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Research Article

Regularity of Weak Solutions to a Class of Nonlinear Parabolic Equations in Fractional Sobolev Spaces

Huimin Cheng, Feng Zhou*

School of Mathematics and Statistics, Shandong Normal University, Jinan, China
E-mail: zhoufeng@u.nus.edu

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Abstract: In this article, we study regularity of weak solutions to a class of nonlinear parabolic equations in divergence form. The main purpose is to present a regularity estimate with more general conditions on coefficients, N-functions and non-homogeneous terms in the fractional Sobolev spaces. By deriving a higher integrability estimate of weak solutions, we obtain the desired regularity estimate. In addition, the results of this article expand the regularity theory of parabolic equations in fractional Sobolev spaces and Besov spaces.

Keywords: nonlinear parabolic equations, regularity, fractional sobolev spaces

MSC: 35K55, 35B65

1. Introduction

In this article, we aim to study regularity of weak solutions to a class of nonlinear parabolic equation in divergence form

$$u_t - \operatorname{div} A(\nabla u,\, x,\, t) = \operatorname{div}\left[\frac{\Phi(|F|)}{|F|^2}F\right] \text{ in } \Omega_T := \Omega \times (t_0,\, t_0+T], \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain, $t_0 \in \mathbb{R},\ T>0,\ u \in C\left((t_0,\, t_0+T];\, L^2(\Omega)\right) \cap L^{\Phi}_{\text{loc}}((t_0,\, t_0+T];\, W^{1,\,\Phi}_{\text{loc}}(\Omega))$, and $\Phi(t)$ is an N-function which will be explained in Section 2. In (1), $F(x,\, t) = (f_1(x,\, t),\, \cdots,\, f_n(x,\, t))$ is a given vector-valued function. Moreover, we assume that $A(\nabla u,\, x,\, t)$ satisfies the following parabolic conditions

$$\text{(A1) } r\frac{\Phi\left[\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{1}{2}}\right]}{\mu^2+|\xi|^2+|\eta|^2}|\xi-\eta|^2 \le \left\langle D_\xi A(\xi,\, x,\, t)(\xi-\eta),\, (\xi-\eta)\right\rangle,$$

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$$\text{(A2)}\ |A(\xi,\ x,\ t)| + \left(\mu^2+|\xi|^2\right)^{\frac{1}{2}} |D_\xi A(\xi,\ x,\ t)| \leq \Lambda \frac{\Phi\left[\left(\mu^2+|\xi|^2\right)^{\frac{1}{2}}\right]}{(\mu^2+|\xi|^2)^{\frac{1}{2}}},$$

where $0<\mu<1$, r, and Λ are constants, $\xi,\ \eta \in \mathbb{R}^n$, $\langle \cdot,\ \cdot \rangle$ means inner product, and $D_\xi A(\xi,\ x,\ t)$ is the derivative of the first variable to $A(\xi,\ x,\ t)$. Based on (A1) and (A2), we calculate that

$$[A(\xi,\ x,\ t)-A(\eta,\ x,\ t)](\xi-\eta) \geq \bar{r} \frac{\Phi\left[\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{1}{2}}\right]}{\mu^2+|\xi|^2+|\eta|^2} |\xi-\eta|^2, \tag{2}$$

where $\bar{r}$ is a positive constant. Meanwhile, given $0<\alpha<1$, we assume there are non-negative measurable functions $g_k(x,\ t) \in L_{\text{loc}}^{\frac{n+2}{\alpha}}(\Omega_T)$ such that

$$\sum_k \|g_k(x,\ t)\|^2_{L^{\frac{n+2}{\alpha}}(\Omega_T)} < +\infty, \tag{3}$$

and

$$\text{(A3)}\ \ |A(\xi,\ x,\ t)-A(\xi,\ y,\ s)| \leq \left[|x-y|^2+|t-s|\right]^{\frac{\alpha}{2}} (g_k(x,\ t)+g_k(y,\ s)) \frac{\Phi\left[\left(\mu^2+|\xi|^2\right)^{\frac{1}{2}}\right]}{(\mu^2+|\xi|^2)^{\frac{1}{2}}}$$

for any $\xi \in \mathbb{R}^n$ and almost every $(x,\ t),\ (y,\ s) \in \Omega_T$ satisfying $2^{-k}\text{diam}(\Omega_T) \leq (|x-y|^2+|t-s|)^{\frac{1}{2}} \leq 2^{-k+1}\text{diam}(\Omega_T)$ with $k \in \mathbb{N}$.

We denote that the equation (1) has some special examples. The most typical example is

$$u_t - \text{div}(|Du|^{p-2}Du) = \text{div}(|F|^{p-2}F). \tag{4}$$

Misawa [1] and Acerbi [2] obtained the Calderón-Zygmund estimates of weak solutions to (4), respectively. If $\frac{\Phi(|F|)}{|F|^2}F = f$, then (1) becomes

$$u_t - \text{div}\ a(Du,\ x,\ t) = \text{div}\, f. \tag{5}$$

Byun [3] analyzed the global Calderón-Zygmund estimates for the weak solutions to (5). Many authors presented various kinds of regularity estimates for weak solutions to

$$u_t - \text{div}\left[a\left((\mathscr{A}Du \cdot Du)^{\frac{1}{2}}\right)\mathscr{A}Du\right] = \text{div}\, f \tag{6}$$

in [4, 5] and [6]. It should be mentioned that Yao [7] obtained the regularity of weak solution to (6) in Besov spaces.

Moreover, Kuusi and Mingione [8] dealt with non-homogeneous, measure data and possible degenerate parabolic equations. For the research of parabolic systems, Bögelein, Foss and Mingione [9] studied a class of quasilinear parabolic systems with p-growth

$$\partial_t u = \mathrm{div}\ a(z,\ u,\ Du),$$

where $z = (x,\ t)$. They filled a gap in the partial regularity theory of quasilinear parabolic systems. Mingione presents a great number of regularity theories of parabolic equations and systems in [10–13]. In recent works [14–16], the regularities of weak solutions to parabolic equations are presented. These conclusions and methods enrich the research perspectives. Based on [17], it is proposed that the nonlinear function of the spatial gradient of the weak solutions possesses higher differentiability.

In [18–20], the authors studied regularity theories in fractional Sobolev spaces. Ambrosio [18] investigated the regularity properties of weak solutions to the strongly degenerate equation

$$u_t - \mathrm{div}\left[(|Du|-1)_+^{p-1}\frac{Du}{|Du|}\right] = f \text{ in } \Omega_T = \Omega\times(0,\ T),$$

where $p \geq 2$ and $(\cdot)_+$ stands for the positive part. In [19], the authors dealed with the Dirichlet problem for the elliptic p-Laplace equation

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{7}$$

where $\Omega \subset \mathbb{R}^n$ is a convex polyhedral domain and $p > 2$. The authors proved global regularity results for weak solutions u under suitable assumptions in fractional order Sobolev spaces. In [20], one has higher-order fractional regularity for the viscosity solutions of uniformly elliptic equations of the form

$$F(D^2u) = f \text{ in } B_1.$$

The assumption of the Carathéodory vector field in this article is inspired by the research [21]. The authors obtained Besov regularity of the weak solution to

$$u_t - \mathrm{div}\, a(Du,\ x,\ t) = \mathrm{div}\, F,\ (x,\ t) \in \Omega_T.$$

They assumed that $a(\xi,\ x,\ t)$ is a Carathéodory function, and there are constants $r_1,\ r_2 > 0$ such that

$$r_1|\xi|^{p-2}|\eta|^2 \leq \langle D_\xi a(\xi,\ x,\ t)\eta,\ \eta\rangle$$

and

$$|a(\xi,\ x,\ t)|+|\xi|\left|D_{\xi}a(\xi,\ x,\ t)\right|\leq r_2|\xi|^{p-1}$$

hold for any $\xi,\ \eta\in\mathbb{R}^n,\ (x,\ t)\in\mathbb{R}^n\times\mathbb{R}$ and $p\geq 2$.

The N-function in this article was studied by many authors. For instance, Behn and Diening [22] studied global regularity of solutions to nonlinear elliptic systems

$$\begin{cases} -\mathrm{div}\left((\delta+|u|)^{p-2}u\right)=f \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega, \end{cases}$$

where $\delta u=\dfrac{1}{2}(\nabla u+(\nabla u)^T)$ denotes the symmetric part of ∇u. Barletta [23] gained the existence and regularity of solutions to nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}\,(B(x,\ u,\ \nabla u))=f(x,\ u,\ \nabla u) \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega. \end{cases} \tag{8}$$

The author considered the growth conditions on the terms appearing in (8) require to replace the customary Sobolev space with an Orlicz space. In [24], the authors studied properties of the local weak solution $u\in W^{1,\ \varphi}(\Omega)$ and $\pi\in L^{\varphi^*}(\Omega)$ of the generalized Stokes problem

$$-\mathrm{div}\,\mathscr{B}(Du)+\nabla u=-\mathrm{div}\,G \quad \text{in } \Omega$$

for given $G:\Omega\to\mathbb{R}^{2\times 2}_{sym}$, where u stands for the velocity of a fluid and π for its pressure.

Inspired by the above insight, this article aims to obtain a fractional regularity of solutions to (1). By introducing

$$H_{\Phi}(\xi)=\left[\frac{\Phi\left[(\mu^2+|\xi|^2)^{\frac{1}{2}}\right]}{\mu^2+|\xi|^2}\right]^{\frac{1}{2}}\xi, \tag{9}$$

we present the main result of this article.

Theorem 1 Let $0<\alpha<1$. Assume that (A1)-(A3) hold. Φ is an N-function satisfying (G1) and (G2). If u is a weak solution to the equation (1) with $\dfrac{\Phi(|F|)}{|F|^2}F\in L^2\left((t_0,\ t_0+T];\ W^{\alpha,\ 2}(\Omega)\right)$, then $H_{\Phi}(\nabla u)\in W^{\alpha,\ 2}(\Omega_T)$ locally.

The classical Orlicz-Sobolev spaces $W^{1,\ \Phi}(\Omega_T)$ and fractional Sobolev spaces $W^{\alpha,\ 2}(\Omega_T)$ appeared in Theorem 1 are introduced in Section 2. We expect to acquire the fractional Sobolev regularity of weak solutions and expand the theory of parabolic equation in divergence form. Meanwhile, we study the weak solutions in $W^{1,\ \Phi}(\Omega_T)$ with variable index $\Phi(t)$. In addition, we obtain the followings.

Corollary 2 Let $0<\alpha<1$. Assume that (A1)-(A3) hold. Φ is an N-function satisfying (G1) and (G2). If u is a weak solution to the equation (1) with $\dfrac{\Phi(|F|)}{|F|^2}F\in L^2\left((t_0,\ t_0+T];\ B^{\alpha}_{2,\ 2}(\Omega)\right)$, then $H_{\Phi}(\nabla u)\in B^{\alpha}_{2,\ 2}(\Omega_T)$ locally.

Corollary 3 Let $0 < \alpha < 1$. Assume that (A1)-(A3) hold. Φ is an N-function satisfying (G1) and (G2). If u is a weak solution to the equation (1) with $\frac{\Phi(|F|)}{|F|^2}F \in L^2\left((t_0,\ t_0+T];\ F^{\alpha}_{2,\ 2}(\Omega)\right)$, then $H_\Phi(\nabla u) \in F^{\alpha}_{2,\ 2}(\Omega_T)$ locally.

In these corollaries, $B^{\alpha}_{2,\ 2}(\Omega)$ and $F^{\alpha}_{2,\ 2}(\Omega)$ represent Besov spaces and Triebel-Lizorkin spaces, respectively.

