Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

A Semi-Analytical Solution to a Generalized Nonlinear Van Der Pol Equation in Plasma By MsDTM

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Received: 7 September 2024; **Revised:** 16 September 2024; **Accepted:** 19 September 2024

Abstract: This article examines the effectiveness of the multistage differential transform method (MsDTM) in solving equations with a very strong nonlinear term. It introduces MsDTM as a method for solving the generalized nonlinear Van der Pol equation, which features strong nonlinearity. The generalized nonlinear Van der Pol equation arises in plasma and describes the propagation of various nonlinear phenomena, such as wave propagation in astrophysical plasma. MsDTM demonstrates greater accuracy compared to other analytical and numerical methods, such as the 4th-order Runge-Kutta Method (4thRKM), due to its ability to enhance accuracy through two factors: the number of iterations and the time step size. Most numerical methods rely solely on reducing the time step size to improve accuracy, but for some types of equations, this requires an impractically small time step size, causing the method to fail. In contrast, MsDTM offers an additional means of improving accuracy by increasing the number of iterations. The paper successfully applies MsDTM to solve the Van der Pol equation and presents the results, demonstrating that the method is highly effective for equations with very strong nonlinearity.

***Keywords*:** van der pol equation, plasma, multistage differential transform method, numerical simulation

MSC: 34A34,34C15,35Q60

1. Introduction

Computation in applied mathematics is a crucial topic for finding solutions in various forms, such as analytical, semi-analytical, exact, numerical, and equivalent. The literature presents a variety of methods, including Fejér-quadrature collocation algorithm [1], Jacobi rational operational method [2], A potent collocation approach [3], Petrov-Galerkin Lucas polynomials procedure [4], Galerkin algorithm [5], Legendre-Galerkin spectral method [6], modified shifted Chebyshev polynomials [7–10], Alleviated shifted Gegenbauer spectral method [11], polynomial coefficients using a Bernstein polynomial basis [12], among others. This article is devoted to studying the multistage differential transform method (MsDTM) which is a numerical method. The MsDTM has been used to solve systems of equations and fractional equations. The main objective of this paper is to examine the effectiveness of MsDTM in solving equations with very strong nonlinear term. To achieve this, we will select an application from field of plasma physics to support our investigation.

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Plasma is one of the states of matter and has several significant applications, including wave propagation in radio, optics, solar energy, and other important industrial, space, and physics applications [13].

As an application, this article studies the generalized nonlinear Van der Pol equation with strong nonlinearity using the highly accurate MsDTM. The generalized nonlinear Van der Pol equation models a range of real-world phenomena where self-sustained oscillations and nonlinear damping occur. It is particularly relevant for describing nonlinear wave propagation in plasma, both in astrophysical and laboratory environments. Examples of plasma phenomena modeled by the Van der Pol equation include nonlinear oscillations in plasma waves, wave propagation in astrophysical plasma, plasma instabilities and turbulence, and radio frequency oscillations in plasma devices.

The equation was introduced by Balthazar Van der Pol, a pioneer of radio telecommunication, in 1927 through experimental work in the laboratory to describe oscillations in electric circuits. In the mid-20th century, the Van der Pol equation was mathematically formalized [14, 15]. Between 1945 and 1949, Cartwright and Littlewood [16] and Levinson [17] confirmed the existence of singular solutions for the Van der Pol equation. The general nonlinear Van der Pol equation is as follows:

$$U_{tt}-\xi(1-U^2-\alpha U^4-\beta U^6)U_t+U=0, \tag{1}$$

where the function $U(t)$ represents the electric and magnetic fields, ξ is a perturbation term indicating the nonlinearity and the strength of the damping on the oscillations ($\xi \ll 1$), and α and β are non-negative parameters that are determined the amplitude of the wave. If $\alpha=\beta=0$, the equation reduces to

$$U_{tt}-\xi(1-U^2)U_t+U=0. \tag{2}$$

Assume the equation (1) is subject to initial conditions; $U(0)=A$ and $U_t(0)=B$. Let's define $\Psi=(\xi,\ \alpha,\ \beta,\ A,\ B)$.

