_{\mathfrak{r}}^{F}|(q,\, t)\right)_{\varphi} \subsetneqq \left(\gamma_{\mathfrak{r}}^{F}(q,\, t)\right)_{\|\cdot\|_{p-qN}}$.
Proof. Assume $\overline{d} \in \left(|\gamma_{\mathfrak{r}}^{F}|(q,\, t)\right)_{\varphi}$, one gets

$$\sum_{l\in\mathscr{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\overline{d_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} \leq \sum_{l\in\mathscr{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{d_z}|,\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} < \infty.$$

So, $\overline{d} \in \left(\gamma_{\mathfrak{r}}^{F}(q,\, t)\right)_{\|\cdot\|_{p-qN}}$. Let $\overline{f} = \left(\frac{(\overline{-1})^z}{\mathfrak{r}_z q_z}\right)_{z\in\mathscr{N}}$, then $\overline{f} \in \left(\gamma_{\mathfrak{r}}^{F}(q,\, t)\right)_{\|\cdot\|_{p-qN}}$ and $\overline{f} \notin \left(|\gamma_{\mathfrak{r}}^{F}|(q,\, t)\right)_{\varphi}$. □
In this part we give the suffient settings on $\gamma_{\mathfrak{r}}^{F}(q,\, t)$ to be a ***p-q.B*** 𝔭𝔰𝔰𝔣.
Theorem 3.5 $\gamma_{\mathfrak{r}}^{F}(q,\, t)$ is a ***p-m*** 𝔭𝔰𝔰𝔣, whenever
(o1) $(t_l) \in \mathfrak{N}_+ \cap \ell_\infty$ and $t_0 > 1$.
(o2) $(\mathfrak{r}_z q_z)_{z\in\mathscr{N}} \in \mathfrak{D}_-$ or, $(\mathfrak{r}_z q_z)_{z\in\mathscr{N}} \in \mathfrak{N}_+ \cap \ell_\infty$ and one has $A \geq 1$ such that $\mathfrak{r}_{2z+1}q_{2z+1} \leq A\mathfrak{r}_z q_z$.
Proof. Let $\overline{d},\, \overline{k} \in \gamma_{\mathfrak{r}}^{F}(q,\, t)$, and $\delta \in \mathbf{R}$. Suppose the conditions (o1) and (o2) are satisfied.
The part (a1): Definitely, $\|\overline{d}\|_{p-qN} \geq 0$ and $\|(|\overline{d}|)\|_{p-qN} = 0 \Leftrightarrow \overline{d} = \overline{\vartheta}$.
The parts (1c) and (a3):

$$\begin{aligned}
\|\overline{d}+\overline{k}\|_{p-qN} &= \sum_{l\in\mathscr{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\left(\overline{d_z}+\overline{k_z}\right),\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} \\
&\leq 2^{\beth-1}\left(\sum_{l\in\mathscr{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\overline{d_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} + \sum_{l\in\mathscr{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\overline{k_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\right) \\
&= C_2(\|\overline{d}\|_{p-qN} + \|\overline{k}\|_{p-qN}) < \infty,
\end{aligned}$$

hence, $\overline{d}+\overline{k} \in \gamma_{\mathfrak{r}}^{F}(q,\, t)$.
The parts (1c) and (a2):

$$\|\delta\overline{d}\|_{p-qN}=\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\delta\overline{d_z},\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\leq\sup_l|\delta|^{t_l}\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\overline{d_z},\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}=C_1\|\overline{d}\|_{p-qN}<\infty.$$

So, $\delta\overline{d}\in\gamma_{\mathfrak{r}}^{F}(q,\,t)$. Hence $\gamma_{\mathfrak{r}}^{F}(q,\,t)$ is a linear space. Also

$$\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\overline{(e_b)_z},\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}=\sum_{l=b}^{\infty}\left(\frac{\mathfrak{r}_b q_b}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\leq\sup_{l=b}^{\infty}(\mathfrak{r}_b q_b)^{t_l}\sum_{l=b}^{\infty}\left(\frac{1}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}<\infty.$$

Therefore, $\overline{\mathfrak{e}_b}\in\gamma_{\mathfrak{r}}^{F}(q,\,t)$, for every $b\in\mathcal{N}$.