2. Preliminaries

2.1 *N-Function*

In this article, the relation $\alpha \sim \beta$ denotes that there are two positive constants C' and C'' such that

$$C'\beta \le \alpha \le C''\beta. \tag{10}$$

Definition 4 [24] We say that a convex function $\Phi(t)$: $[0,\ \infty) \to [0,\ \infty)$ is an N-function, if it has the following properties:

(1) There are derivatives $\Phi'(t)$ and $\Phi''(t)$ of $\Phi(t)$ satisfying $\Phi(0) = 0$, $\Phi(1) = 1$, $\Phi'(t) > 0$ for $t > 0$.

(2) (Δ_2-condition) There exists $C_1 > 0$ such that for all $t > 0$ it holds $\Phi(2t) \le C_1\Phi(t)$. By $\Delta_2(\Phi)$ we denote the smallest constant C_1.

Let Φ be an N-function. We note that since $\Phi(t) \le \Phi(2t)$, the Δ_2-condition implies that $\Phi(2t) \sim \Phi(t)$. In this article, we assume that $\Phi(t)$ satisfies the following assumptions.

(G1) Let $\Phi(t)$ be an N-function with $\Phi(st) = C_2\Phi(s)\cdot\Phi(t)$ and $\Phi(t) \in C^2((0,\ +\infty)) \cap C^1([0,\ +\infty))$ such that $\frac{\Phi(t)}{t^2}$ is almost monotone increasing on $(0,\ +\infty)$.

(G2) There is a positive constant C such that

$$C^{-1}\frac{\Phi\left[(\mu^2+|\xi|^2+|\eta|^2)^{\frac{1}{2}}\right]}{\mu^2+|\xi|^2+|\eta|^2} \le \frac{|H_\Phi(\xi)-H_\Phi(\eta)|^2}{|\xi-\eta|^2} \le C\frac{\Phi\left[(\mu^2+|\xi|^2+|\eta|^2)^{\frac{1}{2}}\right]}{\mu^2+|\xi|^2+|\eta|^2} \tag{11}$$

for any $\xi,\ \eta \in \mathbb{R}^n$ and $|\xi-\eta| \ne 0$.

We note the complementary function $\tilde{\Phi}$ of Φ by

$$\tilde{\Phi}(s) = \sup_{t\ge 0}\{st-\Phi(t)\}. \tag{12}$$

By [24], we get that

$$\Phi(t) \sim \Phi'(t)t \text{ and } \tilde{\Phi}\left(\frac{\Phi(t)}{t}\right) \sim \Phi(t), \tag{13}$$

uniformly in $t \ge 0$. For every $\varepsilon > 0$, there is constant $C(\varepsilon,\ \Delta_2(\tilde{\Phi}))$ such that

$$t\,u \le \varepsilon\ \Phi(t) + C_3\left(\varepsilon,\ \Delta_2(\tilde{\Phi})\right)\tilde{\Phi}(u) \tag{14}$$

for all $t,\ u \ge 0$, which is a Young type inequality [24]. In particular,

$$ab \leq \varepsilon a^2 + C_4(\varepsilon) b^2, \tag{15}$$

with $a,\ b > 0$. Similarly, we present a modified version of Hölder inequality as

$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x \leq C_5\,\tilde{\Phi}^{-1}\left(\int_\Omega \tilde{\Phi}(|f(x)|)\,\mathrm{d}x\right) \cdot \Phi^{-1}\left(\int_\Omega \Phi(|g(x)|)\,\mathrm{d}x\right), \tag{16}$$

where $\Phi^{-1}(\Phi(t)) = t$ and $\tilde{\Phi}(t)$ defined by (12).

2.2 *Function spaces*

We need several function spaces in this article. We first recall the Orlicz spaces $L^\Phi(\mathbb{R}^n)$ and its norm ([25]) by

$$L^\Phi(\mathbb{R}^n) = \left\{u \in L^1(\mathbb{R}^n) \middle| \int_{\mathbb{R}^n} \Phi(|u(x)|)\,\mathrm{d}x < \infty\right\},$$

and

$$\|u\|_{L^\Phi(\mathbb{R}^n)} = inf\left\{k > 0 \middle| \int_{\mathbb{R}^n} \Phi\left(\frac{|u(x)|}{k}\right)\mathrm{d}x < 1\right\}.$$

Then the dual space of $L^\Phi(\mathbb{R}^n)$ is the Orlicz space $L^{\tilde{\Phi}}(\mathbb{R}^n)$, where $\tilde{\Phi}$ is defined by (12). Next we introduce the classical Orlicz-Sobolev spaces $W^{1,\ \Phi}(\mathbb{R}^n)$ and its norm as [26]

$$W^{1,\ \Phi}(\mathbb{R}^n) = \left\{u \in L^\Phi(\mathbb{R}^n) \middle| |\nabla u| \in L^\Phi(\mathbb{R}^n)\right\}$$

and

$$\|u\|_{W^{1,\ \Phi}(\mathbb{R}^n)} = \|u\|_{L^\Phi(\mathbb{R}^n)} + \|\nabla u\|_{L^\Phi(\mathbb{R}^n)}.$$

We say that Ω_0 is compactly contained in Ω, denoted $\Omega_0 \Subset \Omega$, if $\overline{\Omega_0}$ is a compact subset of Ω, where Ω_0 is an open subset of Ω. A function $u \in W^{1,\ \Phi}_{\mathrm{loc}}(\Omega)$, if $u \in W^{1,\ \Phi}(\Omega_0)$ for every $\Omega_0 \Subset \Omega$ [27].

We recall the definition of the fractional Sobolev spaces by

$$\|u\|_{W^{\alpha,\ 2}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\Delta_h u|^2}{|h|^{2\alpha+n}}\,\mathrm{d}x\,\mathrm{d}h\right)^{\frac{1}{2}}, \tag{17}$$

where $0 < \alpha < 1$ and $\Delta_h u = u(x+h,\ t) - u(x,\ t)$. The Besov spaces $B^\alpha_{p,\ q}(\mathbb{R}^n)$ with its norm are defined via [28]

$$\|u\|_{B^{\alpha}_{p,\,q}(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + [u]_{B^{\alpha}_{p,\,q}(\mathbb{R}^n)}\ \left(0 < \alpha < 1,\ 1 \le p < \infty\right),$$

$$[u]_{B^{\alpha}_{p,\,q}(\mathbb{R}^n)} = \begin{cases} \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|\Delta_h u|^p}{|h|^{\alpha p}}\,\mathrm{d}x\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}}, & 1 \le q < \infty; \\ \sup_{h\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|\Delta_h u|^p}{|h|^{\alpha p}}\,\mathrm{d}x\right)^{\frac{1}{p}}, & q = \infty. \end{cases} \tag{18}$$

If $\|u\|_{L^p(\mathbb{R}^n)} < \infty$ and $\|u\|_{B^{\alpha}_{p,\,q}(\mathbb{R}^n)} < \infty$, then $u \in B^{\alpha}_{p,\,q}(\mathbb{R}^n)$. By [29], we define Triebel-Lizorkin spaces $F^{\alpha}_{p,\,q}(\mathbb{R}^n)$ with its norm via

$$\|u\|_{F^{\alpha}_{p,\,q}(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + [u]_{F^{\alpha}_{p,\,q}(\mathbb{R}^n)}\ \left(0 < \alpha < 1,\ 1 \le p < \infty\right),$$

$$[u]_{F^{\alpha}_{p,\,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|\Delta_h u|^q}{|h|^{n+\alpha q}}\,\mathrm{d}h\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}. \tag{19}$$

If $\|u\|_{L^p(\mathbb{R}^n)} < \infty$ and $\|u\|_{F^{\alpha}_{p,\,q}(\mathbb{R}^n)} < \infty$, then u is said to belong to the Triebel-Lizorkin spaces $F^{\alpha}_{p,\,q}(\mathbb{R}^n)$. Furthermore, we note that the $C^{\infty}(\mathbb{R}^n)$ is comprised of a set of all functions that are infinitely continuously differentiable on $\mathbb{R}^n$, and the space $C^{\infty}_0(\mathbb{R}^n)$ represents the space of infinitely differentiable functions with compact support.

The next lemma is a classical embedding theory.

Lemma 5 [30] Assuming $0 < \alpha < 1$. There is a continuous embedding $W^{\alpha,\,2}(\Omega) \subset L^{\frac{2n}{n-2\alpha}}(\Omega)$.

2.3 *Definitions of weak solution and functions of vanishing mean oscillations*

In view of [31], we introduce the definition of the weak solutions to the equation (1).

Definition 6 Supposing $\frac{\Phi(|F|)}{|F|^2}F \in W^{\alpha,\,2}(\Omega_T)$ with $0 < \alpha < 1$. If for any $\varphi \in C^{\infty}_0(\Omega_T)$, there holds

$$\int_{\Omega} u\varphi\,\mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\Omega}\{-u\varphi_t + A(\nabla u,\ x,\ t)\cdot\nabla\varphi\}\,\mathrm{d}x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_{\Omega}\frac{\Phi(|F|)}{|F|^2}F\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{20}$$

where $[t_1,\ t_2] \subset (t_0,\ t_0+T]$. The function $u \in C\left((t_0,\ t_0+T];\ L^2(\Omega)\right) \cap L^{\Phi}_{\mathrm{loc}}((t_0,\ t_0+T];\ W^{1,\,\Phi}_{\mathrm{loc}}(\Omega))$ is called a weak solution to (1). Here the function φ is a test function.

Definition 7 Let

$$V(x,\ t,\ Q_\rho(z)) := \sup_{\xi\in\mathbb{R}^n\setminus\{0\}}\frac{|A(\xi,\ x,\ t) - \overline{A}_{Q_\rho(z)}(\xi)|}{\dfrac{\Phi\left[(\mu^2+|\xi|^2)^{\frac{1}{2}}\right]}{(\mu^2+|\xi|^2)^{\frac{1}{2}}}} \tag{21}$$

and

$$\bar{A}_{Q_\rho(z)}(\xi) := \fint_{Q_\rho(z)} A(\xi,\, x,\, t)\mathrm{d}x\,\mathrm{d}t \tag{22}$$

Assuming that $A(\nabla u,\, x,\, t)$ satisfies

$$\lim_{R\to 0} \sup_{0<\rho\le R} \sup_{Q_\rho(z)\subset\Omega_T} \fint_{Q_\rho(z)} V(x,\, t,\, Q_\rho(z))\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{23}$$

then we say that $A(\nabla u,\, x,\, t)$ has locally uniformly vanishing mean oscillations (abbreviated to VMO), where $z=(y,\, s)\in \mathbb{R}^{n+1}$, $Q_\rho(z)=B_\rho(y)\times(s-\rho^2,\, s+\rho^2)$.

Lemma 8 Assume $0<\alpha<1$ and $A(\xi,\, x,\, t)$ satisfying (A1)-(A3), then $A(\xi,\, x,\, t)$ has locally uniformly VMO.

