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Discussions on the Vertex Euclidean Properties of Graphs

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Abstract: In 2022, the result that the sum of the lengths of any two edges of a triangle is greater than the length of the third edge in Euclidean geometry, is applied to labeling graph theory, a new concept-the vertex Euclidean graph is introduced. A simple graph $G = (V, E)$ is said to be vertex Euclidean if there exists a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that $f(u) + f(v) > f(w)$ for each C_3 subgraph with vertex set $\{u, v, w\}$, where $f(u) < f(v) < f(w)$. The vertex Euclidean deficiency of a graph G, denoted $\mu_{vEuclid}(G)$, is the smallest positive integer m such that $G \cup N_m$ is vertex Euclidean. In this paper, the sufficient condition that the disjoint union of G and H is vertex Euclidean is given, meanwhile, the vertex Euclidean properties of four classes graphs are discussed, the vertex Euclidean deficiency of these graphs are obtained.

***Keywords*:** vertex Euclidean graph, vertex Euclidean deficiency, $Circ(n, 2)$, Zykov sums of a cycle and a m null graph, k-level X-grid, generality triangular snake

MSC: 05C78, 05C25, 05A20

1. Introduction

In Euclidean geometry, there is a famous triangle Theorem.

Triangle Theorem: the sum of the lengths of any two edges of a triangle is greater than the length of the third edge.

If we don't consider the physical distance in Theorem and apply the idea of Theorem to the topological network graph, this suggests a similar graph labeling problem.

Since Rosa [1] introduced the concept of graceful graph labeling, which attracts the attention of the field. The graph labeling is: for a graph G with q edges and p vertices, defining a rule about edges (or vertices) such that the vertices (or edges) have some properties. For example, for a graph G with q edges, it is graceful that there is an injection f from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that every possible difference of the vertex labels of all the edges is the set $\{1, 2, \ldots, q\}$. Some graph labeling concepts and methods have been introduced [2]. These results serve as useful models for a broad range of applications.

Now, we apply the triangle Theorem to graph labeling. Given a simple graph $G(V, E)$, we label the edges with $\{1, 2, \ldots, |E|\}$ such that, in any C_3 subgraph, the sum of any two edge labels is greater than that of the third edge. If such an edge labeling exists, it can be inferred that the graph is *edge Euclidean*. For a simple graph G, it is not edge Euclidean,

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but if adding some P_2, the new graph may be an edge Euclidean graph, and the smallest number of some P_2 is called edge Euclidean deficiency, denoted by $\mu_{edgeEuclid}(G)$. Thus, triangle Theorem is applied to graph labeling.

In 2022, we applied the triangle Theorem in graph labeling [3], introducing that the dual problem of edge Euclidean graph-vertex Euclidean graph, and defining the concept of vertex Euclidean deficiency.

Definition 1.1 Vertex Euclidean graph: Let f be a vertex labeling of a simple graph G. A C_3 subgraph with vertices u, v and w such that $f(u) < f(v) < f(w)$ is said to be vertex Euclidean if $f(u) + f(v) > f(w)$, and we also mention that label set $\{f(u), f(v), f(w)\}$ is vertex Euclidean. A simple graph $G = (V, E)$ is called vertex Euclidean graph if there exists a bijection $f: V \to \{1, 2, \ldots, |V|\}$ such that every C_3 subgraph in G is vertex Euclidean.

A simple graph G is not vertex Euclidean graph, but if N_n is added, where $N_n = \overline{K}_n$ is the null graph with n vertices, the graph $G \cup N_n$ may be a vertex Euclidean graph.

Example 1.1 C_3 is not vertex Euclidean. But an isolated vertex is added, the new graph is vertex Euclidean.

Definition 1.2 Vertex Euclidean deficiency: If a simple graph G is not vertex Euclidean, define its vertex Euclidean deficiency, denoted $\mu_{vEuclid}(G)$, as the smallest positive integer n such that $G \cup N_n$, where $N_n = \overline{K}_n$ is the null graph with n vertices, is vertex Euclidean.

Example 1.2 For C_3, $\mu_{vEuclid}(C_3) = 1$.

First, some general results are presented here.

Theorem 1.1 [3] For a simple graph G, if any vertex $u (u \in V(G))$ is on some subgraph C_3 of G, then $\mu_{vEuclid}(G) \geq 1$.

For C_3, the graph nC_3 denotes the disjoint union of n copies of C_3.

Theorem 1.2 For $n > 1$ is positive integer, $\mu_{vEuclid}(nC_3) = 1$.

Proof. Let the vertices labels on the i-th C_3 $(1 \leq i \leq n)$ be $\{2+3(i-1), 3+3(i-1), 4+3(i-1)\}$. It is easy that the conclusion is correct. □

Theorem 1.3 Suppose that G is vertex Euclidean, then any graph H obtained from G by adding any arbitrary edges without creating new C_3 subgraph is vertex Euclidean.

Notation 1.1 Suppose that a, b are two positive integers, and $a < b$, then for brevity, the set $\{a, a+1, \ldots, b\}$ will be denoted by $[a, b]$.

For graphs G and H, $G + H$ is the disjoint union of G and H.

Theorem 1.4 Suppose that G is a simple graph and $\mu_{vEuclid}(G) = k > 0$, H does not have induced C_3 subgraph. If $|V(H)| \geq k$, then $G + H$ is vertex Euclidean.

Proof. Note that $\mu_{vEuclid}(G) = k > 0$ means there exists a vertex labeling f such that the vertex set is $[k+1, k+|V(G)|]$. Suppose $|V(H)| \geq k$, then the label(s) assigned to the k extra vertice(s) of G can be assigned to any k vertice(s) of H. The remaining vertice(s) of H, if any, can be assigned arbitrarily by integers in $[|V(G| + k + 1, |V(G)| + |V(H)|]$. Thus, $G + H$ is vertex Euclidean. □

Although there are infinitely many graphs G such as $2C_3$, $\mu_{vEuclid}(2C_3) = \mu_{edgeEuclid}(2C_3)$, the vertex label set and the edge set are the same, however the two labeling theories are different.

In the following discussions, we will discuss the vertex Euclidean properties of some graphs, the labels of the vertices on these graphs are not unique, we just provide a vertex labeling rule for each class graphs.

2. The vertex Euclidean properties of generality triangular snakes

In this section, we study generality triangular snake. On triangular snake, Moulton [4] proved that all triangular snakes are graceful. Xi et al. [5] proved that all double triangular snakes are harmonious. In this section, we defined generality triangular snake, and study the vertex Euclidean properties of all generality triangular snakes.

Definition 2.1 [6] Triangular snake: Given a path P_n $(n \geq 2)$, the vertices on P_n are successively denoted by $v_1, v_2, \ldots, v_n$. A triangular snake is obtained from P_n by joining v_i and v_{i+1} to a new vertex u_i for $1 \leq i \leq n-1$.

Definition 2.2 Generality triangular snake: For a path P_n $(n \geq 2)$, the vertices on P_n are successively denoted by $v_1, v_2, \ldots, v_n$. A generality triangular snake is obtained from P_n by joining v_i and v_{i+1} to k_i $(k_i \geq 1)$ new vertices $u_{i,j}$ respectively for $1 \leq i \leq n-1$, $1 \leq j \leq k_i$. It is denoted by $GTS(k_1, k_2, \ldots, k_{n-1}; n)$.

Example 2.1 $GTS(2, 1, 2, 3; 5)$ is shown in Figure 1.