The parts (2c) and (a4): Let $|\overline{d_b}|\leq|\overline{k_b}|$, for $b\in\mathcal{N}$ and $|\overline{k}|\in\gamma_{\mathfrak{r}}^{F}(q,\,t)$. Then

$$\|(|\overline{d}|)\|_{p-qN}=\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{d_z}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\leq\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{k_z}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}=\|(|\overline{k}|)\|_{p-qN}<\infty,$$

so $|\overline{d}|\in\gamma_{\mathfrak{r}}^{F}(q,\,t)$.

The parts (3c) and (a5): Assume $(|\overline{d_z}|)\in\gamma_{\mathfrak{r}}^{F}(q,\,t)$ and $(\mathfrak{r}_z q_z)_{z\in\mathcal{N}}\in\mathfrak{D}_{-}$, we get

$$\begin{aligned}
\|(|\overline{d_{[\frac{z}{2}]}}|)\|_{p-qN}&=\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{d_{[\frac{z}{2}]}}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\\
&=\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{2l}\mathfrak{r}_z q_z|\overline{d_{[\frac{z}{2}]}}|,\,\overline{0}\right)}{\mathfrak{r}_{2l+2}-(2l+2)}\right)^{t_{2l}}+\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{2l+1}\mathfrak{r}_z q_z|\overline{d_{[\frac{z}{2}]}}|,\,\overline{0}\right)}{\mathfrak{r}_{2l+3}-(2l+3)}\right)^{t_{2l+1}}\\
&\leq\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{2l}\mathfrak{r}_z q_z|\overline{d_{[\frac{z}{2}]}}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}+\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{2l+1}\mathfrak{r}_z q_z|\overline{d_{[\frac{z}{2}]}}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\\
&\leq\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\mathfrak{r}_{2l}q_{2l}|\overline{d_l}|+\sum_{z=0}^{l}\left(\mathfrak{r}_{2z}q_{2z}+\mathfrak{r}_{2z+1}q_{2z+1}\right)|\overline{d_z}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\\
&\quad+\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\left(\mathfrak{r}_{2z}q_{2z}+\mathfrak{r}_{2z+1}q_{2z+1}\right)|\overline{d_z}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\\
&\leq 2^{\beth-1}\left(\sum_{l\in\mathcal{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{d_z}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}+\sum_{l\in\mathcal{N}}\left(\frac{2\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{d_z}|,\,\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}\right)
\end{aligned}$$

$$+\sum_{l\in\mathscr{N}}\left(\frac{2\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{d_z}|,\ \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}$$

$$\leq(2^{2\daleth-1}+2^{\daleth-1}+2^{\daleth})\sum_{l\in\mathscr{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z|\overline{d_z}|,\ \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}=C_3\|(|\overline{d_z}|)\|_{p-qN}<\infty,$$

hence $(|\overline{d_{[\frac{z}{2}]}}|)\in\gamma_{\mathfrak{r}}^{F}(q,\ t)$.

Obviously, the parts (a6) and (a7) can be easily proven. $\square$

Theorem 3.6 The sequence space $(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}$ is a ***p-q.B*** $\mathfrak{pssf}$.