The equation (2) was solved by Nayfeh [18] for $A=1,\ B=0$ yielding a periodic solution. Additionally, the same equation was solved numerically using He's homotopy perturbation method [19]. The general form of equation (1) was solved analytically using the ansatz method and the Krylov-Bogoliubov-Mitropolsky method [20]. However, in this work, we aim to demonstrate higher accuracy than previous methods. In this manuscript, we will employ the MsDTM.

The differential transform method (DTM) for one dimension is a numerical method first introduced by Puchov in 1979 [21]. The solution of the target equation is assumed to be in the form of a Taylor expansion, such that $U(t)=\sum_{k=0}^{N}u(k)t^k$, where the initial solution corresponds to $k=0$ and N is the iteration number. The main equation is transformed into a new scheme by applying differential transformation, allowing us to compute $u(k+1)$ from $u(k)$. However, the DTM can only find solutions in small domains due to its local convergence limitations [22]. The DTM was improved by using a multistage technique, where the domain is divided into subdomains, and DTM is applied within each subdomain [23]. MsDTM has proven powerful in solving several problems [24, 25]. In this paper, we aim to demonstrate the capability of MsDTM in solving problems with strong nonlinearity. The significance of using MsDTM is that its accuracy can be enhanced through two factors: the number of iterations and the time step size.

The paper is organized as follows: the next section describes the MsDTM, the third section discusses numerical solutions through various examples, the fourth section examines the accuracy of MsDTM, and the final section summarizes our analysis and results.

2. Algorithm description

The Differential Transform Method (DTM) involves transforming the given equation into an iterative scheme based on the principles of differential transformation. The solution is represented as an infinite series around the initial point. In certain cases, if a closed-form solution can be derived, it results in the exact solution. However, this is rarely achievable for most real-world problems. As a result, the solution is approximated and represented as a finite series. Due to the local convergence of the finite series, the solution is only valid within a small domain. To extend the domain and improve the accuracy of the results, the multistage approach is employed.

The following are key definitions necessary to explain the MsDTM.

2.1 *Preliminaries and definitions*

2.1.1 *The differential transform method*

Assume $U(t)$ is a function of variable t that is continuously differentiable, and consider the Taylor series for $U(t)$ about the point t_0 as follows:

$$U(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \frac{d^k U(t_0)}{dt^k}. \tag{3}$$

The N^{th} Taylor polynomial is defined as

$$\sum_{k=0}^{N} \frac{(t-t_0)^k}{k!} \frac{d^k U(t_0)}{dt^k}. \tag{4}$$

Then,

$$U(t) = \sum_{k=0}^{N} (t-t_0)^k u(k) + R_N, \tag{5}$$

where R_N is the reminder terms and $u(k)$ is the differential transformation defined as:

$$u(k) = \frac{1}{k!} \left[\frac{d^k U(t)}{dt^k} \right]_{t=t_0}, \tag{6}$$

for $k = 0,\ 1,\ 2,\ \ldots \infty$. The point t_0 is the initial point of the the domain T. The list of differential transformation for various functions is provided in Table 1.

Table 1. The one-dimensional differential transformation for basic operations [26]

Original function	Transformed function
$U(t) \pm V(t)$	$u(k) \pm v(k)$
$\frac{d^m U(t)}{dt^m}$	$\frac{(k+m)!}{K!} u(k+m)$
$U(t)V(t)$	$\sum_{l=0}^{k} u(l)v(k-l)$
$U(t)^2$	$\sum_{l=0}^{k} u(l)u(k-l)$
$U(t)^3$	$\sum_{l=0}^{k} (\sum_{m=0}^{l} u(m)u(l-m))u(k-l)$
$U(t)^4$	$\sum_{l=0}^{k} (\sum_{m=0}^{l} (\sum_{d=0}^{m} u(d)u(d-m))u(l-m))u(k-l)$
$U(t)^3 V$	$\sum_{l=0}^{k} (\sum_{m=0}^{l} (\sum_{d=0}^{m} u(d)u(d-m))u(l-m))v(k-l)$
$exp(\lambda t)$	$\frac{\lambda^k}{k!}$

