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Research Article

Improving Model Fitting for Applicable Medical Data: A Novel Exponentiated Transformation of BURR XII

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Abstract: This paper presents a novel extension of the Exponentiated Transformation of the BURR XII (ETBXII) distribution. In this paper, we explore the need for improved model fitting to applicable data and present a literature review on existing distributions that researchers modified. The proposed extension is introduced, and its properties, such as its hazard rate and distribution function, are addressed. The results demonstrate that the suggested extension can better match data than existing distributions and present an empirical application to demonstrate its utility. The conclusion of the manuscript discusses the potential effects of the suggested expansion on statistical modelling in several domains.

***Keywords*:** maximum likelihood estimation (MLE), order statistics, BURR XII, ETBXII

MSC: 62E15, 62N05, 60E05, 62F10, 62N02

1. Introduction

The COVID-19 pandemic latterly resulted from an unknown pneumonia etiology in Wuhan city, Province of Hubei, China in Dec 2019 [1]. Scientifically, the COVID-19 contains different series of nucleic acid viruses-specific, that are familiar to human coronavirus species. They are resembling with 'Beta Corona Virus', which scientists identified in rats [2, 3]. Basically, the Human coronavirus was found to be innocuous respiratory pathogen. In some ways, it conferred worldwide attention after the outbreak of Severe Acute Respiratory syndrome (SARs) and the Middle East Respiratory syndrome (MERs) as an important pathogens in China (2002) and Saudi Arabia (2012) respectively. The coronavirus was found to be zoonosis and it can eventually affect severe diseases in Humans. The 2002 SARS formerly emerged from Himalayan Palm Civet/raccoon whereas 2012 MERs were from "Camelus Dromedarius (Arabian Camel)" [4]. Prior to COVID-19, the first pandemic of the 21st century known to be N1H1 or previously called Sine Flu emerged in 2009 [5, 6].

According to the world health organization 2010, the zoonosis viruses that occurred previously in 1918 are H1NI (Avian Influenza), H2N2 (Asian Flu) in 1957, and H3N2 (Hong Kong Flu) in 1968. These pandemics were jotted down as fatal or mortal. As H1N1 (Avian Influenza) had led up to 20-50 million deaths, H2N2 fattely affected up to 1.5 million population and H3N2 caused a million deaths [6, 7].

In Dec 2019 in China, the unknown pathogenesis of pneumonia patients was detected through throat swabs, and the Chinese Centre for Disease Control and Prevention (CCDC) recognized this as a COVID-19 virus [8]. CCDC considered the pivotal envoy was correlated to the first SARs [9]. As of 30th Jan 2020, World Health Organization stated a global

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Health emergency as an expeditious upsurge of COVID-19 cases in China [10, 11]. On 8th Feb 2020, more than 83,852 were contaminated with COVID-19 and approximately 3000 died due to this pandemic, which was an excessive mortality rate of 4.47% in Wuhan [12]. The COVID-19 cases were considered in the huddle, which ensue in layers and turn out into a series, globally [13]. On 23 Jan 2020, due to the patient's travel history from Wuhan, the no of other countries infected from this were Japan, Maccue Special Administrative Region, Hong Kong Special Administrative Region, Thailand, Taipei Municipality, Republic of Korea, and the United States [13]. As of 20 Feb 2020, Italy declared its first COVID-19 patient, and within 24 h, there were 36 more cases that exist. Then, Italy becomes the most remarkable and had the most severe clusters of COVID-19 in the world. On 27 Jan 2020, 41 new cases were confirmed due to travel history from China. Among those, 27 cases were reported in Asia, 6 in North America, 3 in Europe, and 5 in Oceania [14, 15]. The common symptoms reported with SARs Cov-2 fever were myalgia, dry cough, and respiratory problems [16]. As claimed by WHO, 54.3% were male (median age of 56 years) that infected with COVID-19 [8]. Several researchers presented different probability models on its confirmed cases, the total number of deaths, and the total number of recovered cases. Different methods and techniques were proposed for its aetiology, clinical symptoms, and epidemiology. The number of COVID-19 cases surged exponentially.