Proof. Let $(x,t),\ (y,s)\in\Omega_T$ and $A_k(x,t) := \left\{(y,\, s)\in\Omega_T : 2^{-k}\mathrm{diam}(\Omega_T)\le (|x-y|^2+|t-s|)^{\frac{1}{2}}\le 2^{-k+1}\mathrm{diam}(\Omega_T)\right\}$. According to (21) and assumption (A3), we easily get that

$$\begin{aligned}
\fint_{Q_\rho(y,\,s)} V(x,\, t,\, Q_\rho(y,\, s))\,\mathrm{d}x\,\mathrm{d}t &= \fint_{Q_\rho(y,\,s)} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|A(\xi,\, x,\, t)-\overline{A}_{Q_\rho(y,\,s)}(\xi)|}{\dfrac{\Phi\left[(\mu^2+|\xi|^2)^{\frac{1}{2}}\right]}{(\mu^2+|\xi|^2)^{\frac{1}{2}}}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{|Q_\rho(y,\,s)|^2}\int_{Q_\rho(y,\,s)} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \int_{Q_\rho(y,\,s)} \frac{|A(\xi,\, x,\, t)-A(\xi,\, x',\, t')|\,\mathrm{d}x'\,\mathrm{d}t'}{\dfrac{\Phi\left[(\mu^2+|\xi|^2)^{\frac{1}{2}}\right]}{(\mu^2+|\xi|^2)^{\frac{1}{2}}}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{|Q_\rho(y,\,s)|^2}\int_{Q_\rho(y,\,s)} \sum_k \int_{Q_\rho(y,\,s)\cap A_k(x,\,t)} \frac{|A(\xi,\, x,\, t)-A(\xi,\, x',\, t')|\,\mathrm{d}x'\,\mathrm{d}t'}{\dfrac{\Phi\left[(\mu^2+|\xi|^2)^{\frac{1}{2}}\right]}{(\mu^2+|\xi|^2)^{\frac{1}{2}}}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{c\rho^\alpha}{|Q_\rho(y,\,s)|^2}\sum_k \left|Q_\rho(y,\,s)\cap A_k(x,\,t)\right| \int_{Q_\rho(y,\,s)} g_k(x,\,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{c\rho^\alpha}{|Q_\rho(y,\,s)|}\sum_k \int_{Q_\rho(y,\,s)\,\cap\, A_k(x,\,t)} g_k(x',\,t')\,\mathrm{d}x'\,\mathrm{d}t' \;=\; K_1+K_2.
\end{aligned} \tag{24}$$

In view of Hölder inequality, one gets that

$$K_1 \;\le\; \frac{1}{|Q_\rho(y,\,s)|}\sum_k \left|Q_\rho(y,\,s)\cap A_k(x,\,t)\right| \left[\int_{Q_\rho(y,\,s)} g_k^{\frac{n+2}{\alpha}}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{\alpha}{n+2}} \;\le\; \left[\sum_k \|g_k\|^2_{L^{\frac{n+2}{\alpha}}(Q_\rho(y,\,s))}\right]^{\frac{1}{2}}. \tag{25}$$

According to Hölder inequality again, we obtain that

$$K_2 \leq \frac{c\rho^{\alpha}}{|Q_\rho(y,\, s)|} \sum_k \left[\int_{Q_\rho(y,\, s)\,\cap\, A_k(x,\, t)} g_k^{\frac{n+2}{\alpha}}\, \mathrm{d}x'\, \mathrm{d}t' \right]^{\frac{\alpha}{n+2}} \left| Q_\rho(y,\, s) \cap A_k(x,\, t) \right|^{1-\frac{\alpha}{n+2}} \leq C \left[\sum_k \|g_k\|^2_{L^{\frac{n+2}{\alpha}}(Q_\rho(y,\, s))} \right]^{\frac{1}{2}}. \quad (26)$$

By combining (25) and (26), we have

$$⨍_{Q_\rho(y,\, s)} V(x,\, t,\, Q_\rho(y,\, s))\, \mathrm{d}x\, \mathrm{d}t \leq C \left[\sum_k \|g_k\|^2_{L^{\frac{n+2}{\alpha}}(Q_\rho(y,\, s))} \right]^{\frac{1}{2}}. \quad (27)$$

By the assumption (A3) and Dominated Convergence Theorem, we acquire that

$$\lim_{R\to 0} \sup_{0<\rho\leq R} \sup_{Q_\rho(y,\, s)\subset\Omega_T} ⨍_{Q_\rho(y,\, s)} V(x,\, t,\, Q_\rho(y,\, s))\, \mathrm{d}x\, \mathrm{d}t = 0. \quad (28)$$

This completes the proof of Lemma 8. □

3. Higher integrability

In this section, $Q_z(R^2,\, R) = \left(t - R^2,\, t + R^2\right) \times B_R(x)$ represents a cylinder in Ω_T with the center $z = (x,\, t)$. Sometimes, we write $Q(R^2,\, R)$ in short. Sometimes we write Q_R instead of $Q(R^2,\, R)$. We let $B_x(R)$ denotes a domain in $\Omega \subset \mathbb{R}^n$ with x as the center and R as the radius. Inspired by the conclusion of high integrability in [2], one gets the following conclusion.

Proposition 9 For any $q \in (1,\, +\infty)$, we assume that $\Phi(|F|) \in L^q(\Omega_T)$, and $A(\xi,\, x,\, t)$ satisfies (A1)-(A3). Φ is an N-function satisfying (G1) and (G2). If $u \in W^{1,\,\Phi}(\Omega_T)$ is a weak solution to (1) and $u = 0$ for $(x,\, t) \in \partial\Omega_T$, then $\Phi(|\nabla u|) \in L^q(\Omega_T)$ and

$$\int_{\Omega_T} \Phi^q \left[(\mu^2 + |\nabla u|^2)^{\frac{1}{2}} \right] \mathrm{d}x\, \mathrm{d}t \leq C\Phi \left\{ \left[\left(\int_{\Omega_T} \Phi \left[(\mu^2 + |\nabla u|^2)^{\frac{1}{2}} \right] \mathrm{d}x\, \mathrm{d}t \right)^q + \int_{\Omega_T} \Phi^q(|F|)\, \mathrm{d}x\, \mathrm{d}t + 1 \right]^{\frac{1}{2}} \right\}, \quad (29)$$

where $\partial\Omega_T$ denotes the boundary of Ω_T.

In order to give a proof of Proposition 9, we present the following lemma.

Lemma 10 [32] Let $0 < \mu < 1$, there exists a constant c_l such that if $\xi,\, \eta \in \mathbb{R}^n$, then

$$\Phi \left[(\mu^2 + |\xi|^2)^{\frac{1}{2}} \right] \leq c_l \Phi \left[(\mu^2 + |\eta|^2)^{\frac{1}{2}} \right] + c_l \frac{\Phi \left[(\mu^2 + |\eta|^2 + |\xi|^2)^{\frac{1}{2}} \right]}{\mu^2 + |\eta|^2 + |\xi|^2} |\xi - \eta|^2. \quad (30)$$

We are in a position to give a proof.

Proof of Proposition 9.

Step 1. In this step, we shall approximate the solution u to (1). We let $Q_{2R} = \left(t_0 - (2R)^2,\ t_0 + (2R)^2\right) \times B_{x_0}(2R) \Subset \Omega_T,\ (x_0,\ t_0) \in \Omega_T,\ Q_{2\tilde{R}} \Subset \Omega_T$, and $\tilde{R} > R$. In addition, $Q_{2\tilde{R}}$ and Q_{2R} have the same center. We assume that $\phi_1 :\ \mathbb{R}^n \to \mathbb{R}$ and $\phi_2 :\ \mathbb{R} \to \mathbb{R}$ have compact support in $B_1(0)$ and $(-1,\ 1)$, respectively. We define

$$F_\varepsilon(x,\ t) = \int_{Q_1} F(x+\varepsilon y,\ t+\varepsilon s)\phi_1(y)\phi_2(s)\,\mathrm{d}y\,\mathrm{d}s \tag{31}$$

and

$$A_\varepsilon(\xi,\ x,\ t) = \int_{Q_1} A(\xi,\ x+\varepsilon y,\ t+\varepsilon s)\phi_1(y)\phi_2(s)\,\mathrm{d}y\,\mathrm{d}s, \tag{32}$$

where $Q_1 = B_1 \times (-1,\ 1)$. It is clear that $F_\varepsilon \in C^\infty(Q_{2\tilde{R}},\ \mathbb{R}^n)$, and $A_\varepsilon(\xi,\ \cdot) \in C^\infty(Q_{2\tilde{R}})$. Moreover, there hold

$$\Phi(|F_\varepsilon|) \to \Phi(|F|) \text{ strongly in } L^q(Q_{2\tilde{R}},\ \mathbb{R}^n), \tag{33}$$

$$A_\varepsilon(\xi,\ \cdot) \to A(\xi,\ \cdot) \ \text{ strongly in } L^t(Q_{2\tilde{R}},\ \mathbb{R}^n)\ (t < \infty). \tag{34}$$

We should mention that A_ε satisfies conditions of A. We let u_ε be the unique solution to

$$\begin{cases} (u_\int)_t - divA_\int(\nabla u_\int,\ x,\ t) = div\left[\dfrac{\Phi(|F_\int|)}{|F_\int|^2}F_\int\right] \text{ in } Q_{2\tilde{R}}, \\ u_\int = u \text{ on } \partial Q_{2\tilde{R}}. \end{cases} \tag{35}$$

Since u_ε and u are weak solutions, we get

$$(u_\varepsilon - u)_t - \mathrm{div}\left[A_\varepsilon(\nabla u_\varepsilon,\ x,\ t) - A(\nabla u,\ x,\ t)\right] = \mathrm{div}\left(\frac{\Phi(|F_\varepsilon|)}{|F_\varepsilon|^2}F_\varepsilon - \frac{\Phi(|F|)}{|F|^2}F\right). \tag{36}$$

We choose $u_\varepsilon - u$ as a test function to (36), and hence obtain

$$\sup_{t_0-(2\tilde{R})^2 \le t < t_0+(2\tilde{R})^2} \int_{B_{x_0}(2\tilde{R})} |u_\varepsilon(x,\ t) - u(x,\ t)|^2\,\mathrm{d}x + \int_{Q_{2\tilde{R}}} \left[A_\varepsilon(\nabla u_\varepsilon,\ x,\ t) - A_\varepsilon(\nabla u,\ x,\ t)\right](\nabla u_\varepsilon - \nabla u)\,\mathrm{d}x\,\mathrm{d}t$$

$$\le C\left|\int_{Q_{2\tilde{R}}} \left[A_\varepsilon(\nabla u,\ x,\ t) - A(\nabla u,\ x,\ t)\right](\nabla u_\varepsilon - \nabla u)\,\mathrm{d}x\,\mathrm{d}t\right| + C\left|\int_{2\tilde{R}} \left(\frac{\Phi(|F_\varepsilon|)}{|F_\varepsilon|^2}F_\varepsilon - \frac{\Phi(|F|)}{|F|^2}F\right)(\nabla u_\varepsilon - \nabla u)\,\mathrm{d}x\,\mathrm{d}t\right|. \tag{37}$$

It follows that

$$\int_{Q_{2\tilde{R}}} [A_\varepsilon(\nabla u_\varepsilon,\, x,\, t) - A_\varepsilon(\nabla u,\, x,\, t)]\,(\nabla u_\varepsilon - \nabla u)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C\int_{Q_{2\tilde{R}}} |A_\varepsilon(\nabla u,\, x,\, t) - A(\nabla u,\, x,\, t)|\,|\nabla u_\varepsilon - \nabla u|\,\mathrm{d}x\,\mathrm{d}t + C\int_{2\tilde{R}} \left|\frac{\Phi(|F_\varepsilon|)}{|F_\varepsilon|^2}F_\varepsilon - \frac{\Phi(|F|)}{|F|^2}F\right|\,|\nabla u_\varepsilon - \nabla u|\,\mathrm{d}x\,\mathrm{d}t. \tag{38}$$

By applying (2) and (14) to handle each term of (38), and using (31), (32), and the relationship between u and u_ε, we derive

$$\int_{Q_{2\tilde{R}}} \Phi\left[(\mu^2 + |\nabla u_\varepsilon|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t \leq C\int_{\Omega_T} \Phi\left[(\mu^2 + |\nabla u|^2)^{\frac{1}{2}}\right] + \Phi(|F|)\,\mathrm{d}x\,\mathrm{d}t \leq C. \tag{39}$$