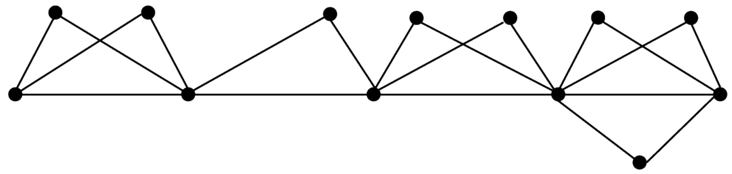

Figure 1. $GTS(2, 1, 2, 3; 5)$

Any vertex on $GTS(k_1, k_2, \ldots, k_{n-1}; n)$ is on C_3, by Theorem 1.1, $\mu_{vEuclid}(GTS(k_1, k_2, \ldots, k_{n-1}; n)) \geq 1$.
Now, we will prove that for any $GTS(k_1, k_2, \ldots, k_{n-1}; n)$, the vertex Euclidean deficiency is always 1.
Theorem 2.1 For any integers $k_i \geq 1$, $n > 1$, $\mu_{vEuclid}(GTS(k_1, k_2, \ldots, k_{n-1}; n)) = 1$.
Proof.
Case 1 $k_1 \geq k_i$ for $1 < i \leq n-1$.
We find a vertex labeling such as $\mu_{vEuclid}(GTS(k_1, k_2, \ldots, k_{n-1}; n)) = 1$.
Label the vertices as follows.

$$f(u_{1,j}) = j+1 \text{ for } 1 \leq j \leq k_1, \quad f(v_1) = k_1+2, \quad f(v_2) = k_1+3.$$

$$f(u_{2,j}) = j+f(v_2) \text{ for } 1 \leq j \leq k_2, \quad f(v_3) = \sum_{i=1}^{2} k_i+4.$$

$$f(u_{3,j}) = j+f(v_3) \text{ for } 1 \leq j \leq k_3, \quad f(v_4) = f(v_3)+k_3+1.$$

$$\vdots \quad \vdots \quad \vdots$$

$$f(u_{n-1,j}) = j+f(v_{n-1}) \text{ for } 1 \leq j \leq k_{n-1}, \quad f(v_n) = f(v_{n-1})+k_{n-1}+1.$$

By the above rules, the labels of all vertices on $GTS(k_1, k_2, \ldots, k_{n-1}; n)$ are obtained, and these vertex labels are different one another. The minimum value is 2, while the maximum value is $f(v_n)$. $f(v_n) = f(v_{n-1})+k_{n-1}+1 = f(v_{n-2})+\sum_{i=n-2}^{n-1} k_i+2 = \cdots = f(v_3)+\sum_{i=3}^{n-1} k_i+n-3 = \sum_{i=1}^{n-1} k_i+n+1$. In $GTS(k_1, k_2, \ldots, k_{n-1}; n)$, there are $\sum_{i=1}^{n-1} k_i+n$ vertices.

Thus, the vertex label set is $\left[2, \sum_{i=1}^{n-1} k_i+n+1\right]$.

For the subgraph C_3, the vertex set is $\{v_1, v_2, u_{1,j}\}$ for $1 \leq j \leq k_1$, $f(u_{1,j}) \geq 2$, $f(v_1)+f(u_{1,j}) \geq k_1+2+2 > k_1+3 = f(v_2)$, these vertex labels satisfy the vertex Euclidean condition.

For the subgraph C_3, the vertex set is $\{v_i, v_{i+1}, u_{i,j}\}$ for $2 \leq i \leq n-1$, $1 \leq j \leq k_i$, in the vertex label set, $f(u_{i,j}) = f(v_i)+j > f(v_2) = k_1+3 > k_1+1$, $f(v_i)+f(u_{i,j}) > f(v_i)+k_i+1+(k_1-k_i) \geq f(v_{i+1})$.

Case 2 $k_{n-1} \geq k_i$ for $1 \leq i < n-1$.

After the manner of the discussions in case 1, we can know that the conclusion is correct.

Case 3 $k_a \geq k_i$ for $a \neq 1, n-1$, $1 \leq i \leq n-1$ and $i \neq a$.

First, label the vertices $u_{a,1}, u_{a,2}, \ldots, u_{a,k_a}, v_a, v_{a+1}$ are successively $2, 3, \ldots, k_a+1, k_a+2, k_a+3$. Next, for $i < a$, label the vertices as follows.

$$f(u_{a-1,j}) = j + f(v_{a+1}) \text{ for } 1 \le j \le k_{a-1}, \quad f(v_{a-1}) = f(v_{a+1}) + k_{a-1} + 1 = \sum_{i=a-1}^{a} k_i + 4.$$

$$f(u_{a-2,j}) = j + f(v_{a-1}) \text{ for } 1 \le j \le k_{a-2}, \quad f(v_{a-2}) = f(v_{a-1}) + k_{a-2} + 1.$$

$$\vdots \quad \vdots \quad \vdots$$

$$f(u_{1,j}) = j + f(v_2) \text{ for } 1 \le j \le k_1, \quad f(v_1) = f(v_2) + k_1 + 1.$$

Finally, for $i > a$, label the vertices as follows:

$$f(u_{a+1,j}) = j + f(v_1) \text{ for } 1 \le j \le k_{a+1}, \quad f(v_{a+2}) = f(v_1) + k_{a+1} + 1.$$

$$f(u_{a+2,j}) = j + f(v_{a+2}) \text{ for } 1 \le j \le k_{a+2}, \quad f(v_{a+3}) = f(v_{a+2}) + k_{a+2} + 1.$$

$$\vdots \quad \vdots \quad \vdots$$

$$f(u_{n-1,j}) = j + f(v_{n-1}) \text{ for } 1 \le j \le k_{n-1}, \quad f(v_n) = f(v_{n-1}) + k_{n-1} + 1.$$

By the above rules, the labels of all vertices on $GTS(k_1, k_2, \ldots, k_{n-1}; n)$ are obtained, and these vertex labels are different one another. The minimum value is 2, while the maximum value is $f(v_n)$.

$$f(v_n) = f(v_{n-1}) + k_{n-1} + 1 = f(v_{n-2}) + \sum_{i=n-2}^{n-1} k_i + 2 = \cdots = f(v_1) + \sum_{i=a+1}^{n-1} k_i + n - a - 1$$

$$= f(v_2) + k_1 + \sum_{i=a+1}^{n-1} k_i + n - a = f(v_3) + \sum_{i=1}^{3} k_i + \sum_{i=a+1}^{n-1} k_i + n - a + 2 = \cdots$$

$$= f(v_{a+1}) + \sum_{i=1}^{a-1} k_i + \sum_{i=a+1}^{n-1} k_i + n - a - 1 + a - 1 = k_a + 3 + \sum_{i=1}^{a-1} k_i + \sum_{i=a+1}^{n-1} k_i + n - 2 = \sum_{i=1}^{n-1} k_i + n + 1.$$

So the vertex label set is $\left[2, \sum_{i=1}^{n-1} k_i + n + 1\right]$.

For the subgraph C_3, the vertex set is $\{v_a, v_{a+1}, u_{a,j}\}$ for $1 \le j \le k_a$, $f(u_{a,j}) \ge 2$, $f(v_a) + f(u_{a,j}) \ge (f(v_a)+1) + 1 > f(v_{a+1})$, these vertex labels satisfy the vertex Euclidean condition.

For the subgraph C_3, the vertex set is $\{v_{a-1},\ v_a,\ u_{a-1,\,j}\}$ for $1 \le j \le k_{a-1}$, $f(v_a)+f(u_{a-1,\,j}) > f(v_a)+k_{a-1}+2f(v_{a-1})$, these vertex labels satisfy the vertex Euclidean condition.

For the subgraph C_3, the vertex set is $\{v_i,\ v_{i+1},\ u_{i,\,j}\}$ for $i < a-1$, $1 \le j \le k_i$, $f(u_{i,\,j}) = f(v_{i+1})+j > f(v_a) = k_a+2$, $f(v_{i+1})+f(u_{i,\,j}) > f(v_{i+1})+k_a+2 > f(v_i)$.