2.1.2 *The multistage differential transform method*

Consider the ordinary differential equation $L(U(t),\ U_t(t),\ U(_{tt}),\ t)$ where $t \in [t_0,\ t_N]$. Let $\Delta_t = (t_N - t_0)/M$, where M is integer and Δ_t is the time step size. Consequently, we obtain subdomains $[t_{m_0},\ t_{m_N}]$, where $m = 1,\ 2,\ 3,\ ....,\ M$. For each subdomain, we apply the DTM with the initial condition at t_{m_0}. The DTM will be applied M times, and the number of iterations is K.

2.2 *The semi-analytical scheme*

For the simplicity, the eq.(1) is transferred to the system of two functions U and V;

$$U_t = V, \tag{7}$$

$$V_t = \xi(1 - U^2 - \alpha U^4 - \beta U^6)V - U. \tag{8}$$

Applying the MsDTM by dividing the domain into M subdomains and we obtain the following scheme

$$U_{m_{k+1}} = \frac{k!}{(k+1)!} V_{m_k}, \tag{9}$$

$$V_{m_{k+1}} = \frac{k!}{(k+1)!} \left(\xi V_{m_k} - \xi \Omega_1 - \alpha\xi\Omega_2 - \beta\xi\Omega_3 - U_{m_k}\right), \tag{10}$$

where

$$\Omega_1 = \sum_{r=0}^{k} V_{m_{k-r}} \left(\sum_{l=0}^{r} U_{m_1} l U_{m_{r-l}} \right),$$

$$\Omega_2 = \sum_{r=0}^{k} V_{m_{k-r}} \left(\sum_{l=0}^{r} U_{m_{r-l}} \left(\sum_{f=0}^{l} U_{m_{l-f}} \left(\sum_{s=0}^{f} U_{m_s} U_{m_{f-s}} \right) \right) \right),$$

$$\Omega_3 = \sum_{r=0}^{k} V_{m_{k-r}} \left(\sum_{l=0}^{r} U_{m_{r-l}} \left(\sum_{f=0}^{l} U_{m_{l-f}} \left(\sum_{s=0}^{f} U_{m_{f-s}} \left(\sum_{w=0}^{s} U_{m_{s-w}} \left(\sum_{d=0}^{w} U_{m_d} U_{m_{w-d}} \right) \right) \right) \right) \right),$$

where $m = 1,\ 2,\ 3,\ .....\ M$.

3. Numerical results

In this section, we will find the solution of the generalized nonlinear Van der Pol equation for different cases. By helping the Mathematica software, we found the numerical solution.

3.1 *Example 1*

Assume the eq.(2) with $A = 1,\ B = 0,\ \xi = 0.1$ and $T = [t_0 = 0,\ t_N = 60]$. First step, the domain $[0,\ 60]$ is divided using the step size $\Delta_t = 60/1,000 = 6\times 10^{-2}$ to obtain subdomains $[t_{m_0},\ t_{m_N}]$ where $t_{m_N} = t_{m+\Delta_t}$, and $m = 1,\ 2,\ ....,\ 1,000$. For first subdomain $m = 1$, $T_1 = [0,\ 0.06]$, we assume $U_{1_0} = A = 1,\ V_{1_0} = B = 0$ and $k = 3$. Then, the DTM is applied in the domain $[0,\ 0.06]$ as following:

$$U_{1_{k+1}} = \frac{k!}{(k+1)!} V_{1_k}, \tag{11}$$

$$V_{1_{k+1}} = \frac{k!}{(k+1)!}\left(\xi V_{1_k} - \xi \sum_{r=0}^{k} V_{1_{k-r}}\left(\sum_{l=0}^{r} U_{1_l} U_{1_{r-l}}\right) - U_{1_k}\right), \tag{12}$$

we obtain $U_{1_0} = 1,\ U_{1_1} = 0,\ U_{1_2} = -0.5,\ U_{1_3} = 0$. Thus, the following Taylor expansion about the point $t_0 = 0$ converges in domain T_1

$$U_1 = \sum_{k=0}^{3} U_{1_k}(t^k - t_{1_0}) = 1. - 0.5t^2,\ \text{ where } t \in [0,\ 0.06].$$