From (37) and (14), we have

$$\int_{Q_{2\tilde{R}}} \frac{\Phi\left[\left(\mu^2 + |\nabla u_\varepsilon|^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right]}{\mu^2 + |\nabla u_\varepsilon|^2 + |\nabla u|^2}|\nabla u_\varepsilon - \nabla u|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C(\delta)\int_{Q2\tilde{R}} \tilde{\Phi}\,[|A_\varepsilon(\nabla u,\, x,\, t) - A(\nabla u,\, x,\, t)|]\,\mathrm{d}x\,\mathrm{d}t + \delta\int_{Q2\tilde{R}} \Phi(|\nabla u_\varepsilon|) + \Phi(|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\ C(\delta)\int_{Q2\tilde{R}} \tilde{\Phi}\left[\left|\frac{\Phi(|F_\varepsilon|)}{|F_\varepsilon|^2}F_\varepsilon - \frac{\Phi(|F|)}{|F|^2}F\right|\right]\mathrm{d}x\,\mathrm{d}t. \tag{40}$$

Let $\varepsilon \to 0$, and then $\delta \to 0$, we acquire that

$$\nabla u_\varepsilon \to \nabla u \ \text{ strongly in } L^\Phi(Q_{2\tilde{R}},\, \mathbb{R}^n). \tag{41}$$

By considering u_ε defined in (35), and applying the regularity theory for the parabolic p-Laplacian equations in [33] together with Fatou lemma, we get

$$\left\{ \fint_{Q_R} \Phi^q \left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}} \right] \mathrm{d}x\,\mathrm{d}t \right\}^{\frac{1}{q}}$$

$$\leq \liminf_{\varepsilon\to 0} \left\{ \fint_{Q_R} \Phi \left[(\mu^2+|\nabla u_\varepsilon|^2)^{\frac{1}{2}} \right] \mathrm{d}x\,\mathrm{d}t \right\}$$

$$\leq C \lim_{\varepsilon\to 0} \Phi \left\{ \left[\fint_{Q_{2R}} \Phi \left[(\mu^2+|\nabla u_\varepsilon|^2)^{\frac{1}{2}} \right] \mathrm{d}x\,\mathrm{d}t + \left[\fint_{Q_{2R}} \Phi^q(|F_\varepsilon|)\,\mathrm{d}x\,\mathrm{d}t + 1 \right]^{\frac{1}{q}} \right]^{\frac{1}{2}} \right\}$$

$$= C\Phi \left\{ \left[\fint_{Q_{2R}} \Phi \left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}} \right] \mathrm{d}x\,\mathrm{d}t + \left[\fint_{Q_{2R}} \Phi^q(|F|)\,\mathrm{d}x\,\mathrm{d}t + 1 \right]^{\frac{1}{q}} \right]^{\frac{1}{2}} \right\}. \tag{42}$$

Step 2. A stopping-time argument. We let $S(t)$ be an N-function satisfying that both $S\circ\Phi^{-1}(t)$ and $\Phi^q\circ S^{-1}(t)$ are also N-functions. We define $\lambda_0 > 1$ such that

$$\Phi^{-1}\left(\lambda_0^2\right) = \Phi^{-1} \left\{ \fint_{Q_{2R}} \Phi \left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}} \right] \mathrm{d}x\,\mathrm{d}t \right\} + S^{-1} \left[\fint_{Q_{2R}} S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t \right] + 1, \tag{43}$$

where $M > 1$ and R will be chosen later. We choose γ and λ such that

$$\frac{R}{\Phi^8(2)} \leq \gamma \leq \frac{R}{\Phi(2)},\ B\lambda_0 := \Phi^{-1}(2)\,2^{10(n+2)}\lambda_0 \leq \lambda. \tag{44}$$

Since $Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\gamma^2,\ \gamma\right) \subset Q_{2R}$, we use (44) to obtain that

$$\Phi^{-1} \left\{ \fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\gamma^2,\ \gamma\right)} \Phi \left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}} \right] \mathrm{d}x\,\mathrm{d}t \right\} + S^{-1} \left[\fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\gamma^2,\ \gamma\right)} S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t \right] < C_2\lambda. \tag{45}$$

We select $z_0 = (x_0,\ t_0) \in Q_R$ such that $(\mu^2+|\nabla u(z_0)|^2)^{\frac{1}{2}} > C_2\lambda$. According to Lebesgue Differentiation Theorem, we get

$$\lim_{\rho\to 0} \Phi^{-1} \left\{ \fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\rho^2,\ \rho\right)} \Phi \left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}} \right] \mathrm{d}x\,\mathrm{d}t \right\} + S^{-1} \left[\fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\rho^2,\ \rho\right)} S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t \right] > C_2\lambda. \tag{46}$$

Via a contradiction argument, there exist some ρ such that $0 < \rho \leq \frac{R}{\Phi(2)}$ and

$$\Phi^{-1}\left\{\fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\rho^2,\ \rho\right)}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\right\}+S^{-1}\left[\fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\rho^2,\ \rho\right)}S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t\right]>C_2\lambda.$$

Combining (44) and (45), we have $\rho<\dfrac{R}{\Phi^8(2)}$. It is possible to select $\rho_{z_0}\leq\dfrac{R}{\Phi(2)}$ such that

$$\Phi^{-1}\left\{\fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\rho_{z_0}^2,\rho_{z_0}\right)}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\right\}+S^{-1}\left[\fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}\rho_{z_0}^2,\rho_{z_0}\right)}S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t\right]=C_2\lambda. \tag{47}$$

In fact, one has $\rho_{z_0}\leq\dfrac{R}{\Phi^8(2)}$. It is easy to show that

$$Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}(\Phi^j(2)\rho_{z_0})^2,\ \Phi^j(2)\rho_{z_0}\right)\subset Q((2R)^2,\ 2R),\ j\in\{0,\ \cdots,\ 5\}. \tag{48}$$

From the choice of ρ_{z_0}, for every $j\in\{0,\ \cdots,\ 5\}$, we get that

$$\begin{aligned}\frac{C_1\lambda}{\Phi^{3j}(2)}\leq&\Phi^{-1}\left\{\fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}(\Phi^j(2)\rho_{z_0})^2,\ \Phi^j(2)\rho_{z_0}\right)}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\right\}\\&+\ S^{-1}\left[\fint_{Q_{z_0}\left(\frac{\lambda^2}{\Phi(\lambda)}(\Phi^j(2)\rho_{z_0})^2,\ \Phi^j(2)\rho_{z_0}\right)}S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t\right]\leq C_2\lambda.\end{aligned} \tag{49}$$

We construct a set $E(\lambda)$ by

$$E(\lambda)=\left\{z\in Q_R:(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>C_2\lambda\right\}.$$

For almost every $z_0\in E(\lambda)$, there is a cube $Q_{z_0}\left(\dfrac{\lambda^2}{\Phi(\lambda)}\rho_{z_0}^2,\ \rho_{z_0}\right)\subset Q_{2R}$ as constructed in (47). In addition, the estimate (49) holds for $j\in\{0,\ ..,\ 5\}$. Then we are able to find a family of disjoint cubes Q_i^0 satisfying that

$$Q_i^0=Q_{z_i}\left(\frac{\lambda^2}{\Phi(\lambda)}\rho_{z_i}^2,\ \rho_{z_i}\right)\subset Q_{2R}.$$

We refer to Figure 1 for the construction of Q_i^0. Then we construct

$$Q_i^1=Q_{z_i}\left(\frac{\lambda^2}{\Phi(\lambda)}(\Phi^3(2)\rho_{z_i})^2,\ \Phi^3(2)\rho_{z_i}\right).$$

By Vitali Covering Theorem, one has $E(\lambda) \subset \bigcup_{i\in\mathbb{N}} Q_i^1 \cup \tilde{N}$, where $\tilde{N}$ is a set with measure zero. We also introduce

$$Q_i^2 = Q_{z_i}\left(\frac{\lambda^2}{\Phi(\lambda)}(\Phi^4(2)\rho_{z_i})^2,\ \Phi^4(2)\rho_{z_i}\right),$$

$$Q_i^3 = Q_{z_i}\left(\frac{\lambda^2}{\Phi(\lambda)}(\Phi^5(2)\rho_{z_i})^2,\ \Phi^5(2)\rho_{z_i}\right),$$

and see Figure 2 for the relationship among those cubes.

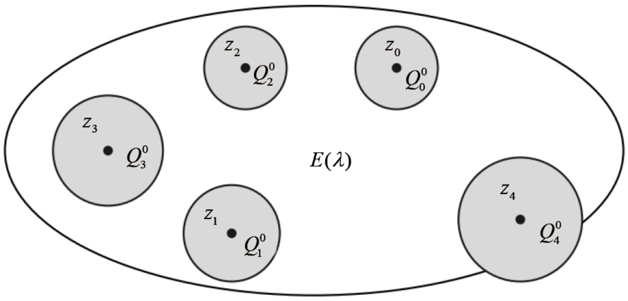

Figure 1. The relationship between $E(\lambda)$ and $\{Q_i^0\}$

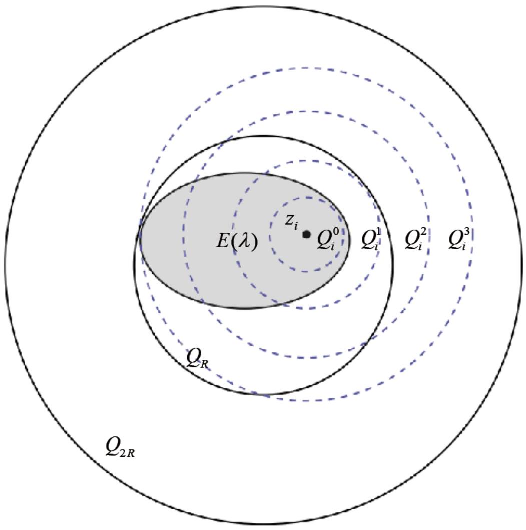

Figure 2. The relationship among $E(\lambda)$, $\{Q_i^0\}$, $\{Q_i^1\}$, $\{Q_i^2\}$ and $\{Q_i^3\}$

Step 3. A comparison argument. We define v_i as a weak solution to

$$\begin{cases} (v_i)_t - \mathrm{div} A_i(\nabla v_i,\ x,\ t) = 0 \text{ in } Q_i^2, \\ v_i = u \text{ on } \partial Q_i^2, \end{cases} \tag{50}$$

where $A_i = \fint_{Q_i^2} A(\nabla u,\ x,\ t)\,\mathrm{d}x\,\mathrm{d}t$. We choose a test function $u - v_i$, we acquire that

$$\int_{Q_i^2} \left[A_i(\nabla v_i,\ x,\ t) - A_i(\nabla u,\ x,\ t)\right](\nabla u - \nabla v_i)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C \int_{Q_i^2} |A_i(\nabla u,\ x,\ t) - A(\nabla u,\ x,\ t)|\,|\nabla u - \nabla v_i|\,\mathrm{d}x\,\mathrm{d}t + C \int_{Q_i^2} \frac{\Phi(|F|)}{|F|} |\nabla u - \nabla v_i|\,\mathrm{d}x\,\mathrm{d}t. \tag{51}$$

Adopting a similar method as in step 1, we get

$$\int_{Q_i^2} \Phi\left[\left(\mu^2 + |\nabla v_i|^2\right)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t \leq C \int_{Q_i^2} \Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right] + \Phi(|F|)\,\mathrm{d}x\,\mathrm{d}t. \tag{52}$$