For the subgraph C_3, the vertex set is $\{v_{a+1},\ v_{a+2},\ u_{a+1,\,j}\}$ for $1 \le j \le k_{a+1}$, $f(v_{a+1}) = k_a+3 > k_{a+1}$, $f(v_{a+1})+f(u_{a+1,\,j}) = f(v_{a+1})+k_a+3 > f(v_{a+1})+k_{a+1} \ge f(v_{a+2})$, these vertex labels satisfy the vertex Euclidean condition.

For the subgraph C_3 that the vertex set is $\{v_i,\ v_{i+1},\ u_{i,\,j}\}$ for $i > a+1$, $1 \le j \le k_i$, $f(u_{i,\,j}) = f(v_i)+j > k_a+2$, $f(v_i)+f(u_{i,\,j}) > (f(v_i)+k_a+1)+1 > f(v_{i+1})$, so $f(v_i)+f(u_{i,\,j}) > f(v_{i+1})$.

This completes the proof. □

Specially, when $k_1 = k_2 = \cdots = k_{n-1} = 1$ and 2 respectively, $GTS(k_1,\ k_2,\ \ldots,\ k_{n-1};\ n)$ is a triangular snake and a double triangular snake.

Corollary 2.2 The vertex Euclidean deficiencies of all triangular snakes are 1.

Corollary 2.3 The vertex Euclidean deficiencies of all double triangular snakes are 1.

3. The vertex Euclidean properties of k-level X-grids

In this section, we introduced k-level X-grids for $k \ge 1$ graph, and investigated the vertex Euclidean properties of k-level X-grids for $k \ge 1$.

Definition 3.1 The $(m-1)$-level X-grid, denoted by $(m-1)-XG(m,\ n)$, where $m,\ n \ge 2$, is the graph with

$$V((m-1)-XG(m,\ n)) = \{v_{i,\,j}\colon 1 \le i \le m,\ 1 \le j \le n\},$$

$$E((m-1)-XG(m,\ n)) = \{(v_{i,\,j},\ v_{i,\,j+1})\colon 1 \le i \le m,\ 1 \le j \le n-1\},$$

$$\cup\{(v_{i,\,j},\ v_{i+1,\,j})\colon 1 \le i \le m-1,\ 1 \le j \le n\},$$

$$\cup\{(v_{i,\,j},\ v_{i+1,\,j+1})\colon 1 \le i \le m-1,\ 1 \le j \le n-1\},$$

$$\cup\{(v_{i,\,j},\ v_{i+1,\,j-1})\colon 1 < i \le m,\ 1 < j \le n\}.$$

Example 3.1 $3-XG(4,\ 5)$ is shown in Figure 2.

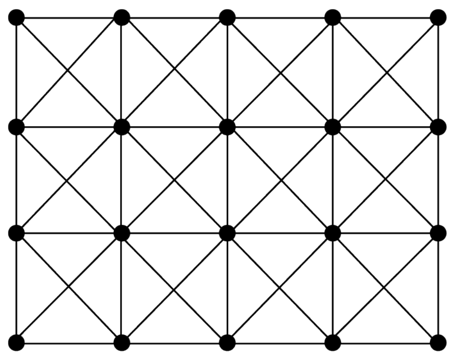

Figure 2. $3-XG(4,\ 5)$

Due to symmetry, we shall assume $m \leq n$. The $(m-1)$-level X-grid is obtained from the rectangular grid $P_m \times P_n$ by adding two diagonals in each of its mn squares.

For graph $(m-1)-XG(m, n)$ where $m \geq 2$.

Theorem 3.1 For $m \geq 2$, $\mu_{vEuclid}((m-1)-XG(m, n)) > 1$.

Proof. Since any vertex on $(m-1)-XG(m, n)$ is on C_3, by Theorem 1.1, $\mu_{vEuclid}((m-1)-XG(m, n)) \geq 1$.

Assume $\mu_{vEuclid}((m-1)-XG(m, n)) = 1$. Then the vertex label set is $[2, |V|+1]$, and V is the vertex set of $(m-1)-XG(m, n)$.

Let the label of $v_{i,j}$ be 2, v_{i_1, j_1}, v_{i_2, j_2}, v_{i_3, j_3} and $v_{i,j}$ are adjacent each other, and $v_{i,j}$ and v_{i_1, j_1} on a diagonal, let the label of v_{i_1, j_1} be a, then the label set of v_{i_2, j_2}, v_{i_3, j_3} can only be $\{a-1, a+1\}$. But $v_{i,j}$, v_{i_2, j_2}, v_{i_3, j_3} are on a C_3 too, $2+a-1 = a+1$, contradiction.

This completes the proof. □

To obtain $\mu_{vEuclid}((m-1)-XG(m, n))$ for $m > 1$, first, we investigate the case of $m = n$.

Lemma 3.2 $\mu_{vEuclid}(1-XG(2, 2)) = 2$.

Proof. Define a vertex labeling f as follows.

$$f(v_{1,1}) = 3, \quad f(v_{2,1}) = 4, \quad f(v_{2,2}) = 5, \quad f(v_{1,2}) = 6.$$

The vertex label set is [3, 6].

On C_3 where the vertex set is $\{v_{1,1}, v_{2,1}, v_{2,2}\}$, $f(v_{1,1}) + f(v_{2,1}) = 3+4 = 7 > 5 = f(v_{2,2})$.

On C_3 where the vertex set is $\{v_{1,1}, v_{2,1}, v_{1,2}\}$, $f(v_{1,1}) + f(v_{2,1}) = 3+4 = 7 > 6 = f(v_{1,2})$.

On C_3 where the vertex set is $\{v_{1,1}, v_{2,2}, v_{1,2}\}$, $f(v_{1,1}) + f(v_{2,2}) = 3+5 = 8 > 6 = f(v_{1,2})$.

On C_3 where the vertex set is $\{v_{2,1}, v_{2,2}, v_{1,2}\}$, $f(v_{2,1}) + f(v_{2,2}) = 4+5 = 9 > 6 = f(v_{1,2})$.

This completes the proof. □

Lemma 3.3 $\mu_{vEuclid}(2-XG(3, 3)) = 2$.

Proof. Define a vertex labeling f as follows.

The labels of $v_{1,1}$, $v_{2,1}$, $v_{2,2}$, $v_{1,2}$ are the same as thats in Lemma 3.2.

Label $v_{3,1}$, $v_{3,2}$, $v_{1,3}$, $v_{2,3}$, $v_{3,3}$ consecutively with 7, 8, 9, 10, 11.

The label set of the $v_{1,1}$, $v_{2,1}$, $v_{2,2}$, $v_{1,2}$, $v_{3,1}$, $v_{3,2}$, $v_{1,3}$, $v_{2,3}$, $v_{3,3}$ is [3, 11].

Clearly, for those subgraphs C_3 that their vertex sets V, $V \subset \{v_{1,1}, v_{2,1}, v_{1,2}, v_{2,2}\}$, the discussions are the same as thats in Lemma 3.2. For the vertex labels on other subgraphs C_3, have

$$f(v_{2,1}) + f(v_{2,2}) = 4+5 = 9 > 7 = f(v_{3,1});\ f(v_{2,1}) + f(v_{2,2}) = 4+5 = 9 > 8 = f(v_{3,2});$$

$$f(v_{2,1}) + f(v_{3,1}) = 4+7 = 12 > 8 = f(v_{3,2});\ f(v_{2,2}) + f(v_{3,1}) = 5+7 = 9 > 8 = f(v_{3,2});$$

$$f(v_{2,2}) + f(v_{1,2}) = 5+6 = 11 > 9 = f(v_{1,3});\ f(v_{2,2}) + f(v_{1,2}) = 5+6 = 11 > 10 = f(v_{2,3});$$

$$f(v_{1,2}) + f(v_{1,3}) = 6+9 = 15 > 10 = f(v_{2,3});\ f(v_{2,2}) + f(v_{1,3}) = 5+9 = 14 > 10 = f(v_{2,3});$$

$$f(v_{2,2}) + f(v_{3,2}) = 5+8 = 13 > 10 = f(v_{2,3});\ f(v_{2,2}) + f(v_{3,2}) = 5+8 = 13 > 11 = f(v_{3,3});$$

$$f(v_{2,2}) + f(v_{2,3}) = 5+10 = 15 > 11 = f(v_{3,3});\ f(v_{3,2}) + f(v_{2,3}) = 8+10 = 18 > 11 = f(v_{3,3}).$$

This completes the proof. □

Lemma 3.4 For $m > 3$, $\mu_{vEuclid}((m-1)-XG(m,\ m)) = 2$.

