Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Fekete-Szegö Functional Problem for Analytic and Bi-Univalent Functions Subordinate to Gegenbauer Polynomials

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Received: 3 September 2024; **Revised:** 21 November 2024; **Accepted:** 21 November 2024

Abstract: This article aims to introduce a new qualitative subclass of bi-univalent and analytic functions that are intricately linked to Gegenbauer polynomials. These polynomials, known for their significant role in various areas of mathematics, provide a robust framework for exploring the properties of analytic functions. In our exploration, we will address the Fekete-Szego problem, which is pivotal in the field of complex analysis. By doing so, we will derive the coefficient bounds $|h_2|$ and $|h_3|$ for functions within this newly defined subclass, thereby enhancing our understanding of their behavior. Furthermore, by concentrating on the specific parameters that were utilized to achieve our primary results, we expect to generate a variety of additional outcomes. These results will not only deepen our insight into the characteristics of these functions but also contribute to the broader discourse on analytic function theory. We anticipate that the findings presented in this article will pave the way for future research and applications, particularly in the realms of mathematical analysis and applied mathematics.

***Keywords*:** univalent function, bell distribution (BD), starlike functions, convex functions, (n, t)-neighborhood, inclusion relations

MSC: 30C45

1. Introduction

Since Legendre discovered orthogonal polynomials in 1784 [1], extensive research has been conducted on their properties and applications. Orthogonal polynomials play a crucial role in various fields of mathematics and physics, particularly in the mathematical analysis of model problems. They frequently arise in the context of solving ordinary differential equations, especially when specific model-related constraints are imposed. These polynomials not only provide a systematic approach to approximating functions but also facilitate the solution of boundary value problems and spectral analysis. Their applications extend beyond pure mathematics into areas such as quantum mechanics, numerical analysis, and approximation theory, highlighting their importance in both theoretical and practical frameworks. The study of orthogonal polynomials continues to evolve, with ongoing research exploring new properties, relationships with other mathematical constructs, and their computational implications.

DOI: https://doi.org/10.37256/cm.5420245636

In contemporary mathematics, orthogonal polynomials hold a significant and multifaceted role, extending their influence far beyond pure mathematical theory into various applied disciplines, particularly in engineering and physics. There is a broad consensus among researchers and practitioners that the utility of these polynomials is undeniable. They are integral to solving approximation theory problems, where they facilitate the representation of complex functions in simpler forms. Moreover, orthogonal polynomials find critical applications in both the theory of differential and integral equations, enhancing our understanding of mathematical models that describe real-world phenomena. Their versatility extends into fields such as mathematical statistics, where they are utilized for constructing estimators and testing hypotheses. In the realm of quantum mechanics, orthogonal polynomials are employed in various applications, including scattering theory, where they help model interactions between particles. They are also vital in automated control systems, where they assist in optimizing system responses. Additionally, in signal analysis, these polynomials contribute to the effective processing and interpretation of signals, while in axisymmetric potential theory, they aid in solving problems related to gravitational and electromagnetic fields. The breadth of their applications underscores the profound impact orthogonal polynomials have across multiple scientific domains [2, 3]. Technically speaking, polynomials P_L and P_m of orders L and m are considered to be orthogonal if and only if they fulfill the requirement that is shown below:

$$\int_b^a w(\rho)P_L(\rho)P_m(\rho)d\rho = 0 \quad \text{for } L \neq m$$

where $w(\rho)$ is a function that can never be negative anywhere in the range $(a,\ b)$. Because of this, the integral is well-defined for any polynomial of finite order $P_L(\rho)$.

The Gegenbauer polynomials could be an example in the category of orthogonal polynomials. They are representatively related to the usual real functions T_R, as can be seen in [4, 5]. Notably, the integral representation of typical real functions and the generating function of Gegenbauer polynomials share algebraic formulas because of their similarity. This demonstrates that they are coupled in a manner typical of the T_R functions that exist. This led to the discovery of several inequalities within the realm of Gegenbauer polynomials, which proved useful.