It follows from Hölder inequality and (49) that

$$\fint_{Q_i^2} \Phi\left[\left(\mu^2 + |\nabla v_i|^2\right)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t$$

$$\leq C \fint_{Q_i^2} \Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t + C\left(\Phi \circ S^{-1}\right) \fint_{Q_i^2} S(M) S(|F|)\,\mathrm{d}x\,\mathrm{d}t \ \leq\ C\Phi(\lambda). \tag{53}$$

According to (53) and the classical L^∞-estimate, there is a constant $T_1 \geq 1$ such that

$$\sup_{Q_i^2} \Phi\left[\left(\mu^2 + |\nabla v_i|^2\right)^{\frac{1}{2}}\right] < C_2 T_1 \Phi(\lambda). \tag{54}$$

Based on (21) and (14), we have

$$\int_{Q_i^2} \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2 + |\nabla v_i|^2\right)^{\frac{1}{2}}\right]}{\mu^2 + |\nabla u|^2 + |\nabla v_i|^2} |\nabla u - \nabla v_i|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C C_3\left(\delta,\ \Delta_2(\tilde{\Phi})\right) \int_{Q_i^2} \tilde{\Phi}\left[V(x,\ t,\ Q_i^2)\right] \Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t$$

$$+\ C\ C_3\left(\delta,\ \Delta(\tilde{\Phi})\right) \int_{Q_i^2} \Phi(|F|)\,\mathrm{d}x\,\mathrm{d}t + C\ \delta \int_{Q_i^2} \Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t. \tag{55}$$

Applying Hölder inequality, one obtains

$$\int_{Q_i^2} \tilde{\Phi}\left[V(x,\, t,\, Q_i^2)\right] \Phi\left[\left(\mu^2+|\nabla u|^2\right)^{\frac{1}{2}}\right] \mathrm{d}x\,\mathrm{d}t$$

$$\leq \left(\widetilde{S\circ\Phi^{-1}}\right)^{-1} \left\{ \fint_{Q_i^2} \widetilde{S\circ\Phi^{-1}} \circ \tilde{\Phi}\left[V(x,\, t,\, Q_i^2)\right] \mathrm{d}x\,\mathrm{d}t \right\} \left(\Phi\circ S^{-1}\right) \left\{ \fint_{Q_i^2} S\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right] \mathrm{d}x\,\mathrm{d}t \right\} |Q_i^2|$$

$$\leq \left(\widetilde{S\circ\Phi^{-1}}\right)^{-1} [V(2R)] \left(\Phi\circ S^{-1}\right) \left\{ \fint_{Q_i^2} S\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right] \mathrm{d}x\,\mathrm{d}t \right\} |Q_i^2|. \tag{56}$$

With the help of (49), we derive that

$$\left(\Phi\circ S^{-1}\right) \left\{ \fint_{Q_i^2} S\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right] \mathrm{d}x\,\mathrm{d}t \right\} \leq C \fint_{Q_i^3} \Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right] \mathrm{d}x\,\mathrm{d}t + C\left(\Phi\circ S^{-1}\right)\left[\fint_{Q_i^3} (1+S(M)S(|F|))\,\mathrm{d}x\,\mathrm{d}t\right].$$

By (21), we also define

$$V(2R) = \sup_{0<r(Q_i^2)<2R}\ \sup_{Q_i^2\subset\Omega_T} \fint_{Q_i^2} V\left(x,\, t,\, Q_i^2\right) \mathrm{d}x\,\mathrm{d}t. \tag{57}$$

Combining (55) and (56), we have

$$\int_{Q_i^2} \frac{\Phi\left[\left(\mu^2+|\nabla u|^2+|\nabla v_i|^2\right)^{\frac{1}{2}}\right]}{\mu^2+|\nabla u|^2+|\nabla v_i|^2} |\nabla u-\nabla v_i|^2\, \mathrm{d}x\,\mathrm{d}t$$

$$\leq C\left\{ C_3\left(\delta,\, \Delta_2(\tilde{\Phi})\right) \left(\widetilde{S\circ\Phi^{-1}}\right)^{-1} [V(2R)] + \delta \right\} \int_{Q_i^3} \Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right] \mathrm{d}x\,\mathrm{d}t$$

$$+\ C C_3\left(\delta,\, \Delta_2(\tilde{\Phi})\right) \left(\widetilde{S\circ\Phi^{-1}}\right)^{-1} [V(2R)] \left(\Phi\circ S^{-1}\right) \left[\fint_{Q_i^3} (1+S(M)S(|F|))\,\mathrm{d}x\,\mathrm{d}t\right] |Q_i^0|$$

$$+\ C C_3\left(\delta,\, \Delta_2(\tilde{\Phi})\right) \int_{Q_i^3} \Phi(|F|)\,\mathrm{d}x\,\mathrm{d}t. \tag{58}$$

Step 4. In this step, we present an estimate of the left side of inequality (60). By Lemma 10, we have

$$\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right] \leq c_l \Phi\left[\left(\mu^2+|\nabla v_i|^2\right)^{\frac{1}{2}}\right] + c_l \frac{\Phi\left[\left(\mu^2+|\nabla u|^2+|\nabla v_i|^2\right)^{\frac{1}{2}}\right]}{\mu^2+|\nabla u|^2+|\nabla v_i|^2} |\nabla u-\nabla v_i|^2, \tag{59}$$

where c_l is a constant. We define $T=(1+2c_l)T_1$, here c_l and T_1 are given in (59) and (54), respectively. By (59) and (54) again, we have

$$\begin{aligned}
&\left|\left\{z\in Q_i^1:\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]>C_2T\Phi(\lambda)\right\}\right| \\
\leq&\left|\left\{z\in Q_i^1:\frac{\Phi\left[\left(\mu^2+|\nabla u|^2+|\nabla v_i|^2\right)^{\frac{1}{2}}\right]}{\mu^2+|\nabla u|^2+|\nabla v_i|^2}|\nabla u-\nabla v_i|^2>C_2T_1\Phi(\lambda)\right\}\right| \\
&+\left|\left\{z\in Q_i^1:\Phi\left[\left(\mu^2+|\nabla v_i|^2\right)^{\frac{1}{2}}\right]>C_2T_1\Phi(\lambda)\right\}\right| \\
=&\left|\left\{z\in Q_i^1:\frac{\Phi\left[\left(\mu^2+|\nabla u|^2+|\nabla v_i|^2\right)^{\frac{1}{2}}\right]}{\mu^2+|\nabla u|^2+|\nabla v_i|^2}|\nabla u-\nabla v_i|^2>C_2T_1\Phi(\lambda)\right\}\right|. \qquad (60)
\end{aligned}$$

Here $|\cdot|$ denotes the Lebesgue measure. Based on (58) and (60), we get

$$\begin{aligned}
&\left|\left\{z\in Q_i^1:\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]>C_2T\Phi(\lambda)\right\}\right| \\
\leq&\frac{1}{C_2T_1\Phi(\lambda)}\int_{Q_i^1}\frac{\Phi\left[\left(\mu^2+|\nabla u|^2+|\nabla v_i|^2\right)^{\frac{1}{2}}\right]}{\mu^2+|\nabla u|^2+|\nabla v_i|^2}|\nabla u-\nabla v_i|^2\,\mathrm{d}x\,\mathrm{d}t \\
\leq&\frac{C}{T\Phi(\lambda)}\left\{C_3\left(\delta,\,\Delta_2(\tilde{\Phi})\right)\left(\widetilde{S\circ\Phi^{-1}}\right)^{-1}[V(2R)]+\delta\right\}\int_{Q_i^3}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t \\
&+\frac{C}{T\Phi(\lambda)}C_3\left(\delta,\,\Delta_2(\tilde{\Phi})\right)\left(\widetilde{S\circ\Phi^{-1}}\right)^{-1}[V(2R)]\left(\Phi\circ S^{-1}\right)\left[\fint_{Q_i^3}(1+S(M)S(|F|))\,\mathrm{d}x\,\mathrm{d}t\right]|Q_i^0| \\
&+\frac{C}{T\Phi(\lambda)}C_3\left(\delta,\,\Delta_2(\tilde{\Phi})\right)\int_{Q_i^3}\Phi(|F|)\,\mathrm{d}x\,\mathrm{d}t. \qquad (61)
\end{aligned}$$

With the help of (47), we obtain

$$\Phi(\lambda)\leq\frac{C}{|Q_i^0|}\int_{Q_i^0}\Phi\left[\left(\mu^2+|\nabla u|^2\right)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t \qquad (62)$$

and

$$S(\lambda) \leq \frac{C}{|Q_i^0|} \int_{Q_i^0} S(M) S(|F|) \,\mathrm{d}x \,\mathrm{d}t. \tag{63}$$

Combining (62) and (63), we acquire

$$|Q_i^0| \leq \frac{C}{\Phi(\lambda)} \int_{Q_i^0} \Phi \left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}} \right] \mathrm{d}x \,\mathrm{d}t + \frac{C}{S(\lambda)} \int_{Q_i^0} S(M) S(|F|) \,\mathrm{d}x \,\mathrm{d}t. \tag{64}$$

By splitting the region of integral, we have

$$\begin{aligned}
&\frac{1}{S(\lambda)} \int_{Q_i^0} S(M) S(|F|) \,\mathrm{d}x \,\mathrm{d}t \\
=&\frac{1}{S(\lambda)} \int_{Q_i^0 \cap \{|F| > \gamma\lambda\}} S(M) S(|F|) \,\mathrm{d}x \,\mathrm{d}t + \frac{1}{S(\lambda)} \int_{Q_i^0 \cap \{|F| \leq \gamma\lambda\}} S(M) S(|F|) \,\mathrm{d}x \,\mathrm{d}t \\
\leq&\frac{1}{S(\lambda)} \int_{Q_i^0 \cap \{|F| > \gamma\lambda\}} S(M) S(|F|) \,\mathrm{d}x \,\mathrm{d}t + C S(M) S(\gamma) \left|Q_i^0\right|.
\end{aligned} \tag{65}$$

By choosing a proper value for γ, we move the last term of (65) to the left side of (64), and obtain

$$\left|Q_i^0\right| \leq \frac{C}{\Phi(\lambda)} \int_{Q_i^0} \Phi \left[(\mu^2 + |\nabla u|^2)^{\frac{1}{2}} \right] \mathrm{d}x \,\mathrm{d}t + \frac{C}{S(\lambda)} \int_{Q_i^0 \cap \{|F| > \gamma\lambda\}} S(M) S(|F|) \,\mathrm{d}x \,\mathrm{d}t. \tag{66}$$

We split the integral again with $\tau > 0$, and get

$$\begin{aligned}
&\frac{1}{\Phi(\lambda)} \int_{Q_i^0} \Phi \left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}} \right] \mathrm{d}x \,\mathrm{d}t \\
=&\frac{1}{\Phi(\lambda)} \int_{Q_i^0 \cap \left\{ \left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}} > \tau\lambda \right\}} \Phi \left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}} \right] \mathrm{d}x \,\mathrm{d}t \\
&+ \frac{1}{\Phi(\lambda)} \int_{Q_i^0 \cap \left\{ \left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}} \leq \tau\lambda \right\}} \Phi \left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}} \right] \mathrm{d}x \,\mathrm{d}t \\
\leq&\frac{1}{\Phi(\lambda)} \int_{Q_i^0 \cap \left\{ \left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}} > \tau\lambda \right\}} \Phi \left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}} \right] \mathrm{d}x \,\mathrm{d}t + C_2 \Phi(\tau) \left|Q_i^0\right|.
\end{aligned} \tag{67}$$