Proof. $m = 4$.

For the vertices on the subgraph $2-XG(3,\ 3)$ containing $v_{1,\ 1}$, the vertex labels are the same as those defined in Lemma 3.3, then these vertex labels satisfy the vertex Euclidean condition.

For the remaining vertices, label the labels as follows.

Label the vertices $v_{4,\ 1}$, $v_{4,\ 2}$, $v_{4,\ 3}$, $v_{1,\ 4}$, $v_{2,\ 4}$, $v_{3,\ 4}$, $v_{4,\ 4}$ consecutively with 12, 13, ..., 18.

Thus, the label set of $v_{4,\ 1}$, $v_{4,\ 2}$, $v_{4,\ 3}$, $v_{1,\ 4}$, $v_{2,\ 4}$, $v_{3,\ 4}$, $v_{4,\ 4}$ is [3, 18].

In C_3 that the vertices are three of $v_{3,\ j_1}$, $v_{4,\ j_2}$, $v_{i_1,\ 3}$, $v_{i_2,\ 4}$ for $1 \le j_1,\ j_2,\ i_1,\ i_2 \le 4$, the vertex label sets are {7, 8, 12}, {7, 8, 13}, {7, 12, 13}, {8, 12, 13}, {8, 13, 14}, {8, 11, 13}, {11, 13, 14}, {8, 11, 14}, {11, 14, 18}, {11, 14, 17}, {11, 17, 18}, {14, 17, 18}, {9, 10, 15}, {9, 10, 16}, {9, 15, 16}, {10, 15, 16}, {10, 11, 17}, {10, 11, 16}, {10, 16, 17}, {11, 16, 17}.

In each set of {7, 8, 12}, {7, 8, 13}, {7, 12, 13}, {8, 12, 13}, {8, 13, 14}, {8, 11, 13}, the maximum value and the minimum value of the vertex labels are denoted by a, b respectively, then $a-b \le 6$, the minimum value of the vertex labels is 7, so the vertex labels in these vertex label sets satisfy the vertex Euclidean condition.

In each set of {11, 14, 18}, {11, 14, 17}, {11, 17, 18}, {14, 17, 18}, {9, 10, 15}, {9, 10, 16}, {9, 15, 16}, {10, 15, 16}, {10, 11, 17}, {10, 11, 16}, {10, 16, 17}, {11, 16, 17}, in each set, the maximum value and the minimum value of the vertex labels are denoted by c, d respectively, then $c-d \le 7$, the minimum value of the vertex labels is 9, so the vertex labels in these vertex label sets satisfy the vertex Euclidean condition.

Now, we study $\mu_{vEuclid}((m-1)-XG(m,\ m))$ by mathematical induction.

Assume $m = k$, $\mu_{vEuclid}((k-1)-XG(k,\ k)) = 2$. And the labels of $v_{k,\ 1}$, $v_{k,\ 2}$, ..., $v_{k,\ k-1}$, $v_{1,\ k}$, $v_{2,\ k}$, ..., $v_{k-1,\ k}$, $v_{k,\ k}$ are successively k^2-2k+4, k^2-2k+5, ..., k^2+2.

Now, we investigate the case of $m = k+1$.

First, for the subgraph $(k-1)-XG(k,\ k)$ that the vertex set is $\{v_{i,\ j} | 1 \le i,\ j \le k\}$, the vertex labels defined are the same as those $m = k$, and the labels of $v_{k,\ 1}$, $v_{k,\ 2}$, ..., $v_{k,\ k-1}$, $v_{1,\ k}$, $v_{2,\ k}$, ..., $v_{k-1,\ k}$, $v_{k,\ k}$ are successively k^2-2k+4, k^2-2k+5, ..., k^2+2.

Next, define the labels of remaining vertices as follows.

Label the vertices $v_{k+1,\ 1}$, $v_{k+1,\ 2}$, ..., $v_{k+1,\ k}$, $v_{1,\ k+1}$, $v_{2,\ k+1}$, ..., $v_{k+1,\ k+1}$ consecutively with $[k^2+3,\ k^2+2k+3]$. Since $k^2+2k+3 = (k+1)^2+2$, the vertex label set is $[3,\ (k+1)^2+2]$.

Finally, we only study the subgraphs C_3 that the vertices are three of $v_{k,\ j_1}$, $v_{k+1,\ j_2}$, $v_{i_1,\ k}$, $v_{i_2,\ k+1}$ for $1 \le j_1,\ j_2,\ i_1,\ i_2 \le k+1$. Their vertex label sets are

(1) $\{k^2-2k+4,\ k^2-2k+5,\ k^2+3\}$, $\{k^2-2k+4,\ k^2-2k+5,\ k^2+4\}$, $\{k^2-2k+4,\ k^2+3,\ k^2+4\}$, $\{k^2-2k+5,\ k^2+3,\ k^2+4\}$, $\{k^2-2k+5,\ k^2-2k+6,\ k^2+4\}$, $\{k^2-2k+5,\ k^2-2k+6,\ k^2+5\}$, $\{k^2-2k+5,\ k^2+4,\ k^2+5\}$, $\{k^2-2k+6,\ k^2+3,\ k^2+4\}$, ..., $\{k^2-k+1,\ k^2-k+2,\ k^2+k\}$, $\{k^2-k+1,\ k^2-k+2,\ k^2+k+1\}$, $\{k^2-k+1,\ k^2+k,\ k^2+k+1\}$, $\{k^2-k+2,\ k^2+k,\ k^2+k+1\}$, $\{k^2-k+2,\ k^2+k+1,\ k^2+k+2\}$, $\{k^2-k+2,\ k^2+2,\ k^2+k+1\}$, $\{k^2-k+2,\ k^2+2,\ k^2+k+2\}$, $\{k^2+2,\ k^2+k+1,\ k^2+k+2\}$ respectively.

(2) $\{k^2-k+3,\ k^2-k+4,\ k^2+k+3\}$, $\{k^2-k+3,\ k^2-k+4,\ k^2+k+4\}$, $\{k^2-k+3,\ k^2+k+3,\ k^2+k+4\}$, $\{k^2-k+4,\ k^2+k+3,\ k^2+k+4\}$, $\{k^2-k+4,\ k^2-k+5,\ k^2+k+4\}$, $\{k^2-k+4,\ k^2-k+5,\ k^2+k+5\}$, $\{k^2-k+4,\ k^2+k+4,\ k^2+k+5\}$, $\{k^2-k+5,\ k^2+k+4,\ k^2+k+5\}$, ..., $\{k^2+1,\ k^2+2,\ k^2+2k+1\}$, $\{k^2+1,\ k^2+2,\ k^2+2k+2\}$, $\{k^2+1,\ k^2+2k+1,\ k^2+2k+2\}$, $\{k^2+2,\ k^2+2k+1,\ k^2+2k+2\}$, $\{k^2+2,\ k^2+2k+2,\ k^2+2k+3\}$, $\{k^2+2,\ k^2+k+2,\ k^2+2k+2\}$, $\{k^2+k+2,\ k^2+2k+2,\ k^2+2k+3\}$ respectively.

