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Research Article

Unveiling of Highly Dispersive Dual-Solitons and Modulation Instability Analysis for Dual-Mode Extension of a Non-Linear Schrödinger Equation

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Abstract: The two-mode equations are nonlinear models that describe the behavior of two-way waves moving simultaneously while being affected by confined phase velocity. This article expands a non-linear Schrödinger equation (NLSE) by constructing it as a dual-mode structure. Applying the modified extended direct algebraic method (MEDAM) yields exact and explicit solutions. The results of this investigation have significant implications for the propagation of solitons in nonlinear optics. There are multiple resulted solutions that comprise singular periodic solutions, Weierstrass elliptic doubly periodic solutions, Jacobi elliptic function (JEF), singular soliton, bright soliton, dark soliton, and rational solutions, moreover, hyperbolic wave solutions. We show our acquired traveling wave solutions' uniqueness and significant addition to current research by contrasting them with the body of existing literature. The method's effectiveness shows that it may be used to address a wide variety of nonlinear problems across multiple disciplines, particularly in the theory of soliton, as the studied model appears in many applications. Additionally, we display the outlines of some of these discovered solution behaviors in 3D and 2D graphs to help with comprehension. Finally, we analyze modulation instability to examine the stability of the discovered solutions.

***Keywords*:** dual waves, nonlinear schrödinger equation, modulation instability, modified extended direct algebraic method

MSC: 35C07, 35C08, 35C09

1. Introduction

Different studies of nonlinear partial differential equations (NLPDEs) have rapidly become pronounced as scientific research advances. Many scientific fields, including physics, engineering, earth sciences, and numerous other technological fields, have employed NLPDEs. These NLPDEs have been the subject of much research for a long time, especially when it comes to accurate and numerical solutions. Among the exact solutions for NLPDEs, numerous solutions for solitary waves have been found, especially in many fields related to physics like plasma theory, nonlinear optics, and fluid mechanics [1–9]. These studies discuss many aspects of optical solitons, including theoretical and experimental

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models, dynamics observations, and nonlinear effects. Bright and solitary solitons in a (2 + 1)-dimensional nonlinear Schrödinger equation with spatio-temporal dispersions were studied by the authors in [10]. The authors focused on the significance of ions in optical systems with dispersive effects by studying their formation, properties, and behavior. Solitons solutions derived from the nonlinear Schrödinger equation were described in reference [11]. The study helped to understand the features and behavior of novel kinds of solitons in this system by identifying them. The authors investigated the Manakov system with asymmetrical and self-similar optical structures in [12]. The study focused on breather stability and interactions, which are essential for nonlinear optical systems and optical communication.

Recently, a category of nonlinear formulas of equations referred to as the two-mode or sometimes named the dual-mode the class has been presented. The NLPDEs family has been recently identified to include second-order NLPDEs in the temporal context [13–17]. Two different nonlinear wave modes propagating are governed by these equations simultaneously. Various kinds of solutions were proposed in some novels of real dual-mode models [18–22]. In some of the previously stated studies, the researchers were able to derive soliton solutions to dual-mode equations under some constraints. Some authors found analytical solutions for nonlinear derivative and quantic Schrödinger's equations by using extended mapping method, the extended and modified direct algebraic methods, and some other methods [23–28]. Furthermore, Alquran [29] investigated how phase velocity affected Schrodinger's equation of dual-mode wave solutions that included various nonlinearities. The motivation behind this work is to explore two-mode waves for the third-order NLSE and study their interactions by controlling some parameters. Various soliton solutions and other wave solutions are established for the suggested model when the modified extended direct algebraic method (MEDAM) is proposed to be used. Additionally, we go over the stability analysis of the solutions that are derived using the modulation instability (MI) analysis idea. Consequently, novel analytical solutions are produced in previously unattainable forms and with more generality. The extracted solutions attest to the existing technique's potency and effectiveness. Also, the nature of the resulting solutions is demonstrated by contour, 3D, and 2D simulations. In this work, the utilized model of the dual-mode model in [23] is represented in its new structure below:

$$i(\Psi_{tt} - s^2\Psi_{xx}) + \left(\frac{\partial}{\partial t} - b\, s\frac{\partial}{\partial x}\right)(\beta_2\Psi_{xx} - i\Psi_{xxx}) + \left(\frac{\partial}{\partial t} - a\, s\frac{\partial}{\partial x}\right)\left(2\beta_2\, \Psi|\Psi|^2 - 6i\,\gamma\,\Psi_x|\Psi|^2\right) = 0, \tag{1}$$

where $\Psi = \Psi(x,\ t)$ represents a function of the complex field in two independent variables x and t that are referring to 2D of space and time coordinates and the imaginary number $i = \sqrt{-1}$. a stands for the dispersive factor, b represents the non-linearity factor while s is the interaction of the phase velocity, under conditions that $|a| \leq \pm 1$, $|b| \leq \pm 1$ and $s \geq 0$. On the other side, the constant coefficients are represented by β_2 and γ. Noting that, sometimes, in water waves and the optical field theories, the variables x and t are exchanged.