In the field of geometric function theory, real functions commonly find themselves playing an important role. This is mostly due to the fact that the relationship $T_R = \overline{co}S_R$ and their significance in the process of determining the boundaries of the coefficients bring them to the forefront. S_R is a representation of the class of single-valued functions that have real coefficients in the unit disk G, and $\overline{co}S_R$ is a representation of the closed convex hull of S_R. This is the reason for its existence. Within the context of the overall comprehension of geometric function theory, real functions are typically considered to play a significant role.

He combination of bi-univalent functions and orthogonal polynomials can be particularly useful in the field of mathematical physics, where both concepts may be utilized to solve complex problems involving boundary value problems and in the study of special functions that arise in various physical contexts. By exploring these applications, researchers can deepen their understanding of both theoretical and practical implications in mathematics and related fields.

Gegenbauer polynomials simplify and improve the analysis of bivalent functions. These polynomials facilitate the estimation of functions' coefficients, the simplification of their representation, and a clearer and more accurate understanding of their geometric characteristics. Weak Boundary Adherence: Gegenbauer polynomials are less appropriate for applications needing strong boundary alignment because of their mathematical characteristics, which might cause them to not align well with boundary requirements in specific circumstances.

Several bi-univalent functions are related to Gegenbauer polynomials in this work. Following this, the study analyzes several properties of the described class. As a result, we have the opportunity to study the links between different subclasses of functions and polynomials. In other words, our key findings are a generalization of earlier findings given in an earlier study, to which we will refer when presenting our findings here.

2. Preliminaries to basic concepts

Let $\mathcal{A}$ denote the class of all analytic functions f defined in the open unit disk $G = \{\vartheta \in \mathbb{C} : |\vartheta| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Thus each $f \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form:

$$f(\vartheta) = \vartheta + h_2\vartheta^2 + h_3\vartheta^3 + \cdots \tag{1}$$

Further, assume that S stands for the sub-collection of the set $\mathcal{A}$ containing of functions f in ϑ satisfying (1) which are univalent in ϑ.

Consider the possibility that the functions f and g are analytical in G. When there is a Schwarz function that is analytic in G, it is conceivable for a function f to be subordinate to another function g, which is represented by the notation $f \prec g$. This is the case if there is a Schwarz function as well.

$$\varpi(0) = 0 \text{ and } |\varpi(\vartheta)| < 1 \ \ (\vartheta \in G)$$

such that

$$f(\vartheta) = g(\varpi(\vartheta))$$

Additional considerations to take into account include the fact that if the function g is univalent in G, then the equivalence that follows is valid:

$$f(\vartheta) \prec g(\vartheta) \text{ if and only if } f(0) = g(0)$$

and

$$f(G) \subset g(G)$$

Everyone is aware of the fact that every function f that belongs to the set S has an inverse function f^{-1}, which is represented by

$$f\left(f^{-1}(\vartheta)\right) = \vartheta \ \ (\vartheta \in G)$$

and

$$f\left(f^{-1}(w)\right) = w\left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - h_2 w^2 + \left(2h_2^2 - h_3\right) w^3 - \left(5h_2^3 - 5h_2 h_3 + h_4\right) w^4 + \cdots = g(w) \tag{2}$$

A simple example of bi-univalent functions $f(\vartheta) = \sqrt{\vartheta - 1}$. Also, a function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(\vartheta)$ and $f^{-1}(\vartheta)$ are univalent in $\mathbb{U}$. Let Σ stands for the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions in the class Σ, see the pioneering work done by Srivastava et al. [6, 7].

Recently published a study on this topic. Over the past few years, a multitude of scholars have rekindled their interest in the study of analytic and bi-univalent functions. For other instances, please refer to the citations [8–11].