For each set in (1), the maximum value and the minimum value of the vertex labels are denoted by a, b respectively, then $a-b \le 2k$, the minimum value of the vertex labels is k^2-2k+4, $k^2-2k+4-2k = (k-2)^2$ for $k > 3$, so the vertex labels in these vertex label sets satisfy the vertex Euclidean condition.

For each set in (2), the maximum value and the minimum value of the vertex labels are denoted by c, d respectively, then $c-d \le 2k+1$, the minimum value of the vertex labels is k^2-k+3, $k^2-k+3-2k-1 = (k-2)(k-1)$ for $k > 3$, so the vertex labels in these vertex label sets satisfy the vertex Euclidean condition too.

This completes the proof. □

Now, we obtained that $\mu_{vEuclid}((m-1)-XG(m, m)) = 2$. Next, with the help of the rules in Lemmas 3.2, 3.3 and 3.4, we study $\mu_{vEuclid}((m-1)-XG(m, n))$.

Theorem 3.5 For any $m, n \geq 2$ and $m \leq n$, $\mu_{vEuclid}((m-1)-XG(m, n)) = 2$.

Proof. 1. From Lemmas 3.2, 3.3 and 3.4, the conclusion holds when $m = n$.

2. $m < n$.

In $(m-1)-XG(m, n)$, first, for the subgraph $(m-1)-XG(m, m)$ that the vertex set is $\{v_{i,\, j} | 1 \leq i,\ j \leq m\}$, define the vertex labels according to the rules in Lemmas 3.2, 3.3 and 3.4.

For the remaining vertices $v_{s,\, t}$ $(1 \leq s \leq m, m+1 \leq t \leq n)$, define the vertices labels as follows.

$$f(v_{s,\, t}) = f(v_{s,\, t-1}) + m,\ 1 \leq s \leq m,\ m+1 \leq t \leq n.$$

Thus, in each C_3 that the vertices are three of $v_{s,\, t}$ $(1 \leq s \leq m, m \leq t \leq n)$, let the maximum value and the minimum value of the vertex labels be a, b respectively, then $a - b \leq 2m - 1$, the minimum value of the vertex labels is $m^2 - m + 3$, $m^2 - m + 3 - 2m + 1 = (m-1)(m-3) + 2 > 0$, so the labels of the vertices on each C_3 satisfy the vertex Euclidean condition.

Theorem holds. □

4. The vertex Euclidean properties of *Circ*(*n*, 2)

Laison et al. introduced circulant graphs when they study prime distance graphs in [7]. In this section, we study the vertex Euclidean properties of a class of the circulant graphs.

Definition 4.1 Circulant graph: For a positive integer $n \geq 3$ and set $S \subseteq \{1, 2, \ldots, n\}$, the circulant graph, denoted by $Circ(n, S)$, is the graph with vertex set$\{v_1, v_2, \ldots, v_n\}$ and an edge between vertices v_i and v_j if and only if $|i-j|$ $(\text{mod } n) \in S$.

If $S = \{1, k\}$ for $1 < k \leq n-1$, which, for simplicity, the circulant graph is written as $Circ(n, k)$.

Example 4.1 If $k = 2$, $Circ(5, 2)$ and $Circ(6, 2)$ are shown in Figure 3.

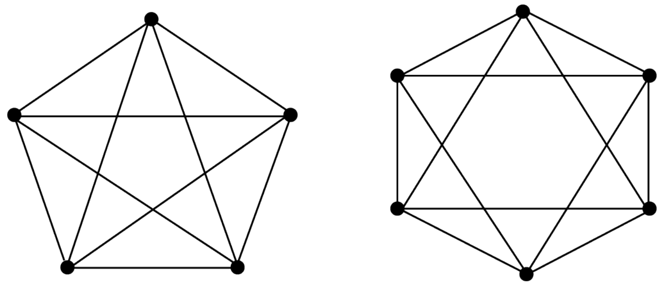

Figure 3. $Circ(5, 2)$ and $Circ(6, 2)$

When $k = 2$, there exists C_3 in $Circ(n, 2)$, which is under study $Circ(n, 2)$. Because all vertices on some subgraph C_3 of $Circ(n, 2)$ $(n \geq 4)$, by Theorem 1.1, $\mu_{vEuclid}(Circ(n, 2)) \geq 1$. In order to obtain $\mu_{vEuclid}(Circ(n, 2))$, first, we investigate several special cases of n.

Theorem 4.1 $\mu_{vEuclid}(Circ(4, 2)) = 2$.

Proof. Assume $\mu_{vEuclid}(Circ(4, 2)) = 1$, then 2 is the label of some vertex, which is denoted by u. In $Circ(4, 2)$, each vertex is adjacent to the other vertices, so the two vertices labeled by 2 and 3 respectively, must with the vertex labeled by 5 are a C_3, contradiction. Hence, $\mu_{vEuclid}(Circ(4, 2)) \geq 2$. □

Now, we need to describe a vertex Euclidean labeling f with $[3, 6]$. Label the vertices in $\{v_1, v_2, v_3, v_4\}$ consecutively with $[3, 6]$, then the minimum value of the sums of the labels of any two adjacent vertices is $3+4=7$, thus, for any three vertices v_i, v_j and v_k that are on a C_3, the sum of any two vertex labels is greater than the label of the third vertex. Hence, $\mu_{vEuclid}(Circ(4, 2)) = 2$.

Theorem 4.2 $\mu_{vEuclid}(Circ(5, 2)) = 3$.

Proof. Assume $\mu_{vEuclid}(Circ(5, 2)) = 1$, then 2 is the label of some vertex, which is denoted by u. In $Circ(5, 2)$, each vertex is adjacent to the other vertices, so the two vertices labeled by 2 and 3 respectively, must with the vertex labeled by 6 are a C_3, contradiction. Hence, $\mu_{vEuclid}(Circ(5, 2)) \geq 2$. □

Similarly, if $\mu_{vEuclid}(Circ(5, 2)) = 2$, then 3 is the label of some vertex, which is denoted by u. In $Circ(5, 2)$, each vertex is adjacent with the other vertices, so the two vertices labeled by 3 and 4 respectively, must with the vertex labeled by 7 are a C_3, contradiction. Hence, $\mu_{vEuclid}(Circ(5, 2)) \geq 3$.

Label the vertices in $\{v_1, v_2, \ldots, v_5\}$ consecutively with $[4, 8]$, then the minimum value of the sums of the labels of any two adjacent vertices is $4+5=9$, thus, for any three vertices v_i, v_j and v_k that are on a C_3, the sum of any two vertex labels is greater than the label of the third vertex. Hence, $\mu_{vEuclid}(Circ(5, 2)) = 3$.

Theorem 4.3 $\mu_{vEuclid}(Circ(6, 2)) = 2$.

Proof. Assume $\mu_{vEuclid}(Circ(6, 2)) = 1$. Without loss of generality, let the label of v_1 be 2, then the labels of the remaining vertices are 3, 4, 5, 6, 7 respectively.

Assume the vertex u labeled by 3 is adjacent to v_1, then v_1 and u are on two C_3, thus, on some C_3, the label of the third vertex is at least 5, $2+3=5$, contradiction. Hence, the vertex labeled by 3 is not adjacent to v_1. i.e. v_4 is labeled by 3. So the vertex w labeled by 4 is must adjacent to v_1, and v_1, w are on two C_3, thus, on some C_3, the label of the third vertex is at least 6, $2+4=6$, contradiction. Thus, we obtain that $\mu_{vEuclid}(Circ(6, 2)) \geq 2$.