By combining (66) and (67) and selecting an appropriate value for τ, we acquire

$$\frac{1}{\Phi(\lambda)}\int_{Q_i^0}\Phi\left[\left(\mu^2+|\nabla u|^2\right)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t$$

$$\leq\frac{2}{\Phi(\lambda)}\int_{Q_i^0\cap\left\{\left(\mu^2+|\nabla u|^2\right)^{\frac{1}{2}}>\tau\lambda\right\}}\Phi\left[\left(\mu^2+|\nabla u|^2\right)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t+\frac{C}{S(\lambda)}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t. \tag{68}$$

Using (49) and (66) consecutively, one gets

$$\frac{1}{\Phi(\lambda)}\int_{Q_i^3}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t \;=\; \frac{|Q_i^3|}{\Phi(\lambda)}\fint_{Q_i^3}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t \;\leq\; C\left|Q_i^3\right|$$

$$\leq\frac{C}{\Phi(\lambda)}\int_{Q_i^0\cap\left\{(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\tau\lambda\right\}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t+\frac{C}{S(\lambda)}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t. \tag{69}$$

Applying Hölder inequality and adopting a similar method as in (69), we derive that

$$\frac{1}{\Phi(\lambda)}\int_{Q_i^3}\Phi(|F|)\,\mathrm{d}x\,\mathrm{d}t\leq\frac{C}{\Phi(M)\Phi(\lambda)}\int_{Q_i^0\cap\left\{(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\tau\lambda\right\}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t$$

$$+\;\frac{C}{\Phi(M)S(\lambda)}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t. \tag{70}$$

Because of $\lambda>1$, (49) and (66), it follows that

$$\frac{|Q_i^0|}{\Phi(\lambda)}\left(\Phi\circ S^{-1}\right)\left(\fint_{Q_i^3}1+S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t\right)\;\leq\;C\left|Q_i^0\right|$$

$$\leq\frac{C}{\Phi(\lambda)}\int_{Q_i^0\cap\left\{(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\tau\lambda\right\}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t+\frac{C}{S(\lambda)}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t. \tag{71}$$

By putting (69), (70), (71), and (61) together, we obtain an estimate by

$$\left|\left\{z\in Q_i^1:\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]>C_2T\Phi(\lambda)\right\}\right|$$

$$\leq G(\delta,\,M,\,V(2R))\left\{\frac{C}{T\Phi(\lambda)}\int_{Q_i^0\cap\left\{(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\tau\lambda\right\}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\right.$$

$$+\frac{CS(M)}{TS(\lambda)}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}S(|F|)\,\mathrm{d}x\,\mathrm{d}t\Bigg\},\tag{72}$$

where

$$G(\delta,\,M,\,V(2R))=C_3\left(\delta,\,\Delta_2(\tilde{\Phi})\right)\,\left(\widetilde{S\circ\Phi^{-1}}\right)^{-1}[V(2R)]+\frac{C_3\left(\delta,\,\Delta_2(\tilde{\Phi})\right)}{\Phi(M)}+\delta.\tag{73}$$

Step 5. Considering Vitali Covering Theorem, we aim to obtain an estimate of the left side of (74). We let $R\leq R_0$. By Step 4, we get

$$\begin{aligned}
&\left|\left\{z\in Q_R:(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\Phi^{-1}(T)\lambda\right\}\right|\\
\leq&\sum_i\left|\left\{z\in Q_i^1:\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]>C_2T\Phi(\lambda)\right\}\right|\\
\leq&G(\delta,\,M,\,V(2R))\\
&\left\{\frac{C}{T\Phi(\lambda)}\int_{Q_i^0\cap\left\{(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\tau\lambda\right\}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t+\frac{CS(M)}{TS(\lambda)}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}S(|F|)\,\mathrm{d}x\,\mathrm{d}t\right\}.
\end{aligned}\tag{74}$$

We integrate the above expression, and get

$$\begin{aligned}
&\int_{B\lambda_0}^{\infty}\frac{\Phi^q(\lambda)}{\lambda}\left|\left\{z\in Q_R:(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\Phi^{-1}(T)\lambda\right\}\right|\mathrm{d}\lambda\\
\leq&\frac{CG(\delta,\,M,\,V(2R))}{T}\int_{B\lambda_0}^{\infty}\frac{\Phi^q(\lambda)}{\lambda\Phi(\lambda)}\int_{Q_{2R}\cap\left\{(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\tau\lambda\right\}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\lambda\\
&+\frac{CG(\delta,\,M,\,V(2R))\,S(M)}{T}\int_{B\lambda_0}^{\infty}\frac{\Phi^q(\lambda)}{\lambda S(\lambda)}\int_{Q_{2R}\cap\{|F|>\gamma\lambda\}}S(|F|)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\lambda.
\end{aligned}\tag{75}$$

According to (10) and (13) and using Fubini Theorem, we obtain

$$\int_E\Phi(|f(x)|)\,\mathrm{d}x\sim\int_E\int_0^{|f(x)|}\frac{\Phi(\lambda)}{\lambda}\,\mathrm{d}\lambda\,\mathrm{d}x\;\sim\;\int_0^{\infty}\frac{\Phi(\lambda)}{\lambda}\int_{\{x\in E:|f(x)|>\lambda\}}\mathrm{d}x\,\mathrm{d}\lambda$$

$$\sim\int_0^{\infty}\frac{\Phi(\lambda)}{\lambda}|\{x\in E:|f(x)|>\lambda\}|\,\mathrm{d}\lambda.$$

Applying a similar approach, we also derive that

$$\int_E \Phi^q(|f(x)|)\,\mathrm{d}x \sim \int_0^\infty \frac{\Phi^q(\lambda)}{\lambda\Phi(t)} \int_{E\cap\{|f(x)|>\lambda\}} \Phi(|f(x)|)\,\mathrm{d}x\,\mathrm{d}\lambda. \tag{76}$$

Splitting the integral and using (76), we acquire

$$\int_{Q_R} \Phi^q\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t$$

$$\leq CB\lambda_0 T^q \frac{\Phi^q(B\lambda_0)}{B\lambda_0}\,|Q_R| + CT^q \int_{B\lambda_0}^\infty \frac{\Phi^q(\lambda)}{\lambda}\left|\left\{z\in Q_R : (\mu^2+|\nabla u|^2)^{\frac{1}{2}} > \Phi^{-1}(T)\lambda\right\}\right|\mathrm{d}\lambda. \tag{77}$$

Based on (75) and (77), we deduce that

$$\int_{Q_R} \Phi^q\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t$$

$$\leq CT^q\Phi^q(B\lambda_0)\,|Q_R|$$

$$+\ T^q\frac{CG(\delta,\,M,\,V(2R))}{T}\int_{B\lambda_0}^\infty \frac{\Phi^q(\lambda)}{\lambda\Phi(\lambda)}\int_{Q_{2R}\cap\left\{(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\tau\lambda\right\}} \Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\lambda$$

$$+\ T^q\frac{CG(\delta,\,M,\,V(2R))S(M)}{T}\int_{B\lambda_0}^\infty \frac{\Phi^q(\lambda)}{\lambda\Phi(\lambda)}\int_{Q_{2R}\cap\{|F|>\gamma\lambda\}} S(|F|)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\lambda$$

$$\leq CT^q\Phi^q(B\lambda_0)\,|Q_R|$$

$$+\ T^q\frac{CG(\delta,\,M,\,V(2R))\Phi(\tau)}{\Phi^q(\tau)}\int_{B\lambda_0}^\infty \frac{\Phi^q(\tau\lambda)}{\tau\lambda\Phi(\tau\lambda)}\int_{Q_{2R}\cap\{(\mu^2+|\nabla u|^2)^{\frac{1}{2}}>\tau\lambda\}} \Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}(\tau\lambda)$$

$$+\ T^q\frac{CG(\delta,\,M,\,V(2R))S(M)S(\gamma)}{\Phi^q(\gamma)}\int_{B\lambda_0}^\infty \frac{\Phi^q(\gamma\lambda)}{\gamma\lambda S(\gamma\lambda)}\int_{Q_{2R}\cap\{|F|>\gamma\lambda\}} S(|F|)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}(\gamma\lambda).$$

Applying (76), we obtain that

$$\int_{Q_R} \Phi^q\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t$$

$$\leq CT^q\Phi^q(B\lambda_0)\,|Q_R| + CG(\delta,\,M,\,V(2R))\,T^q\left\{\int_{Q_{2R}} \Phi^q\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t + \Phi^q(M)\int_{Q_{2R}} \Phi^q(|F|)\,\mathrm{d}x\,\mathrm{d}t\right\}. \tag{78}$$

We choose ε_0 and $\delta > 0$ such that

$$C\delta \le \frac{1}{3}\frac{\varepsilon_0}{2T^q}. \tag{79}$$

By (43), we select $M > 1$ large enough satisfying that

$$\frac{CC_3\left(\delta,\ \Delta_2(\tilde{\Phi})\right)}{\Phi(M)} \le \frac{1}{3}\frac{\varepsilon_0}{2T^q}. \tag{80}$$

By the fact of (57) and (23), let R small enough such that

$$C_3\left(\delta,\ \Delta_2(\tilde{\Phi})\right)\ \left(\widetilde{S\circ\Phi^{-1}}\right)^{-1}[V(R)] \le \frac{1}{3}\frac{\varepsilon_0}{2T^q}. \tag{81}$$

Substituting (79), (80), (81) into (78), we obtain

$$\left\{\fint_{Q_R}\Phi^q\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\right\}^{\frac{1}{q}} \le C\Phi(\lambda_0)+C\left[\fint_{Q_R}\Phi^q(|F|)\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{q}}+\left\{\varepsilon_0\fint_{Q_{2R}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]^2\mathrm{d}x\,\mathrm{d}t\right\}^{\frac{1}{q}}. \tag{82}$$

According to (43), one acquires

$$\begin{aligned}\Phi(\lambda_0) =& C\Phi\left\{\left[\int_{Q_{2R}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t+\left(\Phi\circ S^{-1}\right)\fint_{Q_{2R}}S(M)S(|F|)\,\mathrm{d}x\,\mathrm{d}t+1\right]^{\frac{1}{2}}\right\}\\ \le& C\Phi\left\{\left[\int_{Q_{2R}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t+\left[\fint_{Q_{2R}}\Phi^q(|F|)\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{q}}+1\right]^{\frac{1}{2}}\right\}.\end{aligned} \tag{83}$$

By joining the last estimate and (82), and reabsorbing into the left-hand side of the last integral in (82) via an iteration argument ([34]), one obtains

$$\int_{Q_{2R}}\Phi^q\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t \le C\Phi\left\{\left[\left(\int_{Q_{2R}}\Phi\left[(\mu^2+|\nabla u|^2)^{\frac{1}{2}}\right]\mathrm{d}x\,\mathrm{d}t\right)^q+\int_{Q_{2R}}\Phi^q(|F|)\,\mathrm{d}x\,\mathrm{d}t+1\right]^{\frac{1}{2}}\right\} \tag{84}$$

Then via a covering method, we obtain (29) and complete the proof of Proposition 9. □

4. Proof of theorem 1

Proof. We divide the proof of Theorem 1 in four steps.