Amourah et al. [12, 13] took into account the Gegenbauer polynomials $\Psi_\alpha(\rho,\ \vartheta)$, which are given by

$$\Psi_\alpha(\rho,\ \vartheta) = \frac{1}{(1 - 2\rho\vartheta + \vartheta^2)^\alpha} \tag{3}$$

All of the variables $\rho \in [-1,\ 1]$, $\alpha \in \mathbb{R}\backslash\{0\}$, and $\vartheta \in G$ come into play at this point. In light of the fact that the function Ψ_α is analytic in G, taking into account the fact that ρ is held constant, it is possible to expand it by employing a Taylor series in the manner that is described below:

$$\Psi_\alpha(\rho,\ \vartheta) = \sum_{L=0}^{\infty} C_L^\alpha(\rho)\vartheta^L \tag{4}$$

where the degree-L Gegenbauer polynomial is denoted by $C_L^\alpha(\rho)$. Furthermore, the Gegenbauer polynomials are provided by their generating function (3), which may be stated when $\alpha = 0$.

$$\Psi_0(\rho,\ \vartheta) = \sum_{L=0}^{\infty} C_L^0(\rho)\vartheta^n = 1 - \log\left(1 - 2\rho\vartheta + \vartheta^2\right) \tag{5}$$

for $\alpha = 0$. Additionally, it is important to point out that it is preferable to normalize α to a value that is greater than -0.5 [14]. It is not difficult to observe that the Gegenbauer polynomials are consistent with the recursive relation that is presented below:

$$C_L^\alpha(\rho) = \frac{1}{L}\left[2\rho(L + \alpha - 1)C_{L-1}^\alpha(\rho) - (L + 2\alpha - 2)C_{L-2}^\alpha(\rho)\right] \tag{6}$$

The first three Gegenbauer polynomials are given as follows:

$$C_0^\alpha(\rho) = 1,\ C_1^\alpha(\rho) = 2\alpha\rho \ \text{ and } \ C_2^\alpha(\rho) = 2\alpha(1 + \alpha)\rho^2 - \alpha \tag{7}$$

When we apply these equations (6), we can quickly acquire Legendre polynomials $P_L(\rho) = C_L^{0.5}(\rho)$ and Chebyshev polynomials of the second kind $G_L(\rho) = C_L^1(\rho)$. It is important to note that by putting $\alpha = 0.5$ and $\alpha = 1$, we can achieve this equation.

At the present time, a great number of academics are concentrating their attention on bi-univalent functions that are connected to orthogonal polynomials, see [15–18]. It is essential to highlight the fact that, as far as we are aware, very scant information is available concerning Gegenbauer polynomials in the setting of bi-univalent functions.

The study of some new subclasses of bi-univalent functions that are subordinate to the Gegenbauer polynomial and Jacobi Polynomials is motivated by Amourah et al. [19–23]. Our objective is to get constraints for the coefficients of the functions $|h_2|$ and $|h_3|$, as well as solution to FeketeSzegö functional problems for functions that belong to these new classes.

3. Definition and special cases

We begin this section by defining the family $\mathfrak{G}_\Sigma(\alpha,\ \lambda,\ \delta,\ \rho)$ as follows:

Definition 1 Let $\lambda \geq 0$ and $0 < \delta \leq 1$, a function $f \in \Sigma$ is said to be in the family $\mathfrak{G}_\Sigma(\alpha,\ \lambda,\ \delta,\ \rho)$ if it satisfies the subordinations:

$$\frac{1}{2}\left[\frac{\vartheta^{1-\lambda} f'(\vartheta)}{(f(\vartheta))^{1-\lambda}} + \left(\frac{\vartheta^{1-\lambda} f'(\vartheta)}{(f(\vartheta))^{1-\lambda}}\right)^{\frac{1}{\delta}}\right] \prec \Psi_\alpha(\rho,\ \vartheta)$$

and

$$\frac{1}{2}\left[\frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}} + \left(\frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}}\right)^{\frac{1}{\delta}}\right] \prec \Psi_\alpha(\rho,\ w)$$

where $\rho \in \left(\frac{1}{2},\ 1\right]$, $g(w) = f^{-1}(w)$, which is determined by the equation (2), and Ψ_α represents the generating function of the Gegenbauer polynomial, which is given by the equation (3).