To increase the model fitting to applied data, many existing distributions have been improved by the researchers, for example, Gupta et al. [17] considered the general class of Exponentiated distribution. Nadarajah and Kotz [7] proposed Exponentiated Frechet distribution, Nadarajah and Kotz defined the Beta Gumbel distribution [18]. Mudholkar and Srivastava [19] presented the Exponentiated Weibull distribution. Nadarajah [10] studied the Exponentiated Gumbel distribution. Chen et al. [20] studied a Beta generalized Exponential distribution. Cordeiro and de Castro [21] presented the Kumaraswamy distribution. Nadarajah and Kotz [12] used a Beta Exponentiated distribution. De Wit et al. [2] used the extension of the Weibull Pareto distribution. Recently, Silva et al. [22] used Alpha-Power Exponentiated Inverse Rayleigh distribution with applications. [23] introduced the Exponentiated T-X family of distribution with applications. More recently Ijaz et al. [4, 24, 25] proposed a novel extension of new probability distributions. Moreover, in the application of system reliability, a BURR-II distribution was used by Abdel-Ghaley et al. [26]. Other researchers presented the lifetime distribution (see e.g., [27–30]). The existing literature are inadequate to lifetime distribution with data set having hazard rate shapes (the nonmonotonic) bath-tub or inverse bath-tub. For instance, the Exponential family and Gamma distribution failed to present the increasing/decreasing failure function. Consequently, the need is to present a lifetime distribution, which not only apprehends the monotonic but also the non-monotonic hazard rate of real phenomena. Among other lifetime distributions, the most used distribution that captured the monotonic behavior of real data is known to be the BURR XII distribution. The subject distribution was proposed by Burr in 1942 [31] and become famous because of its large applications. Likewise reliability analysis [32], environmental data analysis [1], finance data modeling [33], and so on. Other applications were found in Bayesian estimation Al Hussaini [34], some statistical inference Mousa [35], and growth reliability by Abdel-Ghaly [32].

This study aims to propose a new model by using DUS transformation. For this purpose, a traditional BURR XII model is used. The subject distribution is considered a basic model for a wide distribution class, and it has a broad application to a complex data set. In order to improve its efficiency and upgrade its adequacy, two new extra parameters are introduced. The proposed model is known as Exponentiated Transformation of BURR XII (ETBXII).

Let T be a random variable with $F(t)$ and $f(t)$ be the cdf and pdf of the basic model (BURR XII) respectively. The DUS transformation can be defined as

$$G(t) = \frac{e^{\varsigma F(t)} - 1}{e^{\varsigma} - 1} \tag{1}$$

with pdf

$$g(t) = \frac{\varsigma f(t) e^{\varsigma F(t)}}{e^{\varsigma} - 1} \tag{2}$$

Since it comprises ς extra shape parameter to normalize the transformation, which makes the ETBUXII model more flexible. Specific statistical and reliable attributes, along with survival and hazard functions, quantile function (qf), reliability and cumulative hazard rate function, a measure of skewness and kurtosis, means and variance moments generating function, central and non-central moments, order statistics (OS), stochastic ordering (SO), characteristic function, factorial generating function, incomplete noncentral moments, conditional moments and mean deviation are derived and considered.

The primary aim is to enhance the flexibility of the model to better fit a wider range of real-world data, particularly in medical and reliability applications. The BURR XII distribution, while useful, has certain limitations in capturing the complexities of certain datasets. By applying an exponentiated transformation, we aim to overcome these limitations and provide a more versatile model that can more accurately reflect the underlying data structures. The Burr XII distribution could be a suitable model for the death rate of COVID-19 patients based on its flexibility in fitting data with different shapes, including skewness and heavy tails. The distribution is often employed in survival analysis and reliability modeling, which are relevant to the study of mortality rates. The death rate of COVID-19 patients may exhibit complex behavior such as skewness towards higher death rates in some cases, and a heavy-tailed distribution of death times (especially in severe cases), which the Burr XII distribution can effectively capture.

In particular, the Burr XII distribution has been used in various epidemiological studies, including modeling survival times and hazard rates, making it a reasonable choice for analyzing death rates in this context. Additionally, its ability to model data that may show varying hazard rates over time could provide a more nuanced understanding of how COVID-19 affects different groups of patients under various conditions.

We of course, carefully validate the model's fit to the empirical data, potentially comparing it against other commonly used distributions like the Weibull or log-normal distributions, to ensure that the Burr XII offers a superior or comparable fit for our specific dataset.

Moreover, uncertainty measures including information generating function and entropy measures namely Renyi, Verma, Tsallis, etc are also derived and represented graphically. The Log-likelihood, density, and trace plots are also ploted for both data sets. Additionally, the model parameter is estimated via maximum likelihood and the Bayesian paradigm. To examine the efficiency of the ETBXII model a simulation analysis is performed. In the end, using COVID-19 data sets to show the flexibility of the suggested model.