Step 1. Assuming $Q_{3R} \subset \Omega_T$ and choosing $\varphi = \Delta_{-h}(\eta^2 \Delta_h u)$ as a test function, where $\eta \in C_0^\infty(\Omega_T)$ is a cut-off function satisfying that

$$0 \le \eta \le 1,\ \eta \equiv 1\ (x,\ t) \in Q_{\frac{R}{2}},\ \eta \equiv 0\ (x,\ t) \in \Omega_T \backslash Q_R,\ |\nabla \eta| \le \frac{C}{R},\ \text{and}\ |\eta_t| < C. \tag{85}$$

By the definition of weak solutions to (1), we get

$$\int_{B_{2R}} \eta^2 (\Delta_h u)^2 \,\mathrm{d}x \Big|_{-4R^2}^{4R^2} + \int_{Q_{2R}} -u\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{2R}} \Delta_h A(\nabla u,\ x,\ t) \cdot \left(2\eta \nabla \eta\, \Delta_h u + \eta^2 \Delta_h \nabla u\right) \mathrm{d}x\,\mathrm{d}t$$

$$= - \int_{Q_{2R}} \Delta_h \left[\frac{\Phi(|F|)}{|F|^2} F \right] \cdot \left(2\eta \nabla \eta\, \Delta_h u + \eta^2 \Delta_h \nabla u\right) \mathrm{d}x\,\mathrm{d}t. \tag{86}$$

The preceding equality (86) is equivalent to

$$\int_{B_{2R}} \eta^2 (\Delta_h u)^2 \,\mathrm{d}x \Big|_{-4R^2}^{4R^2} + \int_{Q_{2R}} [A(\nabla u(x+h,\ t),\ x+h,\ t) - A(\nabla u(x,\ t),\ x+h,\ t)] \cdot \eta^2 \Delta_h \nabla u \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_{Q_{2R}} \eta\, \eta_t\, (\Delta_h u)^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{2R}} [A(\nabla u(x,\ t),\ x+h,\ t) - A(\nabla u(x+h,\ t),\ x+h,\ t)] \cdot 2\eta \nabla \eta\, \Delta_h u \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_{Q_{2R}} [A(\nabla u(x,\ t),\ x,\ t) - A(\nabla u(x,\ t),\ x+h,\ t)] \cdot 2\eta \nabla \eta\, \Delta_h u \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_{Q_{2R}} [A(\nabla u(x,\ t),\ x,\ t) - A(\nabla u(x,\ t),\ x+h,\ t)] \cdot \eta^2 \Delta_h \nabla u \,\mathrm{d}x\,\mathrm{d}t$$

$$- \int_{Q_{2R}} \Delta_h \left[\frac{\Phi(|F|)}{|F|^2} F \right] \cdot 2\eta \nabla \eta\, \Delta_h u \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_{2R}} \Delta_h \left[\frac{\Phi(|F|)}{|F|^2} F \right] \cdot \eta^2 \Delta_h \nabla u \,\mathrm{d}x\,\mathrm{d}t. \tag{87}$$

We write each term of (87) as

$$M_1 + M_2 = M_3 + M_4 + M_5 + M_6 + M_7 + M_8.$$

Step 2. By the assumptions and fundamental inequalities, we shall estimate M_i in a proper form. By a simple calculation, we find that

$$M_1 = \int_{B_{2R}} \eta^2 (\Delta_h u)^2 \,\mathrm{d}x \Big|_{-4R^2}^{4R^2} = 0. \tag{88}$$

According to (2), we easily get

$$M_2 \geq \overline{r} \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2\right)^{\frac{1}{2}}\right]}{\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2}\, |\Delta_h \nabla u|^2\, \eta^2\, \mathrm{d}x\,\mathrm{d}t. \tag{89}$$

By letting $|h| < \delta < R$, we estimate M_3 as

$$M_3 \;\leq\; C|h|^2 \int_{Q_{2R+|h|}} |\nabla u|^2\, \mathrm{d}x\,\mathrm{d}t \;\leq\; C|h|^2 \int_{Q_{2R+|h|}} [\Phi(|\nabla u|)+1]\, \mathrm{d}x\,\mathrm{d}t. \tag{90}$$

To estimate the integral M_4, we consider Lagrange Mean Value Theorem and the assumption (A2). Ones get that

$$M_4 \int_{Q_{2R}} |A(\nabla u(x,\, t),\, x+h,\, t) - A(\nabla u(x+h,\, t),\, x+h,\, t)|\; \eta\; |\Delta_h u|\, \mathrm{d}x\,\mathrm{d}t$$

$$\int_{Q_{2R}} \left|D_\xi A(\nabla u(x,\, t),\, x+h,\, t)\right|\, |\Delta_h \nabla u|\; \eta\; |\Delta_h u|\, \mathrm{d}x\,\mathrm{d}t$$

$$\int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2\right)\right]}{\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2}\, |\Delta_h \nabla u|\; \eta\; |\Delta_h u|\, \mathrm{d}x\,\mathrm{d}t.$$

Using (15) and Lagrange Mean Value Theorem again, we obtain

$$\begin{aligned}
M_4 \leq & \varepsilon \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2\right)\right]}{\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2}\, |\Delta_h \nabla u|^2\, \eta^2\, \mathrm{d}x\,\mathrm{d}t \\
& + C \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2\right)\right]}{\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2}\, |\Delta_h u|^2\, \mathrm{d}x\,\mathrm{d}t \\
\leq & \varepsilon \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2\right)\right]}{\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2}\, |\Delta_h \nabla u|^2\, \eta^2\, \mathrm{d}x\,\mathrm{d}t \\
& + C|h|^2 \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2\right)\right]}{\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2}\, |\nabla u|^2\, \mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Obviously, M_4 is estimated by

$$M_4 \leq \varepsilon \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2\right)\right]}{\mu^2+|\nabla u(x,\, t)|^2+|\nabla u(x+h,\, t)|^2}\, |\Delta_h \nabla u|^2\, \eta^2\, \mathrm{d}x\,\mathrm{d}t + C|h|^2 \int_{Q_{2R+|h|}} \Phi(\mu+|\nabla u|)\, \mathrm{d}x\,\mathrm{d}t. \tag{91}$$

Based on the (A3), we gain that

$$M_5 \le C \int_{Q_{2R}} |A(\nabla u(x,\ t),\ x,\ t) - A(\nabla u(x,\ t),\ x+h,\ t)|\ \eta\ |\Delta_h u|\, \mathrm{d}x\, \mathrm{d}t$$

$$\le C|h|^{\alpha} \int_{Q_{2R}} (g_k(x,\ t) + g_k(x+h,\ t))\, \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right]}{(\mu^2 + |\nabla u|^2)^{\frac{1}{2}}}\ \eta\ |\Delta_h u|\, \mathrm{d}x\, \mathrm{d}t.$$

Via (15) and Lagrange Mean Value Theorem, we have

$$M_5 \le C|h|^{2\alpha} \int_{Q_{2R}} (g_k(x,\ t) + g_k(x+h,\ t))^2\ \Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right] \mathrm{d}x\, \mathrm{d}t + C \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right]}{\mu^2 + |\nabla u|^2}\, |\Delta_h u|^2\, \mathrm{d}x\, \mathrm{d}t$$

$$\le C|h|^{2\alpha} \int_{Q_{2R}} (g_k(x,\ t) + g_k(x+h,\ t))^2\ \Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right] \mathrm{d}x\, \mathrm{d}t + C|h|^2 \int_{Q_{2R+|h|}} \Phi\,(\mu + |\nabla u|)\, \mathrm{d}x\, \mathrm{d}t. \tag{92}$$

In view of (A3), the term M_6 is estimated as

$$M_6 \le C \int_{Q_{2R}} |A(\nabla u(x,\ t),\ x,\ t) - A(\nabla u(x,\ t),\ x+h,\ t)|\ \eta^2\ |\Delta_h \nabla u|\, \mathrm{d}x\, \mathrm{d}t$$

$$\le C|h|^{\alpha} \int_{Q_{2R}} (g_k(x,\ t) + g_k(x+h,\ t))\ \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right]}{(\mu^2 + |\nabla u|^2)^{\frac{1}{2}}}\ \eta^2\ |\Delta_h \nabla u|\, \mathrm{d}x\, \mathrm{d}t.$$

According to (15), we acquire

$$M_6 \le C|h|^{2\alpha} \int_{Q_{2R}} (g_k(x,\ t) + g_k(x+h,\ t))^2\ \Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right] \mathrm{d}x\, \mathrm{d}t + \varepsilon \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right]}{\mu^2 + |\nabla u|^2}\, |\Delta_h \nabla u|^2\ \eta^2\, \mathrm{d}x\, \mathrm{d}t.$$

We finally obtain an estimate of M_6 as

$$\begin{aligned} M_6 \le & C|h|^{2\alpha} \int_{Q_{2R}} (g_k(x,\ t) + g_k(x+h,\ t))^2\ \Phi\left[\left(\mu^2 + |\nabla u|^2\right)^{\frac{1}{2}}\right] \mathrm{d}x\, \mathrm{d}t \\ & +\ \varepsilon \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2 + |\nabla u(x+h,\ t)|^2\right)^{\frac{1}{2}}\right]}{\mu^2 + |\nabla u|^2 + |\nabla u(x+h,\ t)|^2}\, |\Delta_h \nabla u|^2\ \eta^2\, \mathrm{d}x\, \mathrm{d}t. \end{aligned} \tag{93}$$

Using the fact of (15), it is obvious that

$$M_7 \le C|h|^{2\alpha} \int_{Q_{2R}} \left| \frac{\Delta_h \left[\frac{\Phi(|F|)}{|F|^2} F \right]}{|h|^{\alpha}} \right|^2 \mathrm{d}x\,\mathrm{d}t + C\int_{Q_{2R}} |\Delta_h u|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\le C|h|^{2\alpha} \int_{Q_{2R}} \left| \frac{\Delta_h \left[\frac{\Phi(|F|)}{|F|^2} F \right]}{|h|^{\alpha}} \right|^2 \mathrm{d}x\,\mathrm{d}t + C\int_{Q_{2R+|h|}} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\le C|h|^{2\alpha} \int_{Q_{2R}} \left| \frac{\Delta_h \left[\frac{\Phi(|F|)}{|F|^2} F \right]}{|h|^{\alpha}} \right|^2 \mathrm{d}x\,\mathrm{d}t + C\int_{Q_{2R+|h|}} [\Phi(|\nabla u|) + 1] \,\mathrm{d}x\,\mathrm{d}t \tag{94}$$

We use a similar argument as in estimating M_7, and get

$$M_8 |h|^{2\alpha} \int_{Q_{2R}} \left| \frac{\Delta_h \left[\frac{\Phi(|F|)}{|F|^2} F \right]}{|h|^{\alpha}} \right|^2 \mathrm{d}x\,\mathrm{d}t + \varepsilon \int_{Q_{2R}} |\Delta_h \nabla u|^2 \,\eta^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{95}$$

With the help of $0 < \mu < 1$ and (G1), we obtain

$$\varepsilon \int_{Q_{2R}} |\Delta_h \nabla u|^2 \,\eta^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{\varepsilon \mu^2}{\Phi(\mu)} \int_{Q_{2R}} \frac{\Phi(\mu)}{\mu^2} |\Delta_h \nabla u|^2 \,\eta^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\le \frac{\varepsilon \mu^2}{\Phi(\mu)} \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2 + |\nabla u(x+h,\ t)|^2\right)\right]}{\mu^2 + |\nabla u|^2 + |\nabla u(x+h,\ t)|^2} |\Delta_h \nabla u|^2 \,\eta^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{96}$$

Based on the estimates of M_i above, we evidently acquire that

$$\bar{r} \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2 + |\nabla u(x+h,\ t)|^2\right)\right]}{\mu^2 + |\nabla u|^2 + |\nabla u(x+h,\ t)|^2} |\Delta_h \nabla u|^2 \,\eta^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\le \left(2\,\varepsilon + \frac{\varepsilon \mu^2}{\Phi(\mu)}\right) \int_{Q_{2R}} \frac{\Phi\left[\left(\mu^2 + |\nabla u|^2 + |\nabla u(x+h,\ t)|^2\right)\right]}{\mu^2 + |\nabla u|^2 + |\nabla u(x+h,\ t)|^2} |\Delta_h \nabla u|^2 \,\eta^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$+\ C|h|^2\int_{Q_{2R+|h|}}[\Phi(\mu+|\nabla u|)+1]\,\mathrm{d}x\,\mathrm{d}t+C|h|^{2\alpha}\int_{Q_{2R}}\left|\frac{\Delta_h\left[\frac{\Phi(|F|)}{|F|^2}F\right]}{|h|^\alpha}\right|^2\mathrm{d}x\,\mathrm{d}t$$

$$+\ C|h|^{2\alpha}\int_{Q_{2R}}(g_k(x,\ t)+g_k(x+h,\ t))^2\,\Phi(\mu+|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t. \tag{97}$$