Proof. From Theorem 3.5, one has $(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}$ is a ***p-q.N*** $\mathfrak{pssf}$. To explain that $(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}$ is a ***p-q.B*** $\mathfrak{pssf}$, let $\overline{f^a}=(\overline{f_z^a})_{z\in\mathscr{N}}$ be a ***Cs*** in $(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}$, hence for $\lambda\in(0,\ 1)$, one has $m_0\in\mathscr{N}$ for any $m,\ j\geq z_0$, then

$$\|\overline{d^m}-\overline{d^j}\|_{p-qN}=\sum_{l\in\mathscr{N}}\left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\left(\overline{d_z^{z}}-\overline{d_z^{j}}\right),\overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}<\lambda^{\daleth}.$$

That gives

$$\bar{\hbar}\left(\sum_{z=0}^{l}\mathfrak{r}_z q_z\left(\overline{d_z^{m}}-\overline{d_z^{j}}\right),\overline{0}\right)<\lambda.$$

As $(\mathbf{R}^{[0,\ 1]},\ \bar{\hbar})$ is a ***CMs***. So $(\overline{d_z^j})$ is a ***Cs*** in $\mathbf{R}^{[0,\ 1]}$, for fixed $z\in\mathscr{N}$. Therefore, $\|\overline{d^m}-\overline{d^0}\|_{p-qN}<\lambda^{\daleth}$, for every $m\geq m_0$. Clearly from the linearity, $\overline{d^0}\in(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}$. $\square$

4. $\mathscr{M}.\mathscr{O}$s on $(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}$

Under the conditions of theorem 3.5. We discuss $\mathscr{M}.\mathscr{O}$ defined on $(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}$ to be bounded, ***Iv.O***, approximable, ***Fr.O*** and $\mathscr{C}.\mathscr{R}$.

Assume that $\lambda\in\mathbf{R}^{\mathscr{N}}$.

Theorem 4.1 The following are satisfied:

(m1) $\lambda\in\ell_\infty\Longleftrightarrow H_\lambda\in\mathfrak{Bd}\Uparrow_{(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}}$.

(m2) $|\lambda_b|=1$, for any $b\in\mathscr{N}\Longleftrightarrow H_\lambda$ is an ***Iy.O***.

(m3) $H_\lambda\in\mathfrak{Pt}\Uparrow_{(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}}\Longleftrightarrow(\lambda_j)_{j\in\mathscr{N}}\in c_0$.

(m4) $H_\lambda\in\mathfrak{Ct}\Uparrow_{(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}}\Longleftrightarrow(\lambda_j)_{j\in\mathscr{N}}\in c_0$.

(m5) $\mathfrak{Ct}\Uparrow_{(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}}\subsetneqq\mathfrak{Bd}\Uparrow_{(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}}$.

(m6) If $H_\lambda\in\mathfrak{Bd}\Uparrow_{(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}}$. Then we have $\omega_1,\ \omega_2>0$ with $\omega_1<|\lambda_l|<\omega_2$, for $l\in(\ker(\lambda))^c\Longleftrightarrow\mathscr{R}(H_\lambda)$ is $\mathscr{C}.\mathscr{R}$.

(m7) One has $\omega_1,\ \omega_2>0$ with $\omega_1<|\lambda_l|<\omega_2$, for any $l\in\mathscr{N}\Longleftrightarrow H_\lambda\in\mathfrak{Bd}\Uparrow_{(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}}$ is ***Iv.O***.

(m8) If $H_\lambda\in\mathfrak{Bd}\Uparrow_{(\gamma_{\mathfrak{r}}^{F}(q,\ t))_{\|.\|_{p-qN}}}$. Then H_λ is ***Fr.O***$\Longleftrightarrow$ (o1) $\ker(\lambda)\subsetneqq\mathscr{N}\cap\mathfrak{I}$ and (o2) $|\lambda_l|\geq\rho$, for any $l\in(\ker(\lambda))^c$.

Proof. The part (m1): $(\Longrightarrow)$: If $\lambda \in \ell_\infty$. One gets $\alpha > 0$ under $|\lambda_j| \leq \alpha$, for any $j \in \mathscr{N}$. For $\overline{d} \in (\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}$, we obtain

$$\|H_\lambda \overline{d}\|_{p-qN} = \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{j=0}^{l} \lambda_j \mathfrak{r}_j q_j \overline{d_j},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} \leq \sup_l \alpha^{t_l} \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{j=0}^{l} \mathfrak{r}_j q_j \overline{d_j},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} = \sup_l \alpha^{t_l} \|\overline{d}\|_{p-qN}.$$

Hence, $H_\lambda \in \mathfrak{B}\mathfrak{d} \Uparrow_{(\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}}$.