2. Proposed distribution

The baseline distribution is considered to be a BURR XII distribution with two shape parameters γ and φ and one scale parameter λ, by following Pandu R. T. one can easily introduce the location (scale) parameter having cdf as

$$F(t|\gamma,\varphi) = 1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma} \tag{3}$$

And corresponding pdf as

$$f(t|\gamma,\varphi) = \frac{\gamma\varphi}{\lambda}t^{\varphi-1}\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-(\gamma+1)} \tag{4}$$

The pdf and cdf of proposed model can conveniently be gained by using transformation given in (1) and (2) as:

$$g(t|\gamma,\,\varphi,\,\varsigma)=\frac{\varsigma\left[\gamma\varphi t^{\varphi-1}\left(1+\dfrac{t^{\varphi}}{\lambda}\right)^{-\gamma-1}\right]e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}}{\lambda\left(e^{\varsigma}-1\right)} \tag{5}$$

$$G(t|\gamma,\,\varphi,\,\varsigma)=\frac{e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}-1}{e^{\varsigma}-1} \tag{6}$$

One can notice the cdf $G(t|\gamma,\ \varphi)$ of Subsequently mentioned distribution is known as ETBXII distribution is differentiable and it ranges from 0 to ∞ as

$$\underset{t\to 0}{Lim}\,G(t)=0 \tag{7}$$

$$\underset{t\to\infty}{Lim}\,G(t)=1 \tag{8}$$

The pdf plot of ETBXII with different parametric values can be shown in Figure 1.

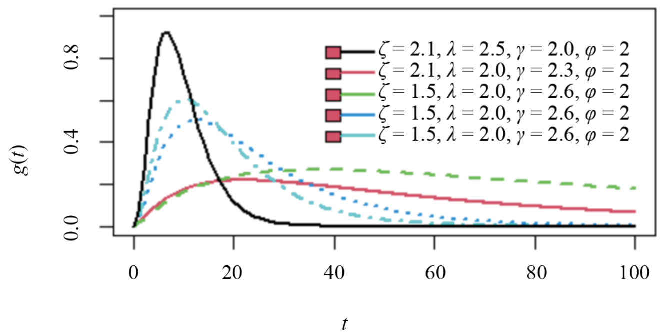


Figure 1. Pdf plot of ETBXII distribution with different parametric values

2.1 *Reliability function*

The $R(t)=1-G(t)$ regulates reliability function and defined as the probability of a lifetime of a certain device (without failure) during a particular time interval (say 'z'). the reliability function of ETBXII distribution is given as

$$R(t|\gamma,\,\varphi,\,\varsigma)=P(T>t)=1-G(t)$$

$$R(t|\gamma,\,\varphi,\,\varsigma)=\frac{e^{\varsigma}\left(1-e^{\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}\right)}{e^{\varsigma}-1} \tag{9}$$

2.2 *Hazard rate function*

In lifetime analysis, another useful function is Hazard Rate Function defined as the expeditious failure rate of a random variable T is the probability that device tumble as it has persevered to the present time (say 'z') is given as

$$H(t|\gamma,\ \varphi,\ \varsigma) = \frac{g(t|\gamma,\ \varphi,\ \varsigma)}{1-G(t|\gamma,\ \varphi,\ \varsigma)}$$

we get,

$$H(t|\gamma,\ \varphi,\ \varsigma) = \frac{\varsigma\left[\frac{\gamma\varphi}{\lambda}t^{\varphi-1}\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-(\gamma+1)}\right]e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}}{2-e^{\varsigma\left[1-(1+t^{\varphi})^{-\gamma}\right]}},\quad \gamma,\ \varphi,\ \varsigma>0 \tag{10}$$

The odd ratio is defined as

$$\Psi_t(t|\gamma,\ \varphi,\ \varsigma) = \frac{R_t(t|\gamma,\ \varphi,\ \varsigma)}{H_t(t|\gamma,\ \varphi,\ \varsigma)}$$

$$\Psi_t(t|\gamma,\ \varphi) = \frac{e^{\varsigma}\left[2-e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}\right]}{\left[\frac{\varsigma\gamma\varphi}{\lambda}t^{\varphi-1}\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-(\gamma+1)}\right]} \tag{11}$$

2.3 *Cumulative hazard rate function*

The cumulative Hazard Rate Function is defined as

$$CH(t) = \log\left|2-e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}\right| \tag{12}$$

2.4 *Extreme values of a model*

Let T_1, T_2, ..., T_n be a r.s of size 'n' from a subject model having $g(t|\gamma,\ \varphi)$ and $G(t|\gamma,\ \varphi)$, pdf and cdf respectively. In probability and statistics, the long stand area of confine model is the sample extremes.

For ETBXII model, $T_{n,\,n}$ is considered to be maximum order statistics as,

$$\underset{n\to\infty}{Lim}\,P(T_{n,\,n}\leq a_n+b_n z) = e^{-e^{-z}},\ -\infty<z<\infty \tag{13}$$

where $a_n = G^{-1}\left(1-\frac{1}{n}\right)$ and $b_n = -\frac{1}{ng(a_n)}$, if

$$\underset{n\to\infty}{Lim}\frac{d}{dy}\left(\frac{1}{H\{g(g(t|.))\}}\right)=0 \tag{14}$$

Proposition 1 For ETBXII model, Let $T_{n,\,n}$ be the largest order statistics, then

$$\underset{n\to\infty}{Lim}P(T_{n,\,n}\leq a_n+b_nz)=e^{-e^{-z}},\; -\infty<z<\infty \tag{15}$$

where $a_n=G^{-1}\left(1-\frac{1}{n}\right)$ and $b_n=-\frac{1}{ng(a_n)}$, if $g(.)$ and $G(.)$ are given in Equations (5) and (6).