Now, we need to describe a vertex Euclidean labeling f with $[3, 8]$. Specifically, let

$$f(v_{2i-1}) = 2+i, \quad 1 \leq i \leq 3,$$

$$f(v_2) = 6, \quad f(v_4) = 8, \quad f(v_6) = 7.$$

On $Circ(6, 2)$, there are eight C_3, their vertex sets are $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$, $\{v_5, v_6, v_1\}$, $\{v_6, v_1, v_2\}$, $\{v_1, v_3, v_5\}$, $\{v_4, v_2, v_6\}$ respectively, the corresponding vertex label sets are $\{3, 4, 6\}$, $\{4, 6, 8\}$, $\{4, 5, 8\}$, $\{5, 7, 8\}$, $\{3, 5, 7\}$, $\{3, 6, 7\}$, $\{3, 4, 5\}$, $\{6, 7, 8\}$. By these vertex label sets, we can see that the sum of the labels of any two vertex labels is greater than the third vertex on each C_3, so $\mu_{vEuclid}(Circ(6, 2)) = 2$. □

Theorem 4.4 $\mu_{vEuclid}(Circ(7, 2)) = 2$.

Proof. Assume $\mu_{vEuclid}(Circ(7, 2)) = 1$. Without loss of generality, let the label of v_1 be 2, then the labels of the remaining vertices are 3, 4, 5, 6, 7, 8 respectively. From the discussions in Theorem 4.3, we can know that the vertex labeled by 3 is not adjacent to v_1, so the vertex labeled by 3 can only be v_4 or v_5. Without loss of generality, let the label of v_5 be 3.

1. If the label of v_2 is 4, then the two vertices labeled by 2 and 4 respectively are on two C_3, $2+4=6$, but among the remaining numbers, only 5 is smaller than 6, contradiction. Thus, 4 can only be a vertex label for one of v_3, v_4 and v_6.

2. If 4 is a label of v_3 or v_6, then the vertex labeled by 4 is adjacent to v_1 and v_5. Thus, in C_3 on which there are the vertices labeled by 2, 4, the third vertex is only labeled by 5, in C_3 on which there are the vertices labeled by 3, 4, the third vertex is only labeled by 6, so then, the vertex labeled by 5 is adjacent to v_1, in C_3 on which there are the vertices labeled by 2, 5, the label of the third vertex is 7 or 8, contradiction.

3. If the label of v_4 is 4, then the two vertices labeled by 3, 4 are on two C_3, $3+4=7$. in the vertex labels, only 8 is greater than 7, contradiction. □

Overall, $\mu_{vEuclid}(Circ(7, 2)) > 1$.

Now, define a vertex labeling f as follows.

Specifically, let $f(v_1) = 3$, $f(v_2) = 6$, $f(v_3) = 5$, $f(v_4) = 4$, $f(v_5) = 8$, $f(v_6) = 9$, $f(v_7) = 7$.

On $Circ(7, 2)$, there are seven C_3, their vertex sets are $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$, $\{v_5, v_6, v_7\}$, $\{v_6, v_7, v_1\}$, $\{v_7, v_1, v_2\}$ respectively, the corresponding vertex label sets are $\{3, 5, 6\}$, $\{4, 5, 6\}$, $\{4, 5, 8\}$, $\{4, 8, 9\}$, $\{7, 8, 9\}$, $\{3, 7, 9\}$, $\{3, 6, 7\}$. On these C_3, have $3+5 > 6$, $4+5 > 6$, $4+5 > 8$, $4+8 > 9$, $7+8 > 9$, $3+7 > 9$, $3+6 > 7$. Hence, $\mu_{vEuclid}(Circ(7, 2)) = 2$.

Now, the results for $n = 4$, 5, 6 and 7 are obtained, next, we study $\mu_{vEuclid}(Circ(n, 2))$ when $n > 7$.

Theorem 4.5 For $n > 7$, $\mu_{vEuclid}(Circ(n, 2)) = 1$.

Proof. By Theorem 1.1, we know that $\mu_{vEuclid}(Circ(n, 2)) \geq 1$. Now, we prove that $\mu_{vEuclid}(Circ(n, 2)) = 1$ for $n > 7$ according to the parity of n.

1. $n \geq 8$ is even.

1.1 $n = 8$.

Define a vertex labeling f as follows.

$$f(v_1) = 2,\ f(v_2) = 5,\ f(v_3) = 4,\ f(v_4) = 8,\ f(v_5) = 9,\ f(v_6) = 3,\ f(v_7) = 7,\ f(v_8) = 6.$$

On $Circ(8, 2)$, there are eight C_3. By these vertex labels, we obtain that the vertex label sets on these eight C_3 are $\{2, 5, 4\}$, $\{5, 4, 8\}$, $\{4, 8, 9\}$, $\{3, 8, 9\}$, $\{3, 7, 9\}$, $\{3, 6, 7\}$, $\{2, 6, 7\}$, $\{2, 5, 6\}$ respectively.

Since $2+4 > 5$, $5+4 > 8$, $4+8 > 9$, $3+8 > 9$, $3+7 > 9$, $3+6 > 7$, $2+6 > 7$, $2+5 > 6$, thus, we obtain that $\mu_{vEuclid}(Circ(8, 2)) = 1$.

1.2 $n > 8$.

Now, we find a vertex labeling f such that the vertex labels set is $[2, n+1]$ and $\mu_{vEuclid}(Circ(n, 2)) = 1$. First, divide the vertices on $Circ(n, 2)$ into two parts, and then label them separately. Next, we prove that these vertex labels satisfy the vertex Euclidean condition. Specifically, let

1. $f(v_1) = 2$, $f(v_2) = 5$, $f(v_3) = 4$, $f(v_4) = 8$, $f(v_{n-3}) = 3$, $f(v_{n-2}) = 9$, $f(v_{n-1}) = 7$, $f(v_n) = 6$.
2. $f(v_i) = 2i$, $5 \leq i \leq \dfrac{n}{2}$.
3. $f(v_{n+1-i}) = 2i+1$, $5 \leq i \leq \dfrac{n}{2}$.

Thus, we obtain that the vertex labels set is $[2, n+1]$.

For those C_3 on which the vertex sets are $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_{n-3}, v_{n-2}, v_{n-1}\}$, $\{v_{n-2}, v_{n-1}, v_n\}$, $\{v_{n-1}, v_n, v_1\}$, $\{v_n, v_1, v_2\}$ respectively, have

$$f(v_1)+f(v_3) = 2+4 = 6 > 5 = f(v_2), \quad f(v_2)+f(v_3) = 5+4 = 9 > 8 = f(v_4),$$

$$f(v_{n-3})+f(v_{n-1}) = 3+7 = 10 > 9 = f(v_{n-2}), \quad f(v_{n-1})+f(v_n) = 7+6 = 13 > 9 = f(v_{n-2}),$$

$$f(v_1)+f(v_n) = 2+6 = 8 > 7 = f(v_{n-1}), \quad f(v_1)+f(v_2) = 2+5 = 7 > 6 = f(v_n).$$

So the vertex labels on these C_3 satisfy the vertex Euclidean condition.

For those four C_3 on which the vertex sets are $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$, $\{v_{n-5}, v_{n-4}, v_{n-3}\}$, $\{v_{n-4}, v_{n-3}, v_{n-2}\}$ respectively, have

$$f(v_3)+f(v_4)=4+8=12>10=f(v_5),\quad f(v_4)+f(v_5)=8+10=18>12=f(v_6),$$

$$f(v_{n-3})+f(v_{n-4})=3+11=14>13=f(v_{n-5}),\quad f(v_{n-2})+f(v_{n-3})=9+3=12>11=f(v_{n-4}).$$

So the vertex labels on these four C_3 satisfy the vertex Euclidean condition.