The phase velocity (s), dispersion parameter (b), and nonlinearity parameter (a) are important factors that shape the behavior of solutions in the context of the NLSE and its dual-mode extension. The phase velocity s affects the dual waves' propagation speed and direction, which changes how near or far apart the waves move in relation to one another. The waves may converge or further diverge as s rises. The wave stability and its inclination to hold its shape are influenced by the dispersion parameter b, which regulates how the wave packet spreads over time. On the other hand, when appropriately balanced with b, the nonlinearity parameter a controls the strength of wave interactions, resulting in phenomena like soliton creation. There are several practical uses for the dual-mode extension of the NLSE, especially in fluid dynamics and fiber optics. It simulates how optical solitons propagate in birefringent fibers, or fibers with different refractive indices, where two modes-for example, polarizations-interact and affect one another's behavior. Comprehending the maintenance of stability and integrity of data transmissions across extended distances is crucial for telecommunications networks. The dual-mode NLSE in fluid dynamics explains how waves interact in shallow water settings, including how solitons and bidirectional wave patterns arise in tidal flows and tsunami wave simulations. These examples show how the dual-mode NLSE contributes to the prediction and control of wave behaviors in complex media, hence guaranteeing system stability in applications such as precise modeling of oceanic wave dynamics or high-speed data transmission.

This article is composed of the following structure: Section 1 offers a broad introduction and theoretical foundation about the suggested model, while the key points of the suggested approach are presented in Section 2. All of the findings are displayed in Section 3, which also provides an explanation of the solution's many dynamic waveforms. Section 4 deals with the analysis of modulation instability for the extracted solutions. In Section 5, a few derived conclusions are presented graphically in both 2D and 3D formats. Section 6 provides physical interpretations of the obtained solutions. Section 7 presents some conclusion marks at the end.

2. The mathematical framework of the applied method

The basic outlines of the MEDAM, which will be used in Section 3, are presented in this section. By beginning to think about the subsequent NLPDE [30]:

$$\mathscr{F}\left(\Psi,\ \Psi_t,\ \Psi_x,\ \Psi_{xx},\ \Psi_{tt},\ \Psi_{xt},\ \ldots\right)=0, \tag{2}$$

such that $\mathscr{F}$ represents a polynomial in terms of $\Psi(x,\ t)$ with some of the partial derivatives of that $\mathscr{F}$ with respect to time and space.

Step 1 Using the wave transition described below:

$$\Psi(x,\ t)=\mathscr{Q}(\xi)\ e^{i\zeta(x+\Omega t)},\qquad \xi=x-ct, \tag{3}$$

where $\mathscr{Q}$ acts as the amplitude value of the solution. $\Omega,\ c$, and ζ denote a few constants of real values that this work will determine later.

By substituting with Eq. (3) in Eq. (2), a nonlinear ordinary differential equation (NLODE) will have the following construction by rearranging its form as:

$$\mathscr{R}(\mathscr{Q},\ \mathscr{Q}',\ \mathscr{Q}'',\ \mathscr{Q}''',\ \ldots)=0. \tag{4}$$

Step 2 According to the used technique, Eq. (4) produces solutions are following the below form:

$$\mathscr{Q}\left(\xi\right)=\sum_{i=-\mathbb{M}}^{\mathbb{M}}\mathfrak{C}_i\mathscr{H}^i(\xi), \tag{5}$$

where $\mathfrak{C}_i$ $(i=0,\ 1,\ 2,\ ..,\ \mathbb{M})$ are constants of the solutions that their values will be determined through the work mathematical procedures, under the condition that $\mathfrak{C}_{\mathbb{M}}$ and $\mathfrak{C}_{-\mathbb{M}}$ can not be equal to zero, concurrently.

Step 3 In addition, $\mathscr{H}(\xi)$ satisfies the following Eq. (6), according to applying the principle of balance to Eq. (4) that works on its calculations between the highest-order derivative term and the term of the highest non-linear to evaluate the positive integer $\mathbb{M}$:

$$\left(\frac{d\mathscr{H}}{d\xi}\right)^2=\rho_0+\rho_1\mathscr{H}(\xi)+\rho_2\mathscr{H}^2(\xi)+\rho_3\mathscr{H}^3(\xi)+\rho_4\mathscr{H}^4(\xi)+\rho_6\mathscr{H}^6(\xi), \tag{6}$$

where ρ_j is a real constant; $(j = 0,\ 1,\ 2,\ 3,\ 4,\ 6)$. Eq. (6) has the following general solutions:

Case 1 When $\rho_0 = \rho_1 = \rho_3 = \rho_6 = 0$, the following solutions are raised:

$$\mathscr{H}(\xi) = \sqrt{-\frac{\rho_2}{\rho_4}}\operatorname{sech}\left(\sqrt{\rho_2}\xi\right), \quad \rho_2 > 0,\ \rho_4 < 0.$$

$$\mathscr{H}(\xi) = \sqrt{-\frac{\rho_2}{\rho_4}}\sec\left(\sqrt{-\rho_2}\xi\right), \quad \rho_2 < 0,\ \rho_4 > 0.$$

$$\mathscr{H}(\xi) = \sqrt{-\frac{\rho_2}{\rho_4}}\csc\left(\sqrt{-\rho_2}\xi\right), \quad \rho_2 < 0,\ \rho_4 > 0.$$

Case 2 When $\rho_1 = \rho_3 = \rho_6 = 0,\ \rho_0 = \dfrac{\rho_2^2}{4\rho_4}$, the following solutions are raised:

$$\mathscr{H}(\xi) = \sqrt{-\frac{\rho_2}{2\rho_4}}\tanh\left(\sqrt{-\frac{\rho_2}{2}}\xi\right), \quad \rho_2 < 0,\ \rho_4 > 0.$$

$$\mathscr{H}(\xi) = \sqrt{\frac{\rho_2}{2\rho_4}}\tan\left(\sqrt{\frac{\rho_2}{2}}\xi\right), \quad \rho_2 > 0,\ \rho_4 > 0.$$