Proof. For ETBXII model, we have

$$\underset{n\to\infty}{Lim}\frac{d}{dy}\left(\frac{1}{H\{g(g(t|.))\}}\right)=\underset{n\to\infty}{Lim}\frac{d}{dy}\left(\frac{2-e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}}{\varsigma\left[\frac{\gamma\varphi}{\lambda}t^{\varphi-1}\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-(\gamma+1)}\right]e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}}\right),$$

$$=\underset{n\to\infty}{Lim}\frac{d}{dy}\left(\frac{\left(1+\frac{t^{\varphi}}{\lambda}\right)^{(\gamma+1)}\left(2-e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}\right)}{\varsigma\frac{\gamma\varphi}{\lambda}t^{\varphi-1}e^{\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}}\right) \tag{16}$$

$$=\frac{1}{\varsigma\gamma\varphi}\underset{n\to\infty}{Lim}\frac{d}{dy}\left(\frac{(1+t^{\varphi})^{(\gamma+1)}\left(2-e^{\varsigma\left[1-(1+t^{\varphi})^{-\gamma}\right]}\right)}{\varsigma\gamma\varphi t^{\varphi-1}e^{\left[1-(1+t^{\varphi})^{-\gamma}\right]}}\right)$$

$$=0.$$

Hence proved. □

2.5 *Order statistics*

Let T_1, T_2, ..., T_n be Order Statistics of a r.s of size 'n' from ETBXII distribution. Then for $j=1,\,2,\,\ldots,\,n$, the pdf of j^{th} OS, $T_{(j)}$ is defined as

$$g_{(j)}(t|\gamma,\,\varphi,\,\varsigma)=\phi\{1-G(t|\gamma,\,\varphi,\,\varsigma)\}^{n-j}G(t|\gamma,\,\varphi,\,\varsigma)^{j-1}g(t|\gamma,\,\varphi,\,\varsigma) \tag{17}$$

where

$$\phi = \frac{n!}{(j-1)!(n-j)!}$$

Thus, from $g(t|\gamma, \varphi, \varsigma)$, $G(t|\gamma, \varphi, \varsigma)$ and ϕ, the OS of ETBXII model can be defined as

$$g_{(j)}(t|\gamma, \varphi, \varsigma) = \varsigma \frac{\varphi\gamma}{\lambda} \sum_{j=0}^{n-j} \begin{pmatrix} n-j \\ f \end{pmatrix} (-1)^f \left(e^{\varsigma\left[1-\left(1+\frac{t^\varphi}{\lambda}\right)^{-\gamma}\right]} - 1 \right) \cdot \frac{t^{\varphi-1}\left(1+\frac{t^\varphi}{\lambda}\right)^{-(\gamma+1)} e^{\varsigma\left[1-\left(1+\frac{t^\varphi}{\lambda}\right)^{-\gamma}\right]}}{(e^\varsigma - 1)^{j+f}} \tag{18}$$

the cdf of j^{th} OS $T_{(j)}$ can be defined as

$$G_{(j)}(t|\gamma, \varphi, \varsigma) = \sum_{j=0}^{n-j} \begin{pmatrix} n-j \\ f \end{pmatrix} G(t|\gamma, \varphi, \varsigma)^m (1 - G(t|\gamma, \varphi, \varsigma))^{n-m} \tag{19}$$

Therefore, the cdf of ETBXII model can be described as

$$G_{(j)}(t|\gamma, \varphi, \varsigma) = \sum_{m=j}^{n} \sum_{h=0}^{n-m} \begin{pmatrix} n \\ j \end{pmatrix} \begin{pmatrix} n-m \\ h \end{pmatrix} (-1)^h G(t|\gamma, \varphi, \varsigma)^{m+h} \tag{20}$$

Generally, the cdf of $T_{(n)}$ and $T_{(1)}$ are given as

$$G_{(1)}(t|\gamma, \varphi, \varsigma) = 1 - [1 - G(t)]^n,\ G_{(n)}(t) = G^n(t)$$

$$G_{(n)}(t|\gamma, \varphi, \varsigma) = \left[\frac{e^{\varsigma\left[1-\left(1+\frac{t^\varphi}{\lambda}\right)^{-\gamma}\right]} - 1}{e^\varsigma - 1} \right]^n$$

$$G_{(1)}(t|\gamma, \varphi, \varsigma) = 1 - \left[1 - \frac{e^{\varsigma\left[1-\left(1+\frac{t^\varphi}{\lambda}\right)^{-\gamma}\right]} - 1}{e^\varsigma - 1} \right]^n$$

thus, we get

$$G_{(1)}(t|\gamma,\, \varphi,\, \varsigma) = 1 - \left[\frac{e^{\varsigma} - e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}}{e^{\varsigma} - 1} \right]^{n} \tag{21}$$