$(\Longleftarrow)$: Presume $H_\lambda \in \mathfrak{B}\mathfrak{d} \Uparrow_{(\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}}$ and $\lambda \notin \ell_\infty$. Therefore, for any $j \in \mathscr{N}$, one obtains $x_j \in \mathscr{N}$ with $\lambda_{x_j} > j$. So

$$\|H_\lambda \overline{\mathfrak{e}_{x_b}}\|_{p-qN} = \|\lambda \overline{\mathfrak{e}_{x_b}}\|_{p-qN} = \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \lambda_z \mathfrak{r}_z q_z \overline{(e_{x_b})_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}$$

$$= \sum_{l=x_b}^{\infty} \left(\frac{\lambda_{(x_b)} \mathfrak{r}_{(x_b)} q_{x_b}}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} > \sum_{l=x_b}^{\infty} \left(\frac{b \mathfrak{r}_{(x_b)} q_{x_b}}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} > b^{t_0} \|\overline{\mathfrak{e}_{x_b}}\|_{p-qN}.$$

Hence, $H_\lambda \notin \mathfrak{B}\mathfrak{d} \Uparrow_{(\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}}$. So $\lambda \in \ell_\infty$.

The part (m2): $(\Longrightarrow)$: Let $|\lambda_b| = 1$, if $b \in \mathscr{N}$. So

$$\|H_\lambda \overline{f}\|_{p-qN} = \|\lambda \overline{f}\|_{p-qN} = \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \lambda_z \overline{f_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} = \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \overline{f_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} = \|\overline{f}\|_{p-qN},$$

for every $\overline{f} \in (\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}$. One gets, H_λ is an ***Iy.O***.

$(\Longleftarrow)$: If there are some $b = b_0$ with $|\lambda_b| < 1$. That implies

$$\|H_\lambda \overline{\mathfrak{e}_{b_0}}\|_{p-qN} = \|\lambda \overline{\mathfrak{e}_{b_0}}\|_{p-qN} = \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \lambda_z \overline{(e_{b_0})_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} = \sum_{l=b_0}^{\infty} \left(\frac{|\lambda_{b_0}| \mathfrak{r}_{b_0} q_{b_0}}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}$$

$$< \sum_{l=b_0}^{\infty} \left(\frac{\mathfrak{r}_{b_0} q_{b_0}}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} = \|\overline{\mathfrak{e}_{b_0}}\|_{p-qN}.$$

Clearly for $|\lambda_{b_0}| > 1$, we get $\|H_\lambda \overline{\mathfrak{e}_{b_0}}\|_{p-qN} > \|\overline{\mathfrak{e}_{b_0}}\|_{p-qN}$. So it must $|\lambda_j| = 1$, for every $j \in \mathscr{N}$.

The part (m3): $(\Longrightarrow)$: If $H_\lambda \in \mathfrak{P}\mathfrak{t} \Uparrow_{(\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}}$, hence $H_\lambda \in \mathfrak{C}\mathfrak{t} \Uparrow_{(\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}}$. Suppose that $\lim_{p\to\infty} \lambda_p \neq 0$. One finds $\rho > 0$ under which $K_\rho = \{p \in \mathscr{N} : |\lambda_p| \geq \rho\} \nsubseteq \mathfrak{I}$. When $\{\omega_p\}_{p\in\mathscr{N}} \subset K_\rho$. So $\{\overline{\mathfrak{e}_{\omega_p}} : \omega_p \in K_\rho\} \in \ell_\infty^F \cap \mathfrak{I}^c \subset (\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}$. As for any $\omega_r,\ \omega_p \in K_\rho$, one has