For $m = 2$, we have the subdomain $T_2 = [0.06,\ 0.12]$ and the initial condition $U_{2_0} = U_1(0.06) = 0.9982$. Following the previous processes for $t \in [0.06,\ 0.12]$ implies:

$$U_2 = 0.00981452\left(t - \frac{3}{50}\right)^3 - 0.499111\left(t - \frac{3}{50}\right)^2 - 0.059964\left(t - \frac{3}{50}\right) + 0.9982.$$

For $t \in [0.12,\ 0.18]$, we have:

$$U_3 = 0.0192413\left(t - \frac{3}{25}\right)^3 - 0.49649\left(t - \frac{3}{25}\right)^2 - 0.119717\left(t - \frac{3}{25}\right) + 0.992807.$$

Then, we continuo applying DTM at the rest of subdomains until we reach the solution for last subdomain $t \in [59.94,\ 60]$:

$$U_{1,000}(t) = -0.15103\left(t - \frac{2,997}{50}\right)^3 + 0.877946\left(t - \frac{2,997}{50}\right)^2 + 0.567098\left(t - \frac{2,997}{50}\right) - 1.90498.$$

Each subdomain has won equation which is used to plot the solution for each subdomain. The solution is plotted in Figure 1 for $\xi = 0.1$ and in Figure 2 for 0.01.

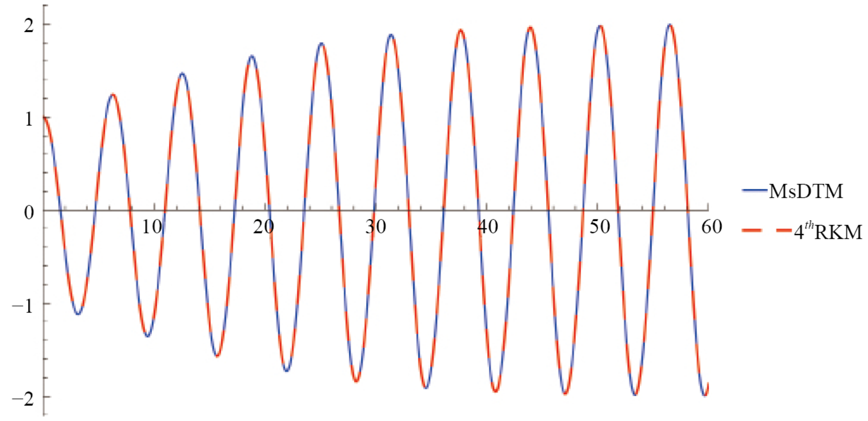

Figure 1. The semi-analytical solution of eq. (1) via MsDTM with $\Psi = (0.1,\ 0,\ 0,\ 1,\ 0)$, Error $= 4.141 \times 10^{-7}$ and the comparison with numerical solution via 4^{th}RKM