2.6 *Quantile function*

Let $\Theta_j(.)$ be quantile function of $T_{(j)}$ (where $0 < q < 1$), then we get

$$\Theta_n(q) = \Theta\left(q^{1/n}\right),\; \Theta_{(1)}(q) = \Theta\left\{1 - (1-q)^{1/n}\right\}$$

As the cdf of $T_{(n)}$ and $T_{(1)}$, are not in explicit form, so we cannot express quantile functions accurately. If, iid random variable are to be considered, then r^{th} ordinary moment of OS can be obtained when $\rho_r < \infty$. By following [36], the r^{th} moment of j^{th} OS can be expressed as

$$\rho_{(j)}^{r} = \left\{T_{(j)}^{r}\right\} = \sum_{m=n-j+1}^{n} (-1)^{m-n+j-1} \begin{pmatrix} m-1 \\ n-j \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} I_m(r) \tag{22}$$

where

$$I_m(r) = \int_{0}^{\infty} rt^{r-1}[1 - G(t)]^{m} dt$$

Proposition 2 For EBURXII model, let $T_{(1)} \leq T_{(2)} \leq \ldots\ldots \leq T_{(n)}$ be OS of a r.s of size 'n', the r^{th} moment of j^{th} OS $T_{(j)}$ of EBURXII pdf can be expressed as

$$\rho_{(j)}^{r} = \Re_{m,\, j,\, n} \int t^{\varphi-1} \frac{\left(1 + \frac{t^{\varphi}}{\lambda}\right)^{-(\gamma+1)} e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}}{(e^{\varsigma} - 1)^{j+f}} dt \tag{23}$$

where

$$\Re_{m,\, j,\, n} = \sum_{m=n-j+1}^{n} \begin{pmatrix} m-1 \\ n-j \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} (-1)^{m-n+j-1}$$

Proof.

$$\rho_{(j)}^{r}=\left\{T_{(j)}^{r}\right\}=\sum_{m=n-j+1}^{n}\left(\begin{array}{c} m-1 \\ n-j \end{array}\right)\left(\begin{array}{c} n \\ m \end{array}\right)(-1)^{m-n+j-1}$$

Now consider,

$$I_m(r)=r\int_{0}^{\infty}t^{r-1}[1-G(t)]^m dt$$

by solving, we get

$$I_m(r)=r\int_{0}^{\infty}t^{r-1}\left[\frac{e^{\varsigma}-e^{\varsigma\left[1-\left(1+\frac{t^{\varphi}}{\lambda}\right)^{-\gamma}\right]}}{e^{\varsigma}-1}\right]^m dt \tag{24}$$

The above equation be in implicit form, so we solve this numerically. □

2.7 *Stochastic ordering*

Let $T_{(1)}\sim EBURRXII-\pi(\gamma_1,\varphi_1,\varsigma_1)$ and $T_{(2)}\sim EBURRXII-\pi(\gamma_2,\varphi_2,\varsigma_2)$
If

$i.\quad \gamma_1=\gamma_2=\gamma,\ \varphi_1=\varphi_2=\varphi$ and $\varsigma_1\le\varsigma_2$

$ii.\quad \gamma_1=\gamma_2=\gamma,\ \varsigma_1=\varsigma_2=\varsigma$ and $\varphi_1\le\varphi_2$, then

$T_{(1)}\le_{lr}T_{(2)}\left(T_{(1)}\le_{lr}T_{(2)}\right)$

2.8 *Moments and measure of dispersion*

For ETBXII distribution, the r^{th} non central moment can be defined as

$$\mu_r'=E(T^r)=\int_{0}^{\infty}t^r dG(t|\gamma,\varphi,\varsigma);\quad r=1,2,\ldots.$$

Theorem 1 For $\gamma,\varphi,\varsigma>0$, the r^{th} non central moment of 't' can be expressed as:

$$\mu_r'(t|\gamma,\varphi,\varsigma)=\sum_{j=0}^{\infty}\frac{1}{j!\lambda(e^{\varsigma}-1)}\left[\frac{e^{r\varsigma(1-z)}}{\varsigma}-\lambda\varsigma^{\frac{1}{\gamma}-1}e^{r\varsigma}\Gamma\left(\frac{r\gamma-1}{\gamma},\varsigma z\right)\right] \tag{25}$$