$$\|H_\lambda \overline{\mathfrak{e}_{\omega_r}} - H_\lambda \overline{\mathfrak{e}_{\omega_p}}\|_{p-qN} = \sum_{l\in\mathcal{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \lambda_z \left(\overline{(e_{\omega_r})_z} - \overline{(e_{\omega_p})_z}\right),\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$\geq \sum_{l\in\mathcal{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \rho \left(\overline{(e_{\omega_r})_z} - \overline{(e_{\omega_p})_z}\right),\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l} \geq \inf_l \rho^{t_l} \|\overline{\mathfrak{e}_{\omega_r}} - \overline{\mathfrak{e}_{\omega_p}}\|_{p-qN}.$$

Hence, $\{\overline{\mathfrak{e}_{\omega_p}} : \omega_p \in K_\rho\} \in \ell_\infty^F$, which cannot have a convergent subsequence under H_λ. That implies $H_\lambda \notin \mathfrak{Ct} \Uparrow_{(\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}}$. So $H_\lambda \notin \mathfrak{Pt} \Uparrow_{(\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}}$, That gives a contradiction. Then $\lim_{p\to\infty} \lambda_p = 0$.

($\Longleftarrow$): Suppose that $\lim_{p\to\infty} \lambda_p = 0$. For $\rho > 0$, we get $K_\rho = \{p \in \mathcal{N} : |\lambda_p| \geq \rho\} \subset \mathfrak{I}$. Then, for any $\rho > 0$, one can see $\dim\left(\left((\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}\right)_{K_\rho}\right) = \dim\left(\mathfrak{R}^{K_\rho}\right) < \infty$. So $H_\lambda \in \mathfrak{Ft}\left(\left((\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}\right)_{K_\rho}\right)$. Assume $\lambda_r \in \mathbf{R}^{\mathcal{N}}$, for every $r \in \mathcal{N}$, where

$$(\lambda_r)_p = \begin{cases} \lambda_p, & p \in K_{\frac{1}{r+1}}, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that, $H_{\lambda_r} \in \mathfrak{Ft}\left(\left((\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}\right)_{P_{\frac{1}{r+1}}}\right)$, as $\dim\left(\left((\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}\right)_{B_{\frac{1}{r+1}}}\right) < \infty$, for every $r \in \mathcal{N}$. Therefore,

$$\|(H_\lambda - H_{\lambda_a})\overline{f}\|_{p-qN} = \sum_{l\in\mathcal{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z (\lambda_z - (\lambda_a)_z)\overline{f_z}, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$= \sum_{l\in\mathcal{N}\cap K_{\frac{1}{a+1}}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z (\lambda_z - (\lambda_a)_z)\overline{f_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l} + \sum_{l\in\mathcal{N}\setminus K_{\frac{1}{a+1}}}^{\infty} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z (\lambda_z - (\lambda_a)_z)\overline{f_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$= \sum_{l\in\mathcal{N}\setminus K_{\frac{1}{a+1}}}^{\infty} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \lambda_z \overline{f_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l} \leq \frac{1}{(a+1)^{t_0}} \sum_{l\in\mathcal{N}\setminus K_{\frac{1}{a+1}}}^{\infty} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \overline{f_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$< \frac{1}{(a+1)^{t_0}} \sum_{l\in\mathcal{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \overline{f_z},\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l} = \frac{1}{(a+1)^{t_0}} \|\overline{f}\|_{p-qN}.$$

Hence, $\|H_\lambda - H_{\lambda_a}\| \leq \dfrac{1}{(a+1)^{t_0}}$. That explains $H_\lambda \in \mathfrak{Pt} \Uparrow_{(\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}}$.

The part (m4): It follows from $\mathfrak{Pt} \Uparrow_{(\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}} \subsetneqq \mathfrak{Ct} \Uparrow_{(\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}}$.