Special cases:

i) Assume that $\lambda = 0$ and $\delta = 1$, a function $f \in \Sigma$ is said to be in the family $\mathfrak{G}_\Sigma(\alpha,\ 0,\ 1,\ \rho)$ if it satisfies the subordination:

$$\frac{\vartheta f'(\vartheta)}{f(\vartheta)} \prec \Psi_\alpha(\rho,\ \vartheta)$$

and

$$\frac{w g'(w)}{g(w)} \prec \Psi_\alpha(\rho,\ w)$$

where $\rho \in \left(\frac{1}{2},\ 1\right]$, $g(w) = f^{-1}(w)$, which is determined by the equation (2), and Ψ_α represents the generating function of the Gegenbauer polynomial, which is given by the equation (3).

ii) Assume that $\lambda = \delta = 1$ a function $f \in \Sigma$ is said to be in the family $\mathfrak{G}_\Sigma(\alpha,\ 1,\ 1,\ \rho)$ if it satisfies the subordination:

$$f'(\vartheta) \prec \Psi_\alpha(\rho, \vartheta)$$

and

$$g'(w) \prec \Psi_\alpha(\rho, w)$$

where $\rho \in \left(\frac{1}{2}, 1\right]$, $g(w) = f^{-1}(w)$, which is determined by the equation (2), and Ψ_α represents the generating function of the Gegenbauer polynomial, which is given by the equation (3).

4. Coefficient bounds of the class $\mathfrak{G}_\Sigma(\alpha, \lambda, \delta, \rho)$

First, we'll give some estimates for the coefficients for class $\mathfrak{G}_\Sigma(\alpha, \lambda, \delta, \rho)$ that were given in Definition 1.

Theorem 1 For (1) be in the class, let $f \in \mathcal{A}$ be in the family $\mathfrak{G}_\Sigma(\alpha, \lambda, \delta, \rho)$. Then

$$|h_2| \leq \frac{2(2\alpha\rho)\sqrt{2\alpha\rho}\delta}{\sqrt{\begin{array}{l}(2\alpha\rho)^2(\lambda+1)[\delta(\lambda+2)(\delta+1)+(\lambda+1)(1-\delta)]- \\ \left(2\alpha(1+\alpha)\rho^2-\alpha\right)(\lambda+1)^2(\delta+1)^2\end{array}}}$$

and

$$|h_3| \leq \frac{2(2\alpha\rho)\delta}{(\lambda+2)(\delta+1)} + \frac{4(2\alpha\rho)^2\delta}{(\lambda+1)^2(\delta+1)^2}$$

Proof. Let $f \in \mathfrak{G}_\Sigma(\alpha, \lambda, \delta, \rho)$. Then there are two analytic functions ϕ, ϱ given by

$$\phi(z) = k_1\vartheta + k_2\vartheta^2 + k_3\vartheta^3 + \cdots \quad (\vartheta \in G) \tag{8}$$

and

$$\varrho(w) = j_1 w + j_2 w^2 + j_3 w^3 + \cdots \quad (w \in G), \tag{9}$$

with $\phi(0) = \varrho(0) = 0$, $|\phi(\vartheta)| < 1$, $|\varrho(w)| < 1$, ϑ, $w \in G$ such that

$$\frac{1}{2}\left[\frac{\vartheta^{1-\lambda}f'(\vartheta)}{(f(\vartheta))^{1-\lambda}} + \left(\frac{\vartheta^{1-\lambda}f'(\vartheta)}{(f(\vartheta))^{1-\lambda}}\right)^{\frac{1}{\delta}}\right] = \Psi_\alpha(\rho, \phi(\vartheta)) \tag{10}$$

and

$$\frac{1}{2}\left[\frac{w^{1-\lambda}g'(w)}{(g(w))^{1-\lambda}}+\left(\frac{w^{1-\lambda}g'(w)}{(g(w))^{1-\lambda}}\right)^{\frac{1}{\delta}}\right]=\Psi_\alpha(\rho,\ \varrho(w)). \tag{11}$$