Proof. By considering the following integral

$$\mu_r'(t|\gamma,\varphi,\varsigma)=\int\limits_0^\infty t^r\frac{\varsigma\left[\frac{\gamma\varphi}{\lambda}t^{\varphi-1}\left(1+\frac{t^\varphi}{\lambda}\right)^{-(\gamma+1)}\right]e^{\varsigma\left[1-\left(1+\frac{t^\varphi}{\lambda}\right)^{-\gamma}\right]}}{e^\varsigma-1}dt$$

After using some transformation as $z=\left(1+\frac{t^\varphi}{\lambda}\right)^{-\gamma}$ then, $dz=-\frac{\varphi\gamma t^{\varphi-1}}{\lambda}\left(1+\frac{t^\varphi}{\lambda}\right)^{-\gamma-1}dt$. □

We get the following outcome after some algebraic manipulation

$$\mu_r'(t|\gamma,\varphi,\varsigma)=\sum_{j=0}^\infty\frac{1}{j!\lambda(e^\varsigma-1)}\int_0^\infty\left(\lambda z^{-\frac{1}{\gamma}}-1\right)^{\frac{\gamma}{\varphi}}e^{\varsigma(1-z)}dz$$

The final outcomes of mean and variance of 't' can be got after integrating the above expression

$$E(t|\gamma,\varphi,\varsigma)=\sum_{j=0}^\infty\frac{1}{j!\lambda(e^\varsigma-1)}\left[\frac{e^{r\varsigma(1-z)}}{\varsigma}-\lambda\varsigma^{\frac{1}{\gamma}-1}e^{r\varsigma}\Gamma\left(\frac{r\gamma-1}{\gamma},\varsigma z\right)\right] \tag{26}$$

and

$$\begin{aligned}\operatorname{var}(t|\gamma,\varphi,\varsigma)=&\sum_{j=0}^\infty\frac{\varsigma^{\frac{1}{\gamma}-3}}{j!(e^\varsigma-1)}\left[\Gamma\left(\frac{3r\gamma-1}{\gamma},\varsigma z\right)-\frac{z(\varsigma z+2)e^{r\varsigma(1-z)}}{\varsigma^2}-\frac{2e^{r\varsigma(1-z)}}{\varsigma^3}\right]\\&-\left[\sum_{j=0}^\infty\frac{1}{j!\lambda(e^\varsigma-1)}\left\{\frac{e^{r\varsigma(1-z)}}{\varsigma}-\lambda\varsigma^{\frac{1}{\gamma}-1}e^{r\varsigma}\Gamma\left(\frac{r\gamma-1}{\gamma},\varsigma z\right)\right\}\right]^2\end{aligned} \tag{27}$$

2.9 *Estimation*

The model under study can be estimated by using maximum likelihood method as.

2.9.1 *Maximum likelihood estimation*

$$L(t|\gamma,\varphi,\varsigma)=\frac{\varsigma^n\left[\gamma^n\varphi^n\sum_{i=1}^n t_i{}^{\varphi-1}\left(1+\sum_{i=1}^n t_i{}^\varphi/\lambda\right)^{-\gamma-1}\right]e^{\varsigma\left[n-\left(1+\sum_{i=1}^n t_i{}^\varphi/\lambda\right)^{-\gamma}\right]}}{\lambda^n(e^\varsigma-1)^n} \tag{28}$$

After taking log likelihood, we get

$$\log L(t|\gamma,\, \varphi,\, \varsigma) = n\log\varsigma + n\log\gamma + n\log\varphi - n\log\lambda + n\log(e^{\varsigma}-1) + (\varphi-1)\log\sum_{i=1}^{n} t_i$$

$$-(\gamma+1)\log\left(1+\sum_{i=1}^{n} t_i^{\varphi}/\lambda\right) + \varsigma\left(n-\left(1+\sum_{i=1}^{n} t_i^{\varphi}/\lambda\right)^{-\gamma}\right) \tag{29}$$

$$\frac{\partial \log L(t|\gamma,\, \varphi,\, \varsigma)}{\partial \varsigma} = \frac{n}{\varsigma} + \left(n-\left(1+\sum_{i=1}^{n} t_i^{\varphi}/\lambda\right)\right) - \frac{e^{\varsigma}}{(e^{\varsigma}-1)} \tag{30}$$

$$\frac{\partial \log L(t|\gamma,\, \varphi,\, \varsigma)}{\partial \gamma} = \frac{n}{\gamma} - \log\left(1+\sum_{i=1}^{n} t_i^{\varphi}/\lambda\right) - \varsigma\frac{\log\left(1+\sum_{i=1}^{n} t_i^{\varphi}/\lambda\right)}{\left(1+\sum_{i=1}^{n} t_i^{\varphi}/\lambda\right)} \tag{31}$$

$$\frac{\partial \log L(t|\gamma,\, \varphi,\, \varsigma)}{\partial \lambda} = -\frac{n}{\lambda} + \frac{(\gamma+1)\left(-\sum_{i=1}^{n} t_i^{\varphi}/\lambda^2\right)}{\left(1+\sum_{i=1}^{n} t_i^{\varphi}/\lambda\right)} + \frac{\gamma\varsigma\left(\sum_{i=1}^{n} t_i^{\varphi}/\lambda^2\right)}{\left(1+\sum_{i=1}^{n} t_i^{\varphi}/\lambda\right)^{\gamma+1}} \tag{32}$$

All of the above equations be in implicit form, so we solve this numerically.