Case 3 When $\rho_3 = \rho_4 = \rho_6 = 0$, the following solutions are raised:

$$\mathscr{H}(\xi) = \frac{\rho_1 \sinh\left(2\sqrt{\rho_2}\xi\right)}{2\rho_2} - \frac{\rho_1}{2\rho_2}, \quad \rho_2 > 0,\ \rho_0 = 0.$$

$$\mathscr{H}(\xi) = \frac{\rho_1 \sin\left(\sqrt{-\rho_2}\xi\right)}{2\rho_2} - \frac{\rho_1}{2\rho_2}, \quad \rho_2 < 0,\ \rho_0 = 0.$$

$$\mathscr{H}(\xi) = \exp\left(\sqrt{\rho_2}\xi\right) - \frac{\rho_1}{2\rho_2}, \quad \rho_2 > 0,\ \rho_0 = \frac{\rho_1^2}{4\rho_2}.$$

Case 4 When $\rho_0 = \rho_1 = \rho_2 = \rho_6 = 0$, the following solution is raised:

$$\mathscr{H}(\xi) = \frac{4\rho_3}{\rho_3^2\xi^2 - 4\rho_4}.$$

Case 5 When $\rho_0 = \rho_1 = \rho_6 = 0$, the following solutions are raised:

$$\mathscr{H}(\xi) = -\frac{\rho_2\left(\tanh\left(\frac{1}{2}\sqrt{\rho_2}\xi\right) + 1\right)}{\rho_3}, \quad \rho_3^2 = 4\rho_2\rho_4,\ \rho_2 > 0.$$

$$\mathscr{H}(\xi) = -\frac{\rho_2\left(\coth\left(\frac{1}{2}\sqrt{\rho_2}(x-\nu t)\right)+1\right)}{\rho_3}, \quad \rho_3^2 = 4\rho_2\rho_4,\ \rho_2 > 0.$$

$$\mathscr{H}(\xi) = \frac{\rho_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho_2}\xi\right)}{2\sqrt{\rho_2\rho_4}\tanh\left(\frac{1}{2}\sqrt{\rho_2}\xi\right)-\rho_3}, \quad \rho_3^2 \neq 4\rho_2\rho_4,\ \rho_2 > 0,\ \rho_4 > 0.$$

$$\mathscr{H}(\xi) = -\frac{\rho_2 \sec^2\left(\frac{1}{2}\sqrt{-\rho_2}\xi\right)}{2\sqrt{-\rho_2\rho_4}\tan\left(\frac{1}{2}\sqrt{-\rho_2}\xi\right)+\rho_3}, \quad \rho_3^2 \neq 4\rho_2\rho_4,\ \rho_2 < 0,\ \rho_4 > 0.$$

Case 6 When $\rho_2 = \rho_4 = \rho_6 = 0$, the following solution is raised:

$$\mathscr{H}(\xi) = \wp\left(\frac{1}{2}\sqrt{\rho_3}(x-\nu t);\ -\frac{4\rho_1}{\rho_3}, -\frac{4\rho_0}{\rho_3}\right), \quad \rho_3 > 0.$$

Case 7 When $\rho_1 = \rho_3 = 0$, the following solutions are raised:

$$\mathscr{H}(\xi) = \sqrt{\frac{2\rho_2 \operatorname{sech}^2\left(\sqrt{\rho_2}\xi\right)}{2\sqrt{\rho_4^2-4\rho_2\rho_6}-\left(\sqrt{\rho_4^2-4\rho_2\rho_6}+\rho_4\right)\operatorname{sech}^2\left(\sqrt{\rho_2}\xi\right)}}.$$

$$\mathscr{H}(\xi) = \sqrt{\frac{2\rho_2 \sec^2\left(\sqrt{-\rho_2}\xi\right)}{2\sqrt{\rho_4^2-4\rho_2\rho_6}-\left(\sqrt{\rho_4^2-4\rho_2\rho_6}-\rho_4\right)\sec^2\left(\sqrt{-\rho_2}\xi\right)}}.$$

Case 8 When $\rho_1 = \rho_3 = \rho_6 = 0$, the following solutions are raised which are shown in Table 1:

Table 1. Solutions of case 8

No.	ρ_0	ρ_2	ρ_4	$\mathscr{H}(\xi)$
1	1	$-1-m^2$	m^2	$sn(\xi,\ m)$ or $cd(\xi,\ m)$
2	m^2-1	$2-m^2$	-1	$dn(\xi,\ m)$
3	$-m^2$	$2m^2-1$	$1-m^2$	$nc(\xi,\ m)$
4	-1	$2-m^2$	m^2-1	$nd(\xi,\ m)$
5	1	$2-4m^2$	1	$dn(\xi\|m)nc(\xi\|m)s(\xi\|m)$
6	$m^4-2m^3+m^2$	$-\frac{4}{m}$	$-m^2+6m-1$	$\frac{mcn(\xi\|m)dn(\xi\|m)}{msn(\xi\|m)^2+1}$
7	$\frac{1}{4}$	$\frac{1}{2}\left(m^2-2\right)$	$\frac{m^4}{4}$	$\frac{cn(\xi\|m)}{dn(\xi\|m)+\sqrt{1-m^2}}$ or $\frac{sn(\xi\|m)}{dn(\xi\|m)+1}$

Step 4 By substituting the solutions that appear to be given in Eq. (5) and Eq. (6) into Eq. (4), we can raise a polynomial $\mathscr{H}(\xi)$; $(i = 0,\ \pm 1,\ \pm 2,\ ...)$ in which the coefficients can be equalized to zero in order to construct an algebraic system of non-linear equations in which Mathematica software program is entered to solve it or some different other software programs. Finally, we can derive several kinds of exact solutions for Eq. (2) as the dual-mode traveling wave solutions.