Combining (8), (9) and (10) yields

$$\frac{1}{2}\left[\frac{\vartheta^{1-\lambda}f'(\vartheta)}{(f(\vartheta))^{1-\lambda}}+\left(\frac{\vartheta^{1-\lambda}f'(\vartheta)}{(f(\vartheta))^{1-\lambda}}\right)^{\frac{1}{\delta}}\right]=1+C_1^\alpha(\rho)k_1\vartheta+\left[C_1^\alpha(\rho)k_2+C_2^\alpha(\rho)k_1^2\right]\vartheta^2+\cdots \tag{12}$$

and

$$\frac{1}{2}\left[\frac{w^{1-\lambda}g'(w)}{(g(w))^{1-\lambda}}+\left(\frac{w^{1-\lambda}g'(w)}{(g(w))^{1-\lambda}}\right)^{\frac{1}{\delta}}\right]=1+C_1^\alpha(\rho)j_1w+\left[C_1^\alpha(\rho)j_2+C_2^\alpha(\rho)j_1^2\right]w^2+\cdots \tag{13}$$

It is quite well-known that if $|\phi(\vartheta)|<1$ and $|\varrho(w)|<1$, $\vartheta,\ w\in G$, then

$$|k_i|\leq 1 \text{ and } |j_i|\leq 1 \text{ for all } i\in\mathbb{N} \tag{14}$$

After simplification, we can compare the coefficients in (12) and (13) and obtain

$$\frac{(\lambda+1)(\delta+1)}{2\delta}h_2=C_1^\alpha(\rho)k_1 \tag{15}$$

$$\frac{(\lambda+2)(\delta+1)}{4\delta}\left(2h_3+(\lambda-1)h_2^2\right)+\frac{(\lambda+1)^2(1-\delta)}{4\delta^2}h_2^2=C_1^\alpha(\rho)k_2+C_2^\alpha(\rho)k_1^2 \tag{16}$$

$$-\frac{(\lambda+1)(\delta+1)}{2\delta}h_2=C_1^\alpha(\rho)j_1 \tag{17}$$

and

$$\frac{(\lambda+2)(\delta+1)}{4\delta}\left((\lambda+3)h_2^2-2h_3\right)+\frac{(\lambda+1)^2(1-\delta)}{4\delta^2}h_2^2=C_1^\alpha(\rho)j_2+C_2^\alpha(\rho)j_1^2 \tag{18}$$

It follows from (15) and (17) that

$$k_1=-j_1 \tag{19}$$

and

$$\left[\frac{(\lambda+1)^2(1+\delta)^2}{2\delta^2}\right] h_2^2 = \left[C_1^\alpha(\rho)\right]^2\left(k_1^2+j_1^2\right). \tag{20}$$

If we add (16) to (18), we find that

$$\left[\frac{(\lambda+2)(\delta+1)(\lambda+1)}{2\delta}+\frac{(\lambda+1)^2(1-\delta)}{2\delta^2}\right] h_2^2 = C_1^\alpha(\rho)\left(k_2+j_2\right)+C_2^\alpha(\rho)\left(k_1^2+j_1^2\right). \tag{21}$$

Since we have the value of $k_1^2+j_1^2$ from (20), we can plug that value into the right-hand side of (21) to get the following conclusion:

$$h_2^2 = \frac{2\left(C_1^\alpha(\rho)\right)^3\delta^2\left(k_2+j_2\right)}{\left(C_1^\alpha(\rho)\right)^2(\lambda+1)[\delta(\lambda+2)(\delta+1)+(\lambda+1)(1-\delta)]-C_2^\alpha(\rho)(\lambda+1)^2(\delta+1)^2}. \tag{22}$$

Further computations using (7), (14) and (22), we obtain

$$|h_2| \leq \frac{2(2\alpha\rho)\sqrt{2\alpha\rho}\,\delta}{\sqrt{\begin{array}{l}(2\alpha\rho)^2(\lambda+1)[\delta(\lambda+2)(\delta+1)+(\lambda+1)(1-\delta)]- \\ \left(2\alpha(1+\alpha)\rho^2-\alpha\right)(\lambda+1)^2(\delta+1)^2\end{array}}} \cdots$$

Next, if we subtract (18) from (16), we can easily see that

$$\frac{(\lambda+2)(\delta+1)}{\delta}\left(h_3-h_2^2\right) = C_1^\alpha(\rho)\left(k_2-j_2\right)+C_2^\alpha(\rho)\left(k_1^2-j_1^2\right) \tag{23}$$