3. Real data application

In order to illustrate the efficiency of proposed model, we used different data sets, which makes it valuable in a range of domains markedly those dealing with lifetime analysis. therefore, this section demonstrates how the proposed models work by applying the subject model to the COVID-19 pandemic lifespan data of countries including, China, Europe, and Italy. This aspect is demonstrated here by comparing suggested model with 5 different models namely, Exponentiated Fréchet Distribution, Gumbel Type-II, Exponentiated Gumbel Type-II, Inverse Weibull Distribution, and traditional Exponentiated BUR XII Distribution. The "best fit" of the proposed model can be demonstrated by choosing the specific measures for comparisons. The standardized goodness of fit measures including Likelihood (LL), Akaike Information Criteria (AIC), Corrected Akaike Information Criteria (CAIC) and Hannan Quinn Information Criteria (HQIC) are implemented on the suggested model and existing mentioned models. The best model might be the one with lower values of those benchmarks. The revealed that the suggested model is better than all existing models in all respects. The value of the considered measures is given in Table 1-5 (the visual illustration of these comparison can be seen in Figure 2).

Data-I: The first data set represents the "no of death" due to second wave of coronavirus known as "Beta Coronavirus" in France. This wave outbreak during the period of 1st July to 15th August 2020. The data is reported in (source), which represent the daily deaths due to this pandemic.

Data-II: The second data set represents the "Total no. of deaths" during second wave of pandemic from the period of 1st July to 15th August 2020 in India.

Data-III: The third data set are taken from (source), which represents the "Total no. of deaths" in Italy during 1st July to 15th August 2020.

Data-IV: This data set are taken from (source), that represents the "Total no. of deaths" in Europe during 1st July to 15th August 2020. The data contain the region of Europe with area (L: 55.3781, −3.436).

Table 1. Descriptive measures for data sets

Datasets	Min	Max	Mean	Median	Q_1	Q_3	Range	SD	Skew	Kutosis
I	0	86	11.13	10	0	16	86	14.15	3.39	16.62
II	379	1,129	714.36	711	553	857	750	182.62	0.12	−0.82
III	2	158	13.42	9	5	13	156	22.67	5.96	38.01
IV	0	57	19.33	17	10	26	57	13.80	1.03	0.65

Table 2. MLEs and goodness of fit for data-I

Models	−2LL	AIC	CAIC	BIC	HQIC
EFD	120.346	133.485	131.564	134.598	133.557
GT-II	118.741	133.575	136.169	135.851	134.374
EGT-II	115.940	132.486	134.001	133.374	132.294
IW	134.217	138.040	138.472	139.146	136.342
EBURXII	117.024	131.277	131.035	132.980	131.996
ETBURXII	116.275	130.289	130.766	131.193	131.864

Table 3. MLEs and goodness of fit for data-II

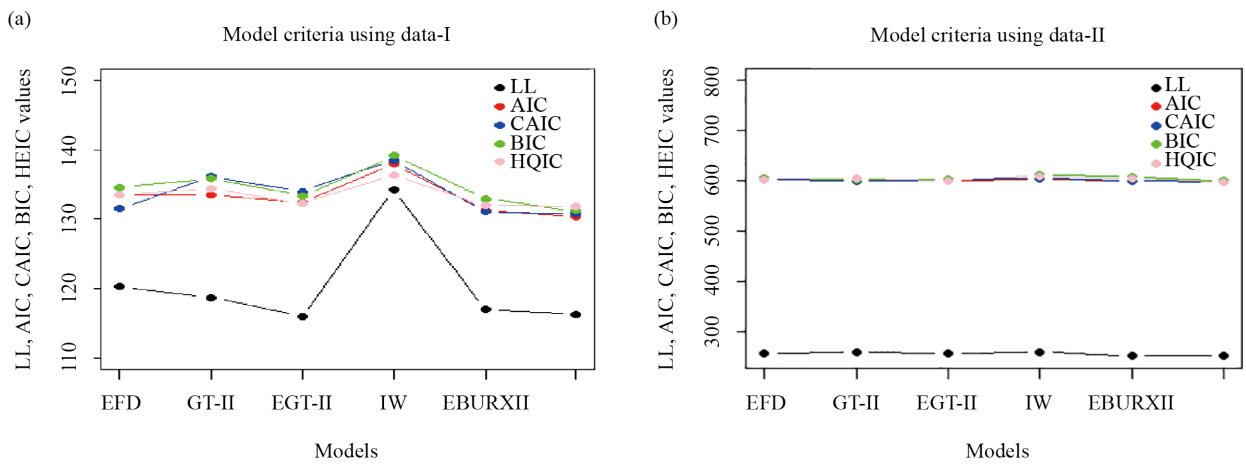
Models	−2LL	AIC	CAIC	BIC	HQIC
EFD	256.327	601.200	601.115	603.872	603.174
GT-II	258.637	600.884	600.285	602.226	604.872
EGT-II	256.372	599.485	600.373	601.374	599.998
IW	259.943	603.237	604.478	611.117	609.375
EBURXII	251.998	599.209	600.184	607.364	604.496
ETBURXII	251.845	598.026	598.180	599.100	597.361