For those C_3 on which the vertex sets $\{v_i, v_{i+1}, v_{i+2}\}$ for $5 \le i \le n-6$, the minimum value of $f(v_i)$ $(5 \le i \le n-4)$ is 10, on each C_3, the difference between any two labels is less than or equal to 4, thereby, the sum of any two vertex labels is greater than the third vertex label in each C_3.

Hence, $\mu_{vEuclid}(Circ(n, 2)) = 1$ for n is even.

2. $n > 8$ is odd.

First, we define a vertex labeling f as follows.

1. $f(v_1)=2$, $f(v_2)=5$, $f(v_3)=4$, $f(v_4)=8$, $f(v_{n-3})=9$, $f(v_{n-2})=3$, $f(v_{n-1})=7$, $f(v_n)=6$.

2. $f(v_i)=2i$, $5 \le i \le \dfrac{n-1}{2}$, $f\left(v_{\frac{n+1}{2}}\right)=n+1$.

3. $f(v_{n+1-i})=2i+1$, $5 \le i \le \dfrac{n-1}{2}$.

The vertex labels set is $[2, n+1]$.

For the those C_3 on which the vertex sets are $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_{n-3}, v_{n-2}, v_{n-1}\}$, $\{v_{n-2}, v_{n-1}, v_n\}$, $\{v_{n-1}, v_n, v_1\}$, $\{v_n, v_1, v_2\}$ respectively, the discussions about the vertex labels are the same as thats in 1.2. For the labeling of other vertices, we will discuss in four cases.

2.1 $n=9$.

There are two C_3 on which the vertex labels are not discussed yet, their vertex label sets are $\{3, 9, 10\}$ and $\{4, 8, 10\}$ respectively. Because $3+9>10$, $4+8>10$, thus, $\mu_{vEuclid}(Circ(9, 2)) = 1$.

2.2 $n=11$.

There are four C_3 on which the vertex labels are not discussed yet, their vertex label sets are $\{3, 9, 11\}$, $\{9, 11, 12\}$, $\{4, 8, 10\}$, $\{8, 10, 12\}$ respectively. Because $3+9>11$, $9+11>12$, $4+8>10$, $8+10>12$, thus, $\mu_{vEuclid}(Circ(11, 2)) = 1$.

2.3 $n=13$.

There are seven C_3 on which the vertex labels are not discussed yet, their vertex label sets are $\{4, 8, 10\}$, $\{8, 10, 12\}$,$\{10, 12, 14\}$, $\{12, 13, 14\}$, $\{3, 9, 11\}$, $\{9, 11, 13\}$, $\{11, 13, 14\}$ respectively. Because $4+8>10$, $8+10>12$, $10+12>14$, $12+13>14$, $3+9>11$, $9+11>12$, $11+13>14$, thus, $\mu_{vEuclid}(Circ(13, 2)) = 1$.

2.4 $n>13$.

First, we discuss the vertex labels are on four C_3 where the vertex label sets are $\{4, 8, 10\}$, $\{8, 10, 12\}$, $\{3, 9, 11\}$, $\{9, 11, 13\}$.

Because $4+8>10$, $8+10=18>12$, $3+9>11$, $9+11=20>13$, these vertex labels satisfy the vertex Euclidean condition.

For the vertex labels on the other C_3, in each C_3, the maximum value and the minimum value of the vertex labels are denoted by a, b respectively, then $a-b \le 4$, the minimum value in these vertex labels is 10, so in each C_3, the vertex labels satisfy the vertex Euclidean condition.

This completes the proof. □

Overall,

Theorem 4.6 For $n > 3$ is integer,

$$\mu_{vEuclid}(Circ(n, 2)) = \begin{cases} 2, & n = 4, 6, 7 \\ 3, & n = 5 \\ 1, & n > 7 \end{cases}$$

5. The vertex Euclidean properties of the zykov sums of a cycle and an *m* null graph

In this section, we study the vertex Euclidean properties of the Zykov sums of a cycle and a m null graph.

Definition 5.1 [8] Zykov sum of two simple graphs G_1 and G_2, denoted $G_1 \oplus G_2$, is defined as the graph with

$$V(G_1 \oplus G_2) = V(G_1) \cup V(G_2),$$

$$E(G_1 \oplus G_2) = E(G_1) \cup E(G_2) \cup \{(u, v): u \in V(G_1), v \in V(G_2)\}.$$

Consequently, the Zykov sum of G_1 and G_2 is formed by adding edges that connect every vertex of G_1 to every vertex of G_2.

In this section, $C_n \oplus N_m$ ($n \geq 3$, $m \geq 1$) is investigated, $C_n \oplus N_m$ is also called m-cone graph [9]. In $C_n \oplus N_m$, the vertices on C_n are successively denoted by $u_1, u_2, \ldots, u_n$, the vertices on N_m are denoted by $v_1, v_2, \ldots, v_m$.

Example 5.1 $C_5 \oplus N_2$ is shown in Figure 4.

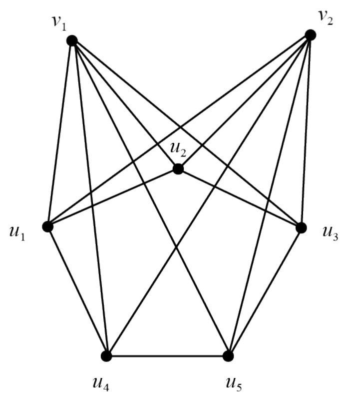

Figure 4. $C_5 \oplus N_2$

Since all vertices are on some subgraph C_3 of $C_n \oplus N_m$, by Theorem 1.1, $\mu_{vEuclid}(C_n \oplus N_m) \geq 1$. Now, we discuss $\mu_{vEuclid}(C_n \oplus N_m)$ based on different values of m, n.

Theorem 5.1 For $C_n \oplus N_1$,

$$\mu_{vEuclid}(C_n \oplus N_1) = \begin{cases} 2, & n = 3 \\ 1, & n > 3 \end{cases}.$$

Proof. 1. $n = 3$.

On $C_3 \oplus N_1$, there are four C_3. Since any vertex on $C_3 \oplus N_1$ is adjacent to other vertices, and any vertex on two C_3, thus, if $\mu_{vEuclid}(C_3 \oplus N_1) = 1$, the vertex labeled by 2 and the vertex labeled by 3 are adjacent and they are on two C_3, $2+3 = 5 > 4$, there must a C_3 on which the vertex labels do not satisfy the vertex Euclidean condition, contradiction. Hence, $\mu_{vEuclid}(C_3 \oplus N_1) > 1$.

Now, define a vertex labeling f as follows: The labels of v_1, u_1, u_2, u_3 are successively 3, 4, 5, 6. Thus, the vertex label sets of four C_3 are $\{3, 4, 5\}$, $\{3, 5, 6\}$, $\{3, 4, 6\}$, $\{4, 5, 6\}$ respectively. Because $3+4 > 5$, $3+4 > 6$, $3+5 > 6$, $4+5 > 6$, thereby, these vertex labels satisfy the vertex Euclidean condition.

2. $n > 3$.

We find a vertex labeling f such that $\mu_{vEuclid}(C_n \oplus N_1) = 1$.

First, we define the labels of v_1, u_1, u_2, u_3, u_4.

Let $f(v_1) = 5$, $f(u_1) = 2$, $f(u_2) = 6$, $f(u_3) = 3$, $f(u_4) = 4$.

Next, after the labels of v_1, u_1, u_2, u_3, u_4 are determined, for other vertices, we define their labels according to the parity of n.

2.1 n is even. Let

$$f(u_i) = 2i-1, \quad 4 \le i \le \frac{n+2}{2}.$$

$$f(u_{n-i}) = 2i+6, \quad 1 \le i \le \frac{n-4}{2}.$$

Thus, we obtain a vertex label set $[2, n+2]$. Now, we investigate the vertex labels on each C_3.