3. Application of the MEDAM methodology

The below NLODEs couple will be created as the real component and the imaginary component of Eq. (1), respectively, by applying the wave transformation that is given in Eq. (3):

$$\begin{aligned}&(-b\beta_2 s-4b\zeta s-\beta_2 c-3c\zeta+\zeta\Omega)\,\mathscr{Q}^{(3)}+(-6a\beta_2 s-24a\gamma\zeta s-6\beta_2 c-18c\gamma\zeta+6\gamma\zeta\Omega)\,\mathscr{Q}^2\mathscr{Q}'\\&+\left(-2\beta_2\zeta^2\Omega+3b\beta_2\zeta^2 s+4b\zeta^3 s+\beta_2 c\zeta^2+c\zeta^3+2c\zeta\Omega-3\zeta^3\Omega+2\zeta s^2\right)\mathscr{Q}'=0,\end{aligned} \tag{7}$$

$$\begin{aligned}&(bs+c)\mathscr{Q}^{(4)}+\left(6a\gamma s\mathscr{Q}^2+\beta_2\zeta\Omega-3b\beta_2\zeta s-6b\zeta^2 s+c^2-2\beta_2 c\zeta-3c\zeta^2+6\gamma c\mathscr{Q}^2+3\zeta^2\Omega-s^2\right)\mathscr{Q}''\\&+(12a\gamma s+12c\gamma)\mathscr{Q}\left(\mathscr{Q}'\right)^2+\left(-2a\beta_2\zeta s-6a\gamma\zeta^2 s+2\beta_2\zeta\Omega+6\gamma\zeta^2\Omega\right)\mathscr{Q}^3\\&+\left(-\beta_2\zeta^3\Omega+b\beta_2\zeta^3 s+b\zeta^4 s-\zeta^4\Omega-\zeta^2\Omega^2+\zeta^2 s^2\right)\mathscr{Q}=0,\end{aligned} \tag{8}$$

Following the process of integration of Eq. (7) with respect to ξ and setting the arbitrary constant's value of the integration to zero, the below ordinary differential equation (ODE) results from entering the result into Eq. (8):

$$\begin{aligned}&(b^2\beta_2 s^2+4b^2\zeta s^2+2b\beta_2 cs+7bc\zeta s-b\zeta s\Omega+\beta_2 c^2+3c^2\zeta-c\zeta\Omega)\mathscr{Q}^{(4)}+\mathscr{E}_1\mathscr{Q}\left(\mathscr{Q}'\right)^2+\mathscr{E}_2\mathscr{Q}+\mathscr{E}_3\mathscr{Q}^3\\&+(-12a^2\beta_2\gamma s^2-48a^2\gamma^2\zeta s^2-24a\beta_2\gamma cs-84a\gamma^2 c\zeta s+12a\gamma^2\zeta s\Omega-12\beta_2\gamma c^2-36\gamma^2 c^2\zeta+12\gamma^2 c\zeta\Omega)\mathscr{Q}^5=0,\end{aligned} \tag{9}$$

where $\mathscr{E}_n$; $(n = 1,\ 2,\ 3)$ are constants coefficients given by:

$$\begin{aligned}\mathscr{E}_1=&12ab\beta_2\gamma s^2+48ab\gamma\zeta s^2+12a\beta_2\gamma cs+36a\gamma c\zeta s-12a\gamma\zeta s\Omega+12b\beta_2\gamma cs+48b\gamma c\zeta s+12\beta_2\gamma c^2\\&+36\gamma c^2\zeta-12\gamma c\zeta\Omega,\\\mathscr{E}_2=&-25b^2\beta_2\zeta^4 s^2-8b^2\beta_2^2\zeta^3 s^2-20b^2\zeta^5 s^2+3b\beta_2 c^2\zeta^2 s+4bc^2\zeta^3 s-22b\beta_2 c\zeta^4 s-8b\beta_2^2 c\zeta^3 s\\&-6b\beta_2 c\zeta^2 s\Omega-15bc\zeta^5 s-12bc\zeta^3 s\Omega-8\beta_2\zeta^4\Omega^2-2\beta_2^2\zeta^3\Omega^2-8b\beta_2\zeta^2 s^3-12b\zeta^3 s^3+28b\beta_2\zeta^4 s\Omega\\&+8b\beta_2^2\zeta^3 s\Omega-b\beta_2\zeta^2 s\Omega^2+25b\zeta^5 s\Omega-4b\zeta^3 s\Omega^2+\beta_2 c^3\zeta^2+c^3\zeta^3+2c^3\zeta\Omega-5\beta_2 c^2\zeta^4-2\beta_2^2 c^2\zeta^3\end{aligned}$$

$$-6\beta_2 c^2\zeta^2\Omega-3c^2\zeta^5-9c^2\zeta^3\Omega+2c^2\zeta s^2+12\beta_2 c\zeta^4\Omega+4\beta_2^2 c\zeta^3\Omega+\beta_2 c\zeta^2\Omega^2+9c\zeta^5\Omega+3c\zeta^3\Omega^2$$

$$-4\beta_2 c\zeta^2 s^2-4c\zeta^3 s^2-2c\zeta s^2\Omega-8\zeta^5\Omega^2+\zeta^3\Omega^3-2\zeta s^4+4\beta_2\zeta^2 s^2\Omega+8\zeta^3 s^2\Omega,$$

$$\mathscr{E}_3=36ab\beta_2\gamma\zeta^2 s^2+4ab\beta_2\zeta^2 s^2+4ab\beta_2^2\zeta s^2+48ab\gamma\zeta^3 s^2-2a\beta_2 c^2 s-8a\gamma c^2\zeta s+16a\beta_2\gamma c\zeta^2 s$$