Table 4. MLEs and goodness of fit for data-III

Models	−2LL	AIC	CAIC	BIC	HQIC
EFD	189.003	311.665	311.783	320.045	322.109
GT-II	196.634	312.174	312.524	319.386	321.115
EGT-II	192.561	311.683	312.935	319.109	319.164
IW	195.827	320.539	326.822	321.115	322.829
EBURXII	180.639	310.187	309.684	316.572	319.118
ETBURXII	178.392	308.028	308.115	313.099	311.744

Table 5. MLEs and goodness of fit for data-IV

Models	−2LL	AIC	CAIC	BIC	HQIC
EFD	162.464	302.119	302.358	288.187	300.432
GT-II	166.295	307.372	307.274	282.284	301.108
EGT-II	158.873	300.274	301.473	282.299	298.466
IW	160.564	304.194	303.287	289.283	300.279
EBURXII	157.267	298.153	299.468	279.124	298.004
ETBURXII	156.945	288.276	288.109	276.265	275.475



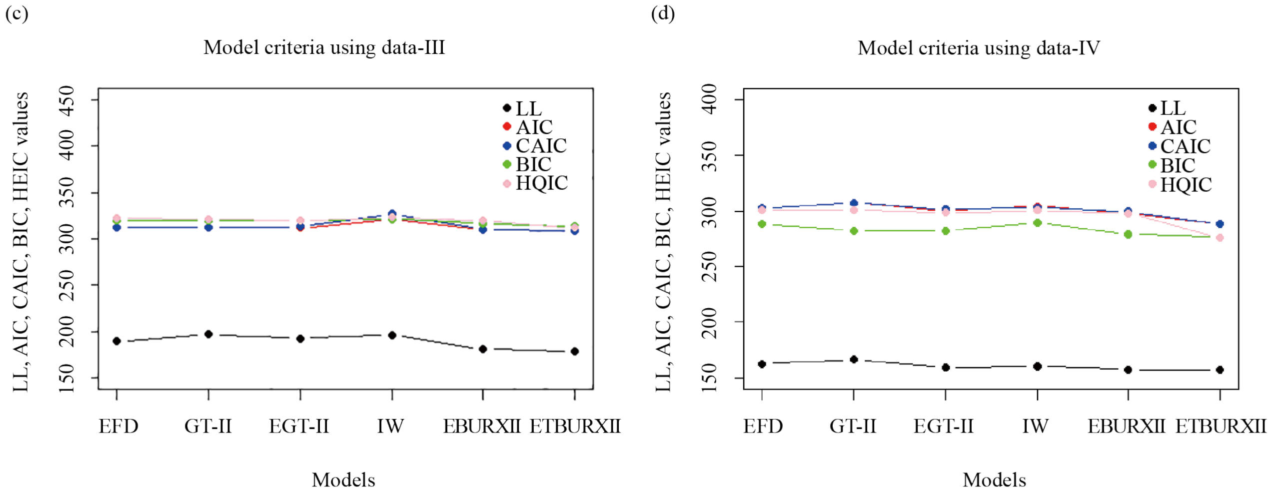


Figure 2. Summary statistics plots for data set I, II, III and IV

4. Conclusions

A novel extension of the Exponentiated Transformation of the Burr XII distribution is presented in this research. The proposed adaptable distribution can accommodate various shapes for different parameter values. It was fitted to real-world data sets to demonstrate the distribution's adaptability and effectiveness in modelling complicated data sets. Regarding goodness-of-fit measurements, the results suggest that the proposed distribution beats other distributions. The proposed distribution's flexibility and versatility make it a valuable tool in various sectors, including finance, engineering, and healthcare. Further work could investigate the proposed distribution's qualities and characteristics and applicability in many fields. Overall, the proposed distribution can contribute to the existing probability distribution literature.

Conflict of interest

The authors declare no competing financial interest.

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