The part (m5): Clearly, $I \notin \mathfrak{Ct} \Uparrow_{(\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}}$ and $I \in \mathfrak{Bd} \Uparrow_{(\gamma_\mathfrak{r}^F(q,\, t))_{\|\cdot\|_{p-qN}}}$. Since $\lambda_I = \sum_{l\in\mathcal{N}} e_l$.

The part (m6): ($\Longrightarrow$): One has $\rho > 0$ under $|\lambda_l| \geq \rho$, for every $l \in (\ker(\lambda))^c$. Let $\overline{m}$ be a limit point of $\mathscr{R}(H_\lambda)$. Therefore, $H_\lambda \overline{f_l} \in (\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}$, for any $l \in \mathscr{N}$ with $\lim_{l\to\infty} H_\lambda \overline{f_l} = \overline{m}$. So $H_\lambda \overline{f_l}$ is a ***Cs***. Therefore,

$$\|H_\lambda \overline{f_a} - H_\lambda \overline{f_b}\|_{p-qN} = \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z (\lambda_z \overline{(f_a)_z} - \lambda_z \overline{(f_b)_z}),\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$= \sum_{l\in\mathscr{N}\cap(\ker(\lambda))^c} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z (\lambda_z \overline{(f_a)_z} - \lambda_z \overline{(f_b)_z}),\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$+ \sum_{l\in\mathscr{N}\setminus(\ker(\lambda))^c} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z (\lambda_z \overline{(f_a)_z} - \lambda_z \overline{(f_b)_z}),\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$\geq \sum_{l\in\mathscr{N}\cap(\ker(\lambda))^c} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z (\lambda_z \overline{(f_a)_z} - \lambda_z \overline{(f_b)_z}),\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$= \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z (\lambda_z \overline{(\alpha_a)_z} - \lambda_z \overline{(\alpha_b)_z}),\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l}$$

$$> \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\rho\sum_{z=0}^{l} \mathfrak{r}_z q_z (\overline{(\alpha_a)_z} - \overline{(\alpha_b)_z}),\, \overline{0}\right)}{\mathfrak{r}_{l+2} - (l+2)} \right)^{t_l} \geq \inf_l \rho^{t_l} \|\overline{\alpha_a} - \overline{\alpha_b}\|_{p-qN},$$

where

$$\overline{(\alpha_a)_j} = \begin{cases} \overline{(f_a)_j}, & j \in (\ker(\lambda))^c, \\ 0, & j \notin (\ker(\lambda))^c. \end{cases}$$

So, $\{\overline{\alpha_l}\}$ is a ***Cs*** in the ***p-q.B*** $(\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}$. One gets $\overline{f} \in (\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}$ under $\lim_{l\to\infty} \overline{\alpha_l} = \overline{f}$. As $H_\lambda \in \mathfrak{B}\mathfrak{d} \Uparrow_{(\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}}$, hence $\lim_{l\to\infty} H_\lambda \overline{\alpha_l} = H_\lambda \overline{f}$. As $\lim_{l\to\infty} H_\lambda \overline{\alpha_l} = \lim_{l\to\infty} H_\lambda \overline{f_l} = \overline{m}$. So $H_\lambda \overline{f} = \overline{m}$. That proves $\overline{m} \in \mathscr{R}(H_\lambda)$. Hence $\mathscr{R}(H_\lambda)$ is $\mathscr{C}.\mathscr{R}$.

($\Longleftarrow$): One obtains $\rho > 0$ with $\|H_\lambda \overline{f}\|_{p-qN} \geq \rho \|\overline{f}\|_{p-qN}$, for any $\overline{f} \in \left((\gamma_{\mathfrak{r}}^F(q,\, t))_{\|.\|_{p-qN}}\right)_{(\ker(\lambda))^c}$. Presume $K = \left\{ l \in (\ker(\lambda))^c : |\lambda_l| < \rho \right\} \neq \emptyset$, so if $a_0 \in K$, then