$$+2a\beta_2^2 c\zeta s+12a\gamma c\zeta^3 s+12a\gamma c\zeta s\Omega+2a\beta_2 s^3+20a\gamma\zeta s^3-20a\beta_2\gamma\zeta^2 s\Omega-4a\beta_2\zeta^2 s\Omega-2a\beta_2^2\zeta s\Omega$$

$$-36a\gamma\zeta^3 s\Omega+2\beta_2\gamma\zeta^2\Omega^2+36b\beta_2\gamma c\zeta^2 s+12b\beta_2 c\zeta^2 s+6b\beta_2^2 c\zeta s+60b\gamma c\zeta^3 s-2\beta_2\zeta^2\Omega^2+8b\beta_2\zeta^2 s\Omega$$

$$+2b\beta_2^2\zeta s\Omega+12b\gamma\zeta^3 s\Omega-2\beta_2 c^3-6\gamma c^3\zeta+18\beta_2\gamma c^2\zeta^2+6\beta_2 c^2\zeta^2+4\beta_2^2 c^2\zeta+24\gamma c^2\zeta^3+14\gamma c^2\zeta\Omega$$

$$-16\beta_2\gamma c\zeta^2\Omega-24\gamma c\zeta^3\Omega+2\beta_2 cs^2+18\gamma c\zeta s^2-2\gamma\zeta s^2\Omega, \tag{10}$$

Applying the principle of balance discussed in the second section, we can precisely solve Eq. (9) as follows:

$$\mathscr{Q}(\xi)=\mathfrak{C}_0+\mathfrak{C}_1\mathscr{H}(\xi)+\frac{\mathfrak{C}_{-1}}{\mathscr{H}(\xi)}. \tag{11}$$

A polynomial in $\mathscr{H}(\xi)$ is resulted by substituting the obtained condition from Eq. (6) with the other of Eq. (11) into Eq. (9). By collecting all terms that have the same powers and adding them then setting them equal to zero, We create a system of algebraic nonlinear equations that is evaluated with the aid of using the Mathematica program in order to produce the next possible scenarios. Providing the requirement that $\mathfrak{C}_1$ and $\mathfrak{C}_{-1}$ cannot both be zero at the same time.

Case 1 If $\rho_0=\rho_1=\rho_3=\rho_6=0$, the set of the following solutions are raised:

$$\mathfrak{C}_{-1}=\mathfrak{C}_0=0,\ \ \mathfrak{C}_1=\pm 2\sqrt{\frac{5\rho_4\mathscr{E}_2}{\rho_2(\rho_2\mathscr{E}_1+\mathscr{E}_3)}},$$

$$b=\frac{1}{2s(\beta_2+4\zeta)}\left(\zeta(\Omega-7c)-2\beta_2 c\mp\sqrt{\zeta^2(c+\Omega)^2-\frac{4\mathscr{E}_2(\beta_2+4\zeta)}{\rho_2^2}}\right),$$

$$a=-\frac{1}{2s(\beta_2+4\gamma\zeta)}\left(\gamma\zeta(7c-\Omega)+2\beta_2 c\pm\sqrt{\gamma^2\zeta^2(c+\Omega)^2-\frac{(\rho_2\mathscr{E}_1+\mathscr{E}_3)(\rho_2\mathscr{E}_1+6\mathscr{E}_3)(\beta_2+4\gamma\zeta)}{300\gamma\mathscr{E}_2}}\right).$$

By using the acquired set of solutions, Eq. (1) can be solved under condition that $\mathscr{E}_2(\rho_2\mathscr{E}_1+\mathscr{E}_3)<0$, giving its derived analytical solutions as follows:

1.1 If $\rho_2>0$ and $\rho_4<0$, the below bright soliton solution is raised:

$$\Psi_{1.1}(x,\ t) = \pm 2\sqrt{-\frac{5\mathscr{E}_2}{\rho_2\mathscr{E}_1 + \mathscr{E}_3}}\operatorname{sech}\left[(x - ct)\sqrt{\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{12}$$

1.2 If $\rho_2 < 0$ and $\rho_4 > 0$, two forms of singular periodic solutions resulted as follows:

$$\Psi_{1.2}(x,\ t) = \pm 2\sqrt{-\frac{5\mathscr{E}_2}{\rho_2\mathscr{E}_1 + \mathscr{E}_3}}\sec\left[(x - ct)\sqrt{-\rho_2}\right]\ e^{i\zeta(x+\Omega t)}, \tag{13}$$

or

$$\Psi_{1.3}(x,\ t) = \pm 2\sqrt{-\frac{5\mathscr{E}_2}{\rho_2\mathscr{E}_1 + \mathscr{E}_3}}\csc\left[(x - ct)\sqrt{-\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{14}$$

Case 2 If $\rho_1 = \rho_3 = \rho_6 = 0$ and $\rho_0 = \dfrac{\rho_2^2}{4\rho_4}$, different sets of solutions are deduced as follows:

(2.1)

$$\mathfrak{C}_0 = 0,\ \mathfrak{C}_{-1} = \mp\frac{1}{4}\sqrt{\frac{5\rho_2\mathscr{E}_2}{\rho_4\left(\mathscr{E}_3 - \rho_2\mathscr{E}_1\right)}},\ \mathfrak{C}_1 = \pm\frac{1}{2}\sqrt{\frac{5\rho_4\mathscr{E}_2}{\rho_2\left(\mathscr{E}_3 - \rho_2\mathscr{E}_1\right)}},$$

$$a = -\frac{1}{2s(\beta_2 + 4\gamma\zeta)}\left(\gamma\zeta(7c - \Omega) + 2\beta_2 c \pm \sqrt{\gamma^2\zeta^2(c+\Omega)^2 - \frac{2\left(\mathscr{E}_3 - \rho_2\mathscr{E}_1\right)\left(2\rho_2\mathscr{E}_1 + 3\mathscr{E}_3\right)\left(\beta_2 + 4\gamma\zeta\right)}{75\gamma\mathscr{E}_2}}\right),$$

$$b = \frac{1}{2s(\beta_2 + 4\zeta)}\left(\zeta(\Omega - 7c) - 2\beta_2 c \pm \sqrt{\zeta^2(c+\Omega)^2 + \frac{\mathscr{E}_2\left(\beta_2 + 4\zeta\right)\left(4\rho_2\mathscr{E}_1 + \mathscr{E}_3\right)}{16\rho_2^2\left(\rho_2\mathscr{E}_1 - \mathscr{E}_3\right)}}\right).$$

(2.2)

$$\mathfrak{C}_0 = \mathfrak{C}_1 = 0,\ \mathfrak{C}_{-1} = \mp\sqrt{\frac{5\rho_2\mathscr{E}_2}{\rho_4\left(4\mathscr{E}_3 - \rho_2\mathscr{E}_1\right)}},$$

$$a = -\frac{1}{2s(\beta_2 + 4\gamma\zeta)}\left(\gamma\zeta(7c - \Omega) + 2\beta_2 c \pm \sqrt{\gamma^2\zeta^2(c+\Omega)^2 - \frac{\left(4\mathscr{E}_3 - \rho_2\mathscr{E}_1\right)\left(\rho_2\mathscr{E}_1 + 6\mathscr{E}_3\right)\left(\beta_2 + 4\gamma\zeta\right)}{300\gamma\mathscr{E}_2}}\right),$$

$$b = \frac{1}{2s(\beta_2 + 4\zeta)}\left(\zeta(\Omega - 7c) - 2\beta_2 c \pm \sqrt{\zeta^2(c+\Omega)^2 - \frac{4\mathscr{E}_2\left(\beta_2 + 4\zeta\right)\left(\rho_2\mathscr{E}_1 + \mathscr{E}_3\right)}{\rho_2^2\left(4\mathscr{E}_3 - \rho_2\mathscr{E}_1\right)}}\right).$$

(2.3)

$$\mathfrak{C}_0 = \mathfrak{C}_{-1} = 0,\ \mathfrak{C}_1 = \mp 2\sqrt{\frac{5\rho_4 \mathscr{E}_2}{\rho_2 (4\mathscr{E}_3 - \rho_2 \mathscr{E}_1)}},$$

$$a = -\frac{1}{2s\beta_2 + 4\gamma\zeta}\left(\gamma\zeta(7c - \Omega) + 2\beta_2 c \pm \sqrt{\gamma^2\zeta^2(c+\Omega)^2 - \frac{(4\mathscr{E}_3 - \rho_2\mathscr{E}_1)(\rho_2\mathscr{E}_1 + 6\mathscr{E}_3)(\beta_2 + 4\gamma\zeta)}{300\gamma\mathscr{E}_2}}\right),$$

$$b = \frac{1}{2s(\beta_2 + 4\zeta)}\left(\zeta(\Omega - 7c) - 2\beta_2 c \pm \sqrt{\zeta^2(c+\Omega)^2 - \frac{4\mathscr{E}_2(\beta_2 + 4\zeta)(\rho_2\mathscr{E}_1 + \mathscr{E}_3)}{\rho_2^2(4\mathscr{E}_3 - \rho_2\mathscr{E}_1)}}\right).$$

(2.4)

$$\mathfrak{C}_0 = 0,\ \mathfrak{C}_{-1} = \mp\sqrt{-\frac{5\rho_2\mathscr{E}_2}{2\rho_4(\mathscr{E}_3 - 2\rho_2\mathscr{E}_1)}},\ \mathfrak{C}_1 = \mp\sqrt{-\frac{10\rho_4\mathscr{E}_2}{\rho_2(\mathscr{E}_3 - 2\rho_2\mathscr{E}_1)}},$$

$$a = -\frac{1}{2s(\beta_2 + 4\gamma\zeta)}\left(\gamma\zeta(7c - \Omega) + 2\beta_2 c \pm \sqrt{\gamma^2\zeta^2(c+\Omega)^2 - \frac{(\mathscr{E}_3 - 2\rho_2\mathscr{E}_1)(3\mathscr{E}_3 - \rho_2\mathscr{E}_1)(\beta_2 + 4\gamma\zeta)}{150\gamma\mathscr{E}_2}}\right),$$

$$b = \frac{1}{2s(\beta_2 + 4\zeta)}\left(\zeta(\Omega - 7c) - 2\beta_2 c \pm \sqrt{\zeta^2(c+\Omega)^2 - \frac{\mathscr{E}_2(\beta_2 + 4\zeta)}{\rho_2^2}}\right).$$

Next, the corresponding solutions to Eq. (1) for the set of solutions (2.1) that was previously mentioned are as follows:

(2.1.1) If $\rho_2 < 0$, $\rho_4 > 0$ and $\mathscr{E}_2(\mathscr{E}_3 - \rho_2\mathscr{E}_1) < 0$, the following singular soliton solution is resulted:

$$\Psi_{2.1,\,1}(x,\,t) = \mp\sqrt{-\frac{5\mathscr{E}_2}{2(\mathscr{E}_3 - \rho_2\mathscr{E}_1)}}\coth\left[(x - ct)\sqrt{-2\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{15}$$