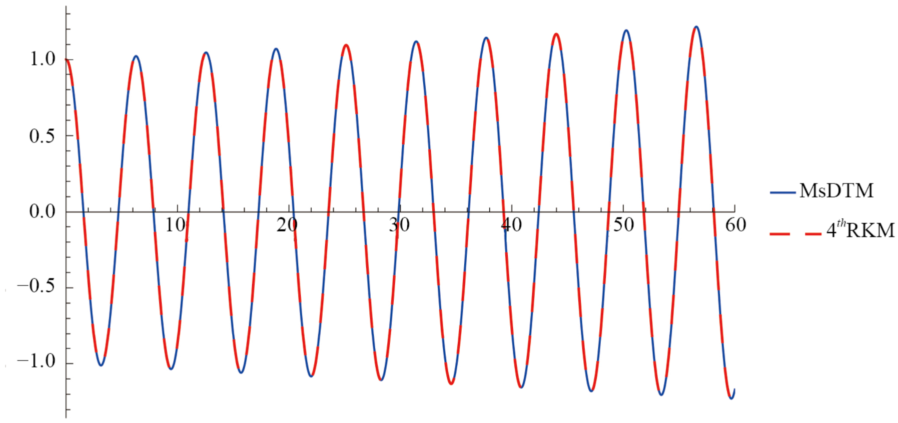

Figure 2. The semi-analytica solution of eq. (1) via MsDTM with $\Psi = (0.01,\ 0,\ 0.1,\ 0)$, Error $= 1.0488 \times 10^{-6}$ and the comparison with numerical solution via 4^{th}RKM

3.2 *Example 2*

In this example, we apply MsDTM on eq. (1) where $\Psi = (0.01,\ 1,\ 1,\ 0.3,\ 0)$. The solutions in Figure 3 as follows

$$U_1 = -0.000450586t^3 - 0.15t^2 + 0.3,$$

$$U_2 = 0.00154918\left(t - \frac{1}{25}\right)^3 - 0.149934\left(t - \frac{1}{25}\right)^2 - 0.011999\left(t - \frac{1}{25}\right) + 0.29976,$$

$$U_3 = 0.00354667\left(t - \frac{2}{25}\right)^3 - 0.149628\left(t - \frac{2}{25}\right)^2 - 0.0239831\left(t - \frac{2}{25}\right) + 0.29904,$$

$$\vdots$$

$$U_{1,000} = 0.0472654\left(t-\frac{999}{25}\right)^3 + 0.115212\left(t-\frac{999}{25}\right)^2 - 0.281009\left(t-\frac{999}{25}\right) - 0.233073.$$

For $\Psi = (0.1,\ 1,\ 1,\ 0.3,\ )$, the solutions in Figure 2 as follows

$$U_1 = -0.00450586t^3 - 0.15t^2 + 0.3,$$

$$U_2 = -0.00251805\left(t-\frac{1}{25}\right)^3 - 0.150421\left(t-\frac{1}{25}\right)^2 - 0.0120185\left(t-\frac{1}{25}\right) + 0.29976,$$

$$U_3 = -0.000524244\left(t-\frac{2}{25}\right)^3 - 0.150604\left(t-\frac{2}{25}\right)^2 - 0.0240611\left(t-\frac{2}{25}\right) + 0.2990384,$$

$$\vdots$$

$$U_{1,000} = 0.24304\left(t-\frac{999}{25}\right)^3 + 0.421201\left(t-\frac{999}{25}\right)^2 - 0.95731\left(t-\frac{999}{25}\right) - 0.808123.$$

For $\Psi = (0.1,\ 1.2,\ 1,\ 0.3,\ 0)$, the solutions in Figure 4 as follows

$$U_1 = -0.00449775t^3 - 0.15t^2 + 0.3,$$

$$U_2 = -0.00251805\left(t-\frac{1}{25}\right)^3 - 0.150421\left(t-\frac{1}{25}\right)^2 - 0.0120185\left(t-\frac{1}{25}\right) + 0.29976,$$

$$U_3 = -0.00051639\left(t-\frac{2}{25}\right)^3 - 0.150602\left(t-\frac{2}{25}\right)^2 - 0.0240609\left(t-\frac{2}{25}\right) + 0.299038,$$

$$\vdots$$

$$U_{1,000} = 0.240655\left(t-\frac{999}{25}\right)^3 + 0.416454\left(t-\frac{999}{25}\right)^2 - 0.944113\left(t-\frac{999}{25}\right) - 0.79728.$$