In view of (19) and (20), we get from (23)

$$h_3 = \frac{C_1^\alpha(\rho)\left(k_2-j_2\right)\delta}{(\lambda+2)(\delta+1)}+\frac{2\left[C_1^\alpha(\rho)\right]^2\left(k_1^2+j_1^2\right)\delta^2}{(\lambda+1)^2(\delta+1)^2}$$

Thus applying (7), we obtain

$$|h_3| \leq \frac{2(2\alpha\rho)\delta}{(\lambda+2)(\delta+1)}+\frac{4(2\alpha\rho)^2\delta^2}{(\lambda+1)^2(\delta+1)^2}$$

We have thus proved the Theorem 1.

5. Fekete-Szegö inequality for the class $\mathfrak{G}_\Sigma(\alpha,\ \lambda,\ \delta,\ \rho)$

In this section, we prove the following Fekete-Szegö (see also [24]) inequality for functions f in the class $\mathfrak{G}_\Sigma(\alpha,\ \lambda,\ \delta,\ \rho)$, using the values of h_3 and h_2^2.

Theorem 2 For (1) in the class and $\eta \in \mathbb{R}$, let $f \in \mathcal{A}$, let be in the family $\mathfrak{G}_\Sigma(\alpha,\ \lambda,\ \delta,\ \rho)$. Then

$$\left|h_3 - \eta h_2^2\right| \leq$$

$$\begin{cases} \dfrac{4|\alpha\rho|\delta}{(\lambda+2)(\delta+1)}, & |\eta-1| \leq K \\ \dfrac{4|2\alpha\rho|^3\delta^2(1-\eta)}{(2\alpha\rho)^2(\lambda+1)[\delta(\lambda+2)(\delta+1)+(\lambda+1)(1-\delta)] - \left(2\alpha(1+\alpha)\rho^2-\alpha\right)(\lambda+1)^2(\delta+1)^2}, & |\eta-1| \geq K, \end{cases}$$

where

$$K = \frac{(2\alpha\rho)^2(\lambda+1)[\delta(\lambda+2)(\delta+1)+(\lambda+1)(1-\delta)] - \left(2\alpha(1+\alpha)\rho^2-\alpha\right)(\lambda+1)^2(\delta+1)^2}{2(2\alpha\rho)^2\delta(\lambda+2)(\delta+1)}.$$

Proof. It follows from (22) and (23) that

$$h_3 - \eta h_2^2$$

$$= \frac{C_1^\alpha(\rho)\,(k_2-j_2)\,\delta}{(\lambda+2)(\delta+1)} + (1-\eta)h_2^2$$

$$= \frac{C_1^\alpha(\rho)\,(k_2-j_2)\,\delta}{(\lambda+2)(\delta+1)} + \frac{2\left(C_1^\alpha(\rho)\right)^3\delta^2\,(k_2+j_2)\,(1-\eta)}{\left(C_1^\alpha(\rho)\right)^2(\lambda+1)[\delta(\lambda+2)(\delta+1)+(\lambda+1)(1-\delta)] - C_2^\alpha(\rho)(\lambda+1)^2(\delta+1)^2}$$

$$= C_1^\alpha(\rho)\left[\left(\Omega(H) + \frac{\delta}{(\lambda+2)(\delta+1)}\right)k_2 + \left(\Omega(H) - \frac{\delta}{(\lambda+2)(\delta+1)}\right)j_2\right]$$

where

$$\Omega(H) = \frac{2\left(C_1^\alpha(\rho)\right)^2\delta(1-\eta)}{\left(C_1^\alpha(\rho)\right)^2(\lambda+1)[\delta(\lambda+2)(\delta+1)+(\lambda+1)(1-\delta)] - C_2^\alpha(\rho)(\lambda+1)^2(\delta+1)^2}$$

According to (7), we find that

$$\left|h_3-\eta h_2^2\right| \leq \begin{cases} \dfrac{2\left|C_1^{\alpha}(\rho)\right|\delta}{(\lambda+2)(\delta+1)}, & 0 \leq |\Omega(H)| \leq \dfrac{1}{(\lambda+2)(\delta+1)} \\ 2\left|C_1^{\alpha}(\rho)\right||\Omega(H)|\delta, & |\Omega(H)| \geq \dfrac{1}{(\lambda+2)(\delta+1)} \end{cases}$$

We have thus proved the Theorem 2.