We choose ε small enough to obtain

$$\int_{Q_{2R}}\frac{\Phi\left[\left(\mu^2+|\nabla u|^2+|\nabla u(x+h,\ t)|^2\right)\right]}{\mu^2+|\nabla u|^2+|\nabla u(x+h,\ t)|^2}\,|\Delta_h\nabla u|^2\,\eta^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C|h|^2\int_{Q_{2R+|h|}}[\Phi(\mu+|\nabla u|)+1]\,\mathrm{d}x\,\mathrm{d}t+C|h|^{2\alpha}\int_{Q_{2R}}\left|\frac{\Delta_h\left[\frac{\Phi(|F|)}{|F|^2}F\right]}{|h|^\alpha}\right|^2\mathrm{d}x\,\mathrm{d}t$$

$$+\ C|h|^{2\alpha}\int_{Q_{2R}}(g_k(x,\ t)+g_k(x+h,\ t))^2\,\Phi(\mu+|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t. \tag{98}$$

Step 3. Combining inequalities (98) and (G2), we acquire

$$\int_{Q_{\frac{R}{2}}}|\Delta_h H_\Phi(\nabla u)|^2\,\mathrm{d}x\,\mathrm{d}t\leq C|h|^2\int_{Q_{2R+|h|}}[\Phi(\mu+|\nabla u|)+1]\,\mathrm{d}x\,\mathrm{d}t$$

$$+\ C|h|^{2\alpha}\int_{Q_{2R}}(g_k(x,\ t)+g_k(x+h,\ t))^2\,\Phi(\mu+|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\ C|h|^{2\alpha}\int_{Q_{2R}}\left|\frac{\Delta_h\left[\frac{\Phi(|F|)}{|F|^2}F\right]}{|h|^\alpha}\right|^2\mathrm{d}x\,\mathrm{d}t. \tag{99}$$

Dividing by $|h|^{2\alpha}$, one obtains that

$$\int_{Q_{\frac{R}{2}}}\left|\frac{\Delta_h H_\Phi(\nabla u)}{|h|^\alpha}\right|^2 \mathrm{d}x\,\mathrm{d}t \le C|h|^{2-2\alpha}\int_{Q_{2R+|h|}}[\Phi(\mu+|\nabla u|)+1]\,\mathrm{d}x\,\mathrm{d}t$$

$$+\ C\int_{Q_{2R}}(g_k(x,\ t)+g_k(x+h,\ t))^2\,\Phi(\mu+|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\ C\int_{Q_{2R}}\left|\frac{\Delta_h\left[\frac{\Phi(|F|)}{|F|^2}F\right]}{|h|^\alpha}\right|^2 \mathrm{d}x\,\mathrm{d}t. \tag{100}$$

Raising (100) to the power of $1/2$, it gives that

$$\left(\int_{Q_{\frac{R}{2}}}\left|\frac{\Delta_h H_\Phi(\nabla u)}{|h|^\alpha}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \le C|h|^{1-\alpha}\left(\int_{Q_{2R+|h|}}[\Phi(\mu+|\nabla u|)+1]\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$+\ C\left(\int_{Q_{2R}}(g_k(x,\ t)+g_k(x+h,\ t))^2\,\Phi(\mu+|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$+\ C\left(\int_{Q_{2R}}\left|\frac{\Delta_h\left[\frac{\Phi(|F|)}{|F|^2}F\right]}{|h|^\alpha}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}. \tag{101}$$

Considering the ball B_δ and taking the L^2 norm with the measure $\frac{\mathrm{d}h}{|h|^n}$ in (101), we obtain

$$\left(\int_{B_\delta}\left(\int_{Q_{\frac{R}{2}}}\left|\frac{\Delta_h H_\Phi(\nabla u)}{|h|^\alpha}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{2}}$$

$$\le C\left(\int_{B_\delta}|h|^{2(1-\alpha)}\left(\int_{Q_{2R+|h|}}[\Phi(\mu+|\nabla u|)+1]\,\mathrm{d}x\,\mathrm{d}t\right)\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{2}}$$

$$+\ C\left(\int_{B_\delta}\left(\int_{Q_{2R}}(g_k(x,\ t)+g_k(x+h,\ t))^2\,\Phi(\mu+|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t\right)\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{2}}$$

$$+\, C\left(\int_{B_\delta}\left(\int_{Q_{2R}}\left|\frac{\Delta_h\left[\frac{\Phi(|F|)}{|F|^2}F\right]}{|h|^\alpha}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{2}} = S_1+S_2+S_3. \tag{102}$$

Step 4. In the last step, we prove that S_i is bounded. By the fact of $u\in W^{1,\,\Phi}(\Omega_T)$, we have

$$S_1\left(\int_{B_\delta}|h|^{2(1-\alpha)-n}\,\mathrm{d}h\right)^{\frac{1}{2}}\left(\int_{Q_{3R}}\left[\Phi\left(\mu+|\nabla u|\right)+1\right]\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\left(\int_0^\delta \rho^{2(1-\alpha)-1}\,\mathrm{d}\rho\right)^{\frac{1}{2}}\left(\int_{Q_{3R}}\left[\Phi\left(\mu+|\nabla u|\right)+1\right]\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\left(\int_{Q_{3R}}\left[\Phi\left(\mu+|\nabla u|\right)+1\right]\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} < +\infty. \tag{103}$$

Since $\dfrac{\Phi(|F|)}{|F|^2}F\in L^2((0,\ T);\ W^{\alpha,\,2}(\Omega))$ and Lemma 5, we have $\dfrac{\Phi(|\nabla u|)}{|\nabla u|^2}\nabla u\in L^{\frac{2n}{n-2\alpha}}_{\mathrm{loc}}(\Omega_T)$. We further obtain $\Phi(|\nabla u|)\in L^{\frac{n}{n-2\alpha}}_{\mathrm{loc}}(\Omega_T)$. We set $r_k=2^{-k}K_0R$, then we get

$$S_2 = C\left(\int_{B_\delta}\left(\int_{Q_{2R}}\left(g_k(x,\ t)+g_k(x+h,\ t)\right)^2\Phi\left(\mu+|\nabla u|\right)\mathrm{d}x\,\mathrm{d}t\right)\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{2}}$$

$$\left(\int_0^{2^{-k_0}K_0R}\int_{\partial B_r}\left(\int_{Q_{2R}}\left(g_k(x,\ t)+g_k(x+h,\ t)\right)^2\Phi\left(\mu+|\nabla u|\right)\mathrm{d}x\,\mathrm{d}t\right)\mathrm{d}S(h)\,\mathrm{d}r\right)^{\frac{1}{2}}$$

$$\left(\sum_{k=k_0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{\partial B_r}\left(\int_{Q_{2R}}\left(g_k(x,\ t)+g_k(x+h,\ t)\right)^2\Phi\left(\mu+|\nabla u|\right)\mathrm{d}x\,\mathrm{d}t\right)\mathrm{d}S(h)\,\mathrm{d}r\right)^{\frac{1}{2}}.$$

Using Young inequality, we obtain

$$\int_{Q_{2R}} (g_k(x,\ t)+g_k(x+h,\ t))^2\, \Phi(\mu+|\nabla u|)\, \mathrm{d}x\, \mathrm{d}t$$

$$\leq C\left[\int_{Q_{2R}} (g_k(x,\ t)+g_k(x+h,\ t))^{2\cdot\frac{n}{2\alpha}}\, \mathrm{d}x\, \mathrm{d}t\right]^{\frac{2\alpha}{n}} \left[\int_{Q_{2R}} \Phi(\mu+|\nabla u|)^{\frac{n}{n-2\alpha}}\, \mathrm{d}x\, \mathrm{d}t\right]^{\frac{n-2\alpha}{n}}$$

$$= C\, \|g_k(x,\ t)\|^2_{L^{\frac{n}{\alpha}}(Q_{3R})}\, \|\Phi(\mu+|\nabla u|)\|_{L^{\frac{n}{n-2\alpha}}(Q_{2R})}.$$

Based on assumption (A3), we easily acquire

$$S_2\, \|\Phi(|\nabla u|)\|^{\frac{1}{2}}_{L^{\frac{n}{n-2\alpha}}(Q_{2R})} \left(\sum_{k=k_0}^{\infty} \int_{r_{k+1}}^{r_k} \int_{\partial B_r} \left(\|g_k(x,\ t)\|^2_{L^{\frac{n}{\alpha}}(Q_{3R})}\right) \mathrm{d}S(h)\, \mathrm{d}r\right)^{\frac{1}{q}}$$

$$\|\Phi(|\nabla u|)\|^{\frac{1}{2}}_{L^{\frac{n}{n-2\alpha}}(Q_{2R})} \left[\|\{g_k\}_k\|_{l^2\left(L^{\frac{n}{\alpha}}(Q_{2R})\right)}\right] < +\infty. \tag{104}$$

According to $\dfrac{\Phi(|F|)}{|F|^2} F \in L^2\left((t_0,\ t_0+T];\ W^{\alpha,\ 2}(\Omega)\right)$ and Fubini Theorem, we get an estimation of S_3 as

$$S_3 = C\left(\int_{B_\delta}\int_{-4R^2}^{4R^2}\left(\int_{Q_{2R}}\left|\frac{\Delta_h\left[\frac{\Phi(|F|)}{|F|^2}F\right]}{|h|^\alpha}\right|^2 \mathrm{d}x\right)\mathrm{d}t\, \frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{2}}$$

$$= C\left(\int_{-4R^2}^{4R^2}\int_{B_\delta}\left(\int_{Q_{2R}}\left|\frac{\Delta_h\left[\frac{\Phi(|F|)}{|F|^2}F\right]}{|h|^\alpha}\right|^2 \mathrm{d}x\right)\frac{\mathrm{d}h}{|h|^n}\, \mathrm{d}t\right)^{\frac{1}{2}}$$

$$= C\, \|F\|_{L^2\left((-4R^2,\ 4R^2);\ W^{\alpha,\ 2}(B_{2R})\right)} < +\infty. \tag{105}$$

From the estimates of S_i, we finally acquire

$$\left(\int_{B_\delta}\left(\int_{Q_{\frac{R}{2}}}\left|\frac{\Delta_h H_\Phi(\nabla u)}{|h|^\alpha}\right|^2 \mathrm{d}x\, \mathrm{d}t\right)\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{2}} < +\infty, \tag{106}$$

which means $H_\Phi(\nabla u) \in W^{\alpha,\ 2}(\Omega_T)$. This completes the proof of Theorem 1 □.

Remark According to the proof of Theorem 1, we derive the results of Corollary 2 and 3, immediately. By the norms of fractional Sobolev spaces, Besov spaces and Triebel-Lizorkin spaces, see (17), (18) and (19), one obtains an obvious conclusion that $W^{\alpha,\,2} = B^{\alpha}_{2,\,2} = F^{\alpha}_{2,\,2}$ with $0 < \alpha < 1$. At this point, the Corollary 2 and 3 are clearly valid.

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Conflict of interest

The authors declare no competing financial interest.

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