For $\Psi = (0.1,\ 1,\ 1.5,\ 0.3,\ 0)$, the solutions in Figure 5 as follows

$$U_1 = -0.00450403t^3 - 0.15t^2 + 0.3,$$

$$U_2 = -0.00251624\left(t-\frac{1}{25}\right)^3 - 0.150421\left(t-\frac{1}{25}\right)^2 - 0.0120184\left(t-\frac{1}{25}\right) + 0.29976,$$

$$U_3 = -0.000522511\left(t-\frac{2}{25}\right)^3 - 0.150604\left(t-\frac{2}{25}\right)^2 - 0.024061\left(t-\frac{2}{25}\right) + 0.299038,$$

$$\vdots$$

$$U_{1,000} = 0.242112\left(t-\frac{999}{25}\right)^3 + 0.411099\left(t-\frac{999}{25}\right)^2 - 0.9349\left(t-\frac{999}{25}\right) - 0.788009.$$

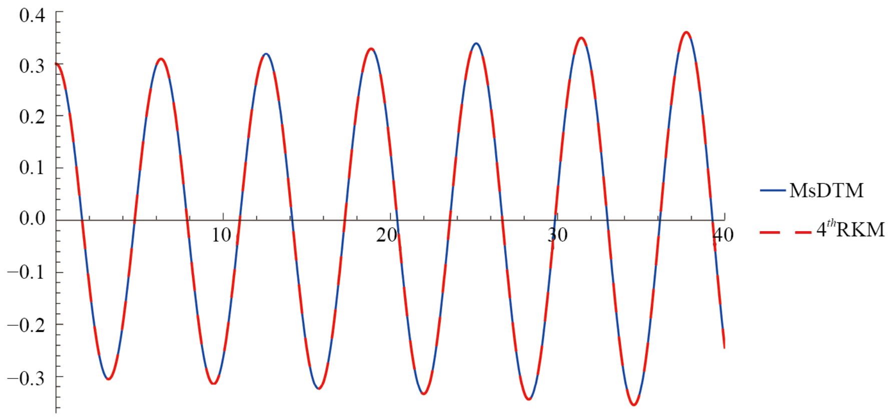

Figure 3. The semi-analytica solution of eq. (1) via MsDTM with $\Psi = (0.01,\ 1,\ 1,\ 0.3,\ 0)$, Error $= 6.64037 \times 10^{-7}$ and the comparison with numerical solution via 4^{th}RKM

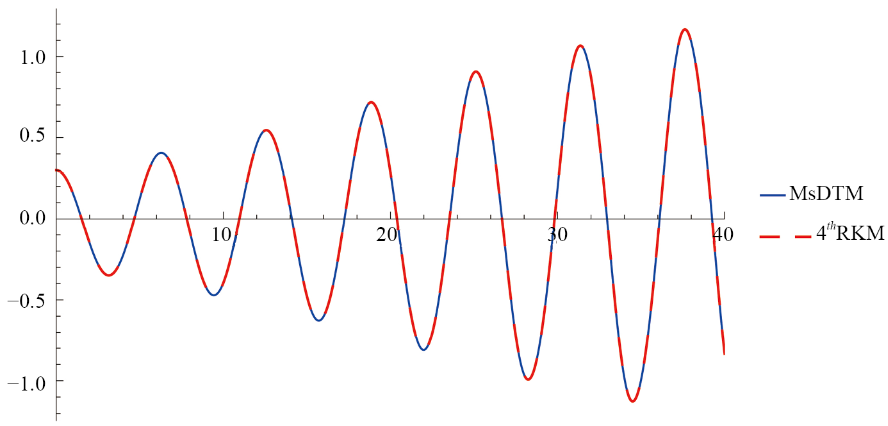

Figure 4. The semi-analytica solution of eq. (1) via MsDTM with $\Psi = (0.1,\ 1,\ 1,\ 0.3,\ 0)$, Error $= 8.24691 \times 10^{-7}$ and the comparison with numerical solution via 4^{th}RKM