6. Corollaries and consequences

Here we find two corollaries where $\lambda=0$, $\delta=1$ and $\lambda=\delta=1$, respectively.

Corollary 1 For (1) in the class, let $f \in \mathcal{A}$ be in the family $\mathfrak{G}_{\Sigma}(\alpha,\ 0,\ 1,\ \rho)$. Then

$$|h_2| \leq \frac{2|2\alpha\rho|\sqrt{2\alpha\rho}}{\sqrt{4\left|(2\alpha\rho)^2-2\alpha(1+\alpha)\rho^2+\alpha\right|}}$$

$$|h_3| \leq \alpha\rho+4(\alpha\rho)^2$$

and

$$\left|h_3-\eta h_2^2\right| \leq \begin{cases} |\alpha\rho|, & |\eta-1| \leq |\mathfrak{G}_{\Sigma}(\alpha,\ 0,\ 1)| \\ \dfrac{|2\alpha\rho|^3(1-\eta)}{(2\alpha\rho)^2-2\alpha(1+\alpha)\rho^2+\alpha}, & |\eta-1| \geq |\mathfrak{G}_{\Sigma}(\alpha,\ 0,\ 1)| \end{cases}$$

where

$$\mathfrak{G}_{\Sigma}(\alpha,\ 0,\ 1)=\frac{(2\alpha\rho)^2-2\alpha(1+\alpha)\rho^2+\alpha}{2(2\alpha\rho)^2}$$

Corollary 2 For (1) in the class, let $f \in \mathcal{A}$ be in the family $\mathfrak{G}_{\Sigma}(\alpha,\ 1,\ 1,\ \rho)$. Then

$$|h_2| \leq \frac{2|\alpha\rho|\sqrt{\alpha\rho}}{\sqrt{2\left|3(2\alpha\rho)^2-4\left(2\alpha(1+\alpha)\rho^2-\alpha\right)\right|}}$$

$$|h_3| \leq \frac{2|\alpha\rho|}{3}+(\alpha\rho)^2$$

and

$$\left|h_3-\eta h_2^2\right| \leq \begin{cases} \dfrac{2|\alpha\rho|}{3}, & |\eta-1| \leq |\mathfrak{G}_\Sigma(\alpha,\ 1,\ 1)| \\ \\ \dfrac{8|\alpha\rho|^3(1-\eta)}{3(2\alpha\rho)^2-4\left(2\alpha(1+\alpha)\rho^2-\alpha\right)}, & |\eta-1| \geq |\mathfrak{G}_\Sigma(\alpha,\ 1,\ 1)| \end{cases}$$

where

$$\mathfrak{G}_\Sigma(\alpha,\ 1,\ 1) = \frac{3(2\alpha\rho)^2-4\left(2\alpha(1+\alpha)\rho^2-\alpha\right)}{12(\alpha\rho)^2}$$

Remark 1 The results obtained from this study have the potential to result in novel discoveries about the classes $\mathfrak{G}_\Sigma(1,\ \lambda,\ \delta,\ \rho)$ with respect to Chebyshev polynomials and $\mathfrak{G}_\Sigma(0.5,\ \lambda,\ \delta,\ \rho)$ with regard to Legendre polynomials.

Concluding Remark The primary purpose of this study is to derive some estimations of the Taylor-Maclaurin coefficients, denoted as $|h_2|$ and $|h_3|$, as well as the inequality of Fekete-Szego, denoted as $\left|h_3-\eta h_2^2\right|$, for functions that belong to a novel class $\mathfrak{G}_\Sigma(\alpha,\ \lambda,\ \delta,\ \rho)$ of analytic and bi-univalent functions $f(\vartheta)$ in the open unit disk G.

Conflict of interest

The authors declare no competing financial interest.

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