On C_3 which vertex set is $\{v_1, u_1, u_2\}$, has $f(v_1)+f(u_1) = 5+2 = 7 > 6 = f(u_2)$.

On C_3 which vertex set is $\{v_1, u_2, u_3\}$, has $f(v_1)+f(u_2) = 5+3 = 8 > 6 = f(u_2)$.

On C_3 which vertex set is $\{v_1, u_3, u_4\}$, has $f(v_1)+f(u_3) = 5+3 = 8 > 7 = f(u_4)$.

On C_3 which vertex set is $\{v_1, u_{n-1}, u_n\}$, has $f(v_1)+f(u_n) = 5+4 = 9 > 8 = f(u_{n-1})$.

On C_3 which vertex set is $\{v_1, u_n, u_1\}$, has $f(u_1)+f(u_n) = 2+4 = 6 > 5 = f(v_1)$.

For other C_3, in each C_3, the difference of the vertex labels between u_i and u_{i+1} is less than or equal to 2, the minimum value of the vertex labels is $f(v_1) = 5$, so, the conclusion is correct for $n > 4$ is even.

2.2 n is odd. Let

$$f(u_i) = 2i-1, \quad 4 \le i \le \frac{n+3}{2}.$$

$$f(u_{n-i}) = 2i+6, \quad 1 \le i \le \frac{n-5}{2}.$$

The discussions are the same as thats in 2.1, we can to know that the conclusion holds for $n > 4$ is odd.

This completes the proof. □

Theorem 5.2 When $m > 1$, then $\mu_{vEuclid}(C_n \oplus N_m) > 1$.

Proof. Assume $\mu_{vEuclid}(C_n \oplus N_m) = 1$. Then there exist a vertex labeling f such that the vertex label set is $[2, m+n+1]$.

1. Let $f(w_1) = 2$, $w_1 \in V(C_n \oplus N_m)$, the vertex w_2 is adjacent with w_1 labeled by a ($a < n+m+1$). Since w_1 and w_2 are at least on two C_3, but on the vertex label set of the third on these C_3 is only $\{a-1, a+1\}$.

1.1 Let $w_1 \in \{v_i | 1 \le i \le m\}$, $w_2 \in \{u_j | 1 \le j \le n\}$. Without loss of generality, let $w_1 = v_1$, $w_2 = u_1$.

At this time, for $u_2, u_3, \ldots, u_n$, no matter how their labels are defined, in $u_1, u_2, \ldots, u_n$, there must be two vertex labels with a difference greater than or equal to 2, resulting in a situation where the vertex Euclidean condition is not satisfied.

1.2 Let $w_1 \in \{u_j | 1 \leq j \leq n\}$, $w_2 \in \{v_i | 1 \leq i \leq m\}$. Without loss of generality, let $w_1 = u_1$, $w_2 = v_1$.

Since u_1 is labeled by 2, v_1 is labeled by a, then the vertex label set is $\{a-1, a+1\}$. Thus, the vertex v_i $(i \geq 2)$ has no label that satisfies the vertex Euclidean condition.

1.3 $w_1, w_2 \in \{u_i | 1 \leq i \leq n\}$. Without loss of generality, let $w_1 = u_1$, $w_2 = u_2$.

At this time, in $v_1, v_2, \ldots, v_m$, there must exist two vertices labeled by $a-1$ and $a+1$ respectively. Thus, there is not a a positive integer is the label of u_n so that the vertex Euclidean condition holds.

2. $a = m+n+1$. □

Since w_1 and w_2 are in two C_3, but there only a positive integer $m+n$ so that $2+m+n > m+n+1$ holds, contradiction.

Overall, $\mu_{vEuclid}(C_n \oplus N_m) > 1$ when $m > 1$.

Theorem 5.3 For $m > 1$, $n > 2$, $\mu_{vEuclid}(C_n \oplus N_m) = 2$.

Proof. 1. $n = 3$.

Define a vertex labeling f as follows.

$$f(v_i) = 2+i, \ 1 \leq i \leq m.$$

$$f(u_i) = m+2+i, \ i = 1, 2, 3.$$

Because $|f(u_i) - f(u_j)| \leq 2$ for $1 \leq i, j \leq 3$ and $i \neq j$, the minimum value of the vertex labels is 3, so these vertex labels satisfy the vertex Euclidean condition.

2. $n > 3$.

We find a vertex labeling f such that $\mu_{vEuclid}(C_n \oplus N_m) = 2$.

First, define the labels of $v_1, v_2, \ldots, v_m, u_1, u_2, u_3$. Let

$$f(v_i) = 2+i, \quad 1 \leq i \leq m.$$

$$f(u_1) = m+3, \quad f(u_2) = m+4, \quad f(u_3) = m+5.$$

From the discussions on the case of $n = 3$, we can to know that these vertex labels satisfy the vertex Euclidean condition.

Next, define the remaining vertex labels according to the parity of n.

2.1 n is odd.

Define the labels of the remaining vertices as follows.

$$f(u_i) = m+2i, \quad 3 \leq i \leq \frac{n+1}{2}.$$

$$f(u_{n-i}) = m+5+2i, \quad 1 \leq i \leq \frac{n-3}{2}.$$

Thus, for any two adjacent vertices on C_n, the difference of their labels is less than or equal to 2, the minimum value in $[3, m+2]$ is 3, so these vertex labels satisfy the vertex Euclidean condition.

2.2 n is even.

Defining the labels of the remaining vertices as follows.

$$f(u_i) = m + 2i, \quad 3 \le i \le \frac{n+2}{2}.$$

$$f(u_{n-i}) = m + 5 + 2i, \quad 1 \le i \le \frac{n-4}{2}.$$

The discussions are the same as thats in 2.1, we can to know that the conclusion is correct.

This completes the proof. □

When $m = 1$, $C_n \oplus N_1$ is also called wheel graph in [10], denoted by W_n; when $m = 2$, $C_n \oplus N_2$ is also called a double cone graph in [11], denoted by $DC(n)$. By Theorems 5.1 and 5.3, have

Corollary 5.4 For $n \ge 3$,

$$\mu_{vEuclid}(W_n) = \begin{cases} 2, & n = 3 \\ 1, & n > 3 \end{cases}.$$

Corollary 5.5 For $n \ge 3$, $\mu_{vEuclid}(DC(n)) = 2$.

6. Conclusions

In this paper, we have studied four classes of graphs, they are $GTS(k_1, k_2, \ldots, k_{n-1}; n)$, $(m-1)-XG(m,n)$, $Circ(n,2)$ and $C_n \oplus N_m$ respectively. On $GTS(k_1, k_2, \ldots, k_{n-1}; n)$, the vertex Euclidean deficiency is 1 for any $k_i \ge 1$, $n > 1$, at the same time, besides these results, we have also obtained the vertex Euclidean deficiencies on triangular snakes, double triangular snakes. On $(m-1)-XG(m,n)$, the vertex Euclidean deficiency is 2 for any $m, n \ge 2$ and $m \le n$. On $Circ(n, 2)$, the vertex Euclidean deficiency is different in n equal to different values. On $C_n \oplus N_m$, the vertex Euclidean deficiency is 2 for $m > 1$, $n > 2$, and the vertex Euclidean deficiencies of W_n and $DC(n)$ are obtained too.

7. Closing remarks

The vertex Euclidean labeling is an new area of graph labeling problems, there haven't been many results yet on vertex Euclidean labeling of graphs. Hence, in the future, we will conduct research on the vertex Euclidean properties of graphs, and like to invite the readers to join us.

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Conflict of interest

The authors have no conflict of interest to disclose.

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