$$\|H_\lambda \overline{\mathfrak{e}_{a_0}}\|_{p-qN} = \|\left(\lambda_b \overline{(e_{a_0})_b}\right)_{b\in\mathscr{N}}\|_{p-qN} = \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\sum_{z=0}^{l} \mathfrak{r}_z q_z \lambda_z \overline{(e_{a_0})_z},\ \bar{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l}$$

$$< \sum_{l\in\mathscr{N}} \left(\frac{\bar{\hbar}\left(\rho \sum_{z=0}^{l} \mathfrak{r}_z q_z \overline{(e_{a_0})_z},\ \bar{0}\right)}{\mathfrak{r}_{l+2}-(l+2)}\right)^{t_l} \leq \sup_l \rho^{t_l} \|\overline{\mathfrak{e}_{a_0}}\|_{p-qN},$$

That explains a contradiction. Therefore, $K=\phi$, one has $|\lambda_l| \geq \rho$, for $l \in (\ker(\lambda))^c$.

The part (m7): $(\Longrightarrow)$: If $\alpha \in \mathbf{R}^{\mathscr{N}}$ under $\alpha_l = \dfrac{1}{\omega_l}$. By Theorem 4.1, one has $H_\omega,\ H_\alpha \in \mathfrak{B}\mathfrak{d} \Uparrow_{(\gamma_{\mathfrak{r}}^F(q,\ t))_{\|\cdot\|_{p-qN}}}$ with $H_\omega.H_\alpha = H_\alpha.H_\omega = I$. So $H_\alpha = H_\omega^{-1}$.

$(\Longleftarrow)$: Let H_ω be ***Iv.O***. So $\mathscr{R}(H_\omega) = \left((\gamma_{\mathfrak{r}}^F(q,\ t))_{\|\cdot\|_{p-qN}}\right)_{\mathscr{N}}$. hence, $\mathscr{R}(H_\omega)$ is $\mathscr{C}.\mathscr{R}$. From the part (m6), one has $\zeta > 0$ with $|\omega_l| \geq \zeta$, for any $l \in (\ker(\omega))^c$. Then $\ker(\omega) = \emptyset$, whenever $\omega_{l_0} = 0$, for any $l_0 \in \mathscr{N}$, hence $e_{l_0} \in \ker(H_\omega)$, which is a contradiction, since $\ker(H_\omega)$ is trivial. So $|\omega_l| \geq \zeta$, for any $l \in \mathscr{N}$. As $H_\omega \in \ell_\infty$. From the part (m1), one gets $\xi > 0$ with $|\omega_l| \leq \xi$, for any $l \in \mathscr{N}$. Hence, one has $\zeta \leq |\omega_l| \leq \xi$, for $l \in \mathscr{N}$.

The part (m8): $(\Longrightarrow)$: Suppose that $\ker(\lambda) \subsetneqq \mathscr{N} \cap \mathfrak{I}^c$, so $\overline{\mathfrak{e}_l} \in \ker(H_\lambda)$, for any $l \in \ker(\lambda)$. That explains a contradiction, since $\dim(\ker(H_\lambda)) = \infty$. Therefore, $\ker(\lambda) \subsetneqq \mathscr{N} \cap \mathfrak{I}$. From the part (m6), one has (o2) is verified.

$(\Longleftarrow)$: From the part (m6), the setting (o2) gives $\mathscr{R}(H_\lambda)$ is $\mathscr{C}.\mathscr{R}$. The condition (o1) means $\dim(\ker(H_\lambda)) < \infty$ and $\dim((\mathscr{R}(H_\lambda))^c) < \infty$. So H_λ is ***Fr.O***.

5. Conclusion

We explained a few topological and geometric properties of multiplication maps acting on $\left(\gamma_{\mathfrak{r}}^F(q,\ t)\right)_{\|\cdot\|_{p-qN}}$. This novel fuzzy function space is providing a new universal solution space for a wide variety of stochastic Fredholm nonlinear dynamical systems.

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Ethics approval and consent to participate

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest

The authors declare no competing financial interest.

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