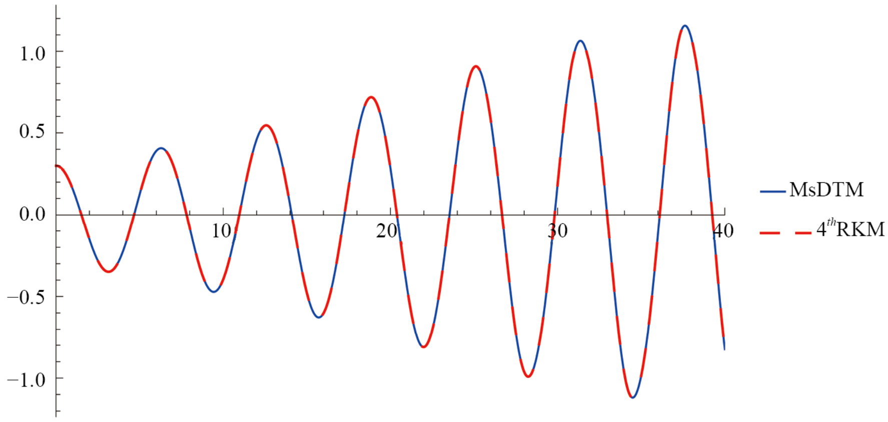

Figure 5. The semi-analytica solution of eq. (1) via MsDTM with $\Psi = (0.1,\ 1.2,\ 1,\ 0.3,\ 0)$, Error $= 1.62967 \times 10^{-6}$ and the comparison with numerical solution via 4^{th}RKM

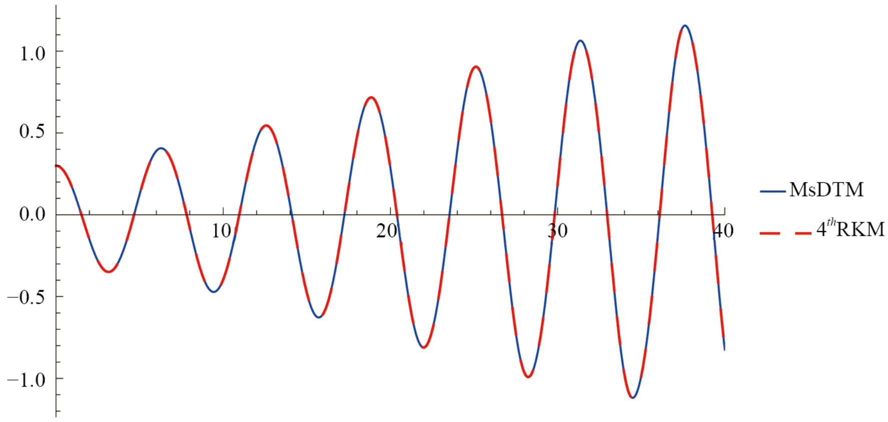

Figure 6. The semi-analytica solution of eq. (1) via MsDTM with $\Psi = (0.1,\ 1,\ 1.5,\ 0.3,\ 0)$, Error $= 5.89819 \times 10^{-7}$ and the comparison with numerical solution via 4^{th}RKM

The numerical results show how oscillations grow, stabilize, and sustain themselves, often leading to complex dynamical behaviors like limit cycles.

4. Discussion of the accuracy

To assess the accuracy of MsDTM, we define the error as the distance between the solution obtained via MsDTM and the numerical solution obtained using the 4^{th}RKM, as given by

$$Error = \max_{x_0 < t < t_N} |4^{th}\text{RKM} - \text{MsDTM}|. \tag{13}$$

Actually, the example (1) has been solved by Nayfe using the Perturbation method (PM) [18], yielding the solution:

$$U(x) = \xi\left(-\frac{9\sin(x)}{32} - \frac{1}{32}\sin(3x) + \frac{3}{8}x\cos(x)\right) + \cos(x),\ + \ldots\ldots,$$

and also by He's homotopy perturbation method (HHPM) [19], with the solution:

$$U(x) = \left(\frac{3a^2}{4\left(9a^2+\xi^2\right)} + \frac{a^2-4}{4\left(a^2+\xi^2\right)}\right)\exp(\xi x) + \frac{\left(4\xi - a^2\xi\right)\sin(ax)}{4\left(a^3+a\xi^2\right)} -$$

$$\frac{(3a\xi)\sin(3ax)}{12\left(9a^2+\xi^2\right)} + \frac{\left(-4a^2-3\xi^2+4\right)\cos(ax)}{4\left(a^2+\xi^2\right)} + \frac{\xi^2\cos(3ax)}{12\left(9a^2+\xi^2\right)} + \cos(ax) + 2,$$

where $a \approx \dfrac{3\pi}{2\xi}$.

The Table 2 shows that the results obtained using MsDTM are more accurate compared to those obtained with PM and HHPM. While PM can yield more accurate results for very small perturbation parameter ξ, MsDTM provides highly accurate results even with large perturbation parameters.

Table 2. The error of using PM, HHPM and MsDTM with $K = 4$ and $\Delta_t = 0.06$

ξ	Error by PM	Error by HHPM	Error by MsDTM
0.01	3.75788×10^{-3}	-1.33061	6.76611×10^{-6}
0.1	1.2528	-1.33333	1.20984×10^{-5}

The definition (13) was also used in [20] to study the error of the analytical solutions for the equation (1) (example (2)) using the ansatz method and the Krylov-Bogoliubov-Mitropolsky (KBM) method. Table 3 compares the errors associated with MsDTM, the ansatz method and the KBM method. MsDTM achieves higher accuracy than both analytical methods.

The accuracy of MsDTM depends on two factors; the time step Δ_t and the number of iterations K. Table 2 demonstrates that the accuracy of MsDTM improves either by decreasing Δ_t or increasing K. This is a signficant advantages of the MsDTM compared to other numerical methods such as Euler method, Runge Kutta method, or finite difference method, which relay solely on the time step Δ_t. As illustrated in the figure, our numerical solution closely matches the results obtained by the 4th RK method, but the MsDTM scheme requires less time to achieve a solution compared to the 4th RK method.

Table 3. The error of using PM, HHPM and MsDTM with $K = 4$ and $\Delta_t = 0.06$

$(\xi,\ \alpha,\ \beta,\ A,\ B)$	Ansatz method	KBM method	MsDTM
(0.01, 1, 1, 0.3, 0)	2.09×10^{-2}	2.149×10^{-6}	6.64037×10^{-7}
(0.1, 1, 1, 0.3, 0)	4.99×10^{-2}	8.042×10^{-4}	8.24691×10^{-7}
(0.1, 1.2, 1, 0.3, 0)	5.96×10^{-2}	7.377×10^{-4}	1.62967×10^{-6}
(0.1, 1, 1.5, 0.3, 0)	6.973×10^{-2}	7.685×10^{-4}	5.89819×10^{-7}

Table 4. The accuracy improvement by changing iteration number K comparing to the change of time step size Δ_t

K with $\Delta_t = 1000$	Error	Δ_t with $K = 4$	Error
2	1.2593×10^{-2}	100	8.80352×10^{-3}
3	5.05061×10^{-5}	500	1.5794×10^{-5}
4	8.24691×10^{-7}	$1,000$	8.24691×10^{-7}
5	8.39729×10^{-7}	$10,000$	8.37739×10^{-7}

5. Conclusions

MsDTM is a versatile tool that can be applied to a wide range of equations in the sciences. The accuracy of MsDTM can be controlled by increasing the number of iterations or decreasing the time step size. Many equations can be readily adapted to the MsDTM scheme. However, a limitation of the method arises when the forcing term in the equation is a complex function for which it is difficult to apply differential transformation. This can reduce the efficiency of the MsDTM in handling certain types of problems. In future work, MsDTM holds the potential to provide accurate solutions to numerous problems in plasma physics, as well as other scientific fields.

Acknowledgement

This Project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (GPIP: 301-665-2024). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

Conflict of interest

The authors declare no competing financial interest.

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