(2.1.2) If $\rho_2 > 0$, $\rho_4 > 0$ and $\mathscr{E}_2(\mathscr{E}_3 - \rho_2\mathscr{E}_1) > 0$, the below singular periodic solution can be reached:

$$\Psi_{2.1,\,2}(x,\,t) = \mp\sqrt{\frac{5\mathscr{E}_2}{2(\mathscr{E}_3 - \rho_2\mathscr{E}_1)}}\cot\left[(x - ct)\sqrt{2\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{16}$$

By applying case (2.2), Eq. (1) is solved, giving some solutions as displayed below:

(2.2.1) If $\rho_2 < 0$, $\rho_4 > 0$ and $\mathscr{E}_2(4\mathscr{E}_3 - \rho_2\mathscr{E}_1) < 0$, a singular soliton solution is determined as:

$$\Psi_{2.2,\,1}(x,\,t) = \mp\sqrt{-\frac{10\mathscr{E}_2}{4\mathscr{E}_3 - \rho_2\mathscr{E}_1}}\coth\left[(x - ct)\sqrt{-\frac{\rho_2}{2}}\right]\ e^{i\zeta(x+\Omega t)}. \tag{17}$$

(2.2.2) If $\rho_2 > 0$, $\rho_4 > 0$ and $\mathscr{E}_2\left(4\mathscr{E}_3 - \rho_2\mathscr{E}_1\right) > 0$, the solution is obtained as a singular periodic solution that is shown below:

$$\Psi_{2.2,\,2}(x,\,t) = \mp\sqrt{\frac{10\mathscr{E}_2}{4\mathscr{E}_3 - \rho_2\mathscr{E}_1}}\cot\left[(x-ct)\sqrt{\frac{\rho_2}{2}}\right]\,e^{i\zeta(x+\Omega t)}. \tag{18}$$

Through applying the case (2.3), Eq. (1) gives the following solutions:
(2.3.1) If $\rho_2 < 0$, $\rho_4 > 0$ and $\mathscr{E}_2\left(4\mathscr{E}_3 - \rho_2\mathscr{E}_1\right) < 0$, the following dark soliton solution is raised as:

$$\Psi_{2.3,\,1}(x,\,t) = \mp\sqrt{-\frac{10\mathscr{E}_2}{4\mathscr{E}_3 - \rho_2\mathscr{E}_1}}\tanh\left[(x-ct)\sqrt{-\frac{\rho_2}{2}}\right]\,e^{i\zeta(x+\Omega t)}. \tag{19}$$

(2.3.2) If $\rho_2 > 0$, $\rho_4 > 0$ and $\mathscr{E}_2\left(4\mathscr{E}_3 - \rho_2\mathscr{E}_1\right) > 0$, the solution produces as singular periodic solution as below:

$$\Psi_{2.3,\,2}(x,\,t) = \mp\sqrt{\frac{10\mathscr{E}_2}{4\mathscr{E}_3 - \rho_2\mathscr{E}_1}}\tan\left[(x-ct)\sqrt{\frac{\rho_2}{2}}\right]\,e^{i\zeta(x+\Omega t)}. \tag{20}$$

The set of solutions (2.4) indicates that Eq. (1) has certain exact solutions, which are as follows:
(2.4.1) If $\rho_2 < 0$, $\rho_4 > 0$ and $\mathscr{E}_2\left(\mathscr{E}_3 - 2\rho_2\mathscr{E}_1\right) > 0$, the following singular soliton solution can be carried out as:

$$\Psi_{2.4,\,1}(x,\,t) = \mp 2\sqrt{\frac{5\mathscr{E}_2}{\mathscr{E}_3 - 2\rho_2\mathscr{E}_1}}\left[(x-ct)\sqrt{-2\rho_2}\right]\,e^{i\zeta(x+\Omega t)}. \tag{21}$$

(2.4.2) If $\rho_2 > 0$, $\rho_4 > 0$ and $\mathscr{E}_2\left(\mathscr{E}_3 - 2\rho_2\mathscr{E}_1\right) < 0$, a singular periodic solution is presented as:

$$\Psi_{2.4,\,2}(x,\,t) = \mp 2\sqrt{-\frac{5\mathscr{E}_2}{\mathscr{E}_3 - 2\rho_2\mathscr{E}_1}}\csc\left[(x-ct)\sqrt{2\rho_2}\right]\,e^{i\zeta(x+\Omega t)}. \tag{22}$$

Case 3 If $\rho_3 = \rho_4 = \rho_6 = 0$, then the following sets of solutions are produced:

$$\mathfrak{C}_{-1} = 0,\ \mathfrak{C}_0 = \rho_1\sqrt{-\frac{\beta_2\rho_2^2(bs+c)^2 + \mathscr{E}_2}{\left(\rho_1^2 - 4\rho_0\rho_2\right)\mathscr{E}_3}},\ \mathfrak{C}_1 = 2\rho_2\sqrt{-\frac{\beta_2\rho_2^2(bs+c)^2 + \mathscr{E}_2}{\left(\rho_1^2 - 4\rho_0\rho_2\right)\mathscr{E}_3}},\ \mathscr{E}_1 = -\frac{\mathscr{E}_3}{\rho_2},$$

(i) $\zeta = \gamma = 0$.
(ii) $\zeta = 0,\ a = -\frac{c}{s}$.
(iii) $\zeta = 0,\ a = -\frac{c}{s}$.;
(iv) $\zeta = \gamma = 0,\ a = -\frac{c}{s}$.
(v) $\gamma = 0,\ a = -\frac{c}{s},\ \Omega = 4bs + 3c$.
These sets of solutions indicate that Eq. (1) has certain exact solutions, which are as follows:

(3.1) If $\rho_0 = 0$ or $\rho_0 > 0$, $\rho_1 = 0$ and $\rho_2 < 0$, the following periodic wave solution is obtained such that $\mathscr{E}_3\left(\beta_2\rho_2^2(bs+c)^2+\mathscr{E}_2\right) < 0$:

$$\Psi_{3.1}(x,\ t) = \sqrt{-\frac{\beta_2\rho_2^2(bs+c)^2+\mathscr{E}_2}{\mathscr{E}_3}}\sin\left[(x-ct)\sqrt{-\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{23}$$

(3.2) If $\rho_0 = 0$, $\rho_2 > 0$ and $\mathscr{E}_3\left(\beta_2\rho_2^2(bs+c)^2+\mathscr{E}_2\right) < 0$, the reached solution is as the following hyperbolic wave solution:

$$\Psi_{3.2}(x,\ t) = \sqrt{-\frac{\beta_2\rho_2^2(bs+c)^2+\mathscr{E}_2}{\mathscr{E}_3}}\sinh\left[2(x-ct)\sqrt{\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{24}$$

(3.3) If $\rho_0 > 0$, $\rho_1 = 0$, $\rho_2 > 0$ and $\mathscr{E}_3\left(\beta_2\rho_2^2(bs+c)^2+\mathscr{E}_2\right) > 0$, the following hyperbolic wave solutions is resulted:

$$\Psi_{3.3}(x,\ t) = \sqrt{\frac{\beta_2\rho_2^2(bs+c)^2+\mathscr{E}_2}{\mathscr{E}_3}}\sinh\left[(x-ct)\sqrt{\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{25}$$

Case 4 If $\rho_0 = \rho_1 = \rho_2 = \rho_6 = 0$, the evaluated sets of solutions are as follows:

$$\mathfrak{C}_{-1} = \mathfrak{C}_0 = \mathscr{E}_2 = 0,\ \mathfrak{C}_1 = (bs+c)\sqrt{-\frac{30\beta_2\rho_4}{\mathscr{E}_1}},\ \rho_3 = 2\sqrt{\frac{\rho_4\mathscr{E}_3}{\mathscr{E}_1}},$$

(i) $\zeta = 0,\ a = -\frac{1}{s}\left(c \pm \frac{\mathscr{E}_1}{30\beta_2(bs+c)}\sqrt{-\frac{1}{2\gamma}}\right)$.

(ii) $\zeta = 0,\ a = b,\ \gamma = -\frac{\mathscr{E}_1^2}{1,800\beta_2^2(bs+c)^4}$.

(iii) $a = b,\ \gamma = -\frac{\mathscr{E}_1^2}{1,800\beta_2^2(bs+c)^4},\ \Omega = 4bs+3c$.

Using the above-obtained set of solutions, a rational wave solution for Eq. (1) is computed :

(4.1) If $\mathscr{E}_1 \neq 0$ and $\beta_2\mathscr{E}_3 < 0$, the obtained rational solution will be as follows:

$$\Psi_{4.1}(x,\ t) = \frac{8\rho_4}{\mathscr{E}_1}\sqrt{-30\beta_2\mathscr{E}_3}\left[\frac{bs+c}{\rho_3^2(x-ct)^2-4\rho_4}\right]e^{i\zeta(x+\Omega t)}. \tag{26}$$

Case 5 If $\rho_0 = \rho_1 = \rho_6 = 0$, the resulted set of solutions is mentioned below:

$$\mathfrak{C}_{-1} = 0,\ \mathfrak{C}_0 = \pm 2\sqrt{-\frac{5\mathscr{E}_2}{\rho_2\mathscr{E}_1+8\mathscr{E}_3}},\ \mathfrak{C}_1 = \pm\frac{2\rho_3}{\rho_2}\sqrt{-\frac{5\mathscr{E}_2}{\rho_2\mathscr{E}_1+8\mathscr{E}_3}},$$

$$a = -\frac{1}{2s(\beta_2+4\gamma\zeta)}\left(\gamma\zeta(7c-\Omega)+2\beta_2 c\right.$$

$$\pm\sqrt{\gamma^2\zeta^2(c+\Omega)^2-\frac{(12\mathscr{E}_3-\rho_2\mathscr{E}_1)(\rho_2\mathscr{E}_1+8\mathscr{E}_3)(\beta_2+4\gamma\zeta)}{1,200\gamma\mathscr{E}_2}},$$

$$b=\frac{1}{2s(\beta_2+4\zeta)}\left(\zeta(\Omega-7c)-2\beta_2c\pm\sqrt{\zeta^2(c+\Omega)^2-\frac{16\mathscr{E}_2(\beta_2+4\zeta)(2\mathscr{E}_3-\rho_2\mathscr{E}_1)}{\rho_2^2(\rho_2\mathscr{E}_1+8\mathscr{E}_3)}}\right).$$

Based on the collected set of solutions, the precise exact solutions to Eq.(1) are resulted having the following structures:

(5.1) If $\rho_2>0$, $\rho_3=2\sqrt{\rho_2\rho_4}$ and $\mathscr{E}_2(\rho_2\mathscr{E}_1+8\mathscr{E}_3)<0$, either dark soliton or singular soliton solutions are produced as:

$$\Psi_{5.1,\,1}(x,\,t)=\mp2\sqrt{-\frac{5\mathscr{E}_2}{\rho_2\mathscr{E}_1+8\mathscr{E}_3}}\tanh\left[\frac{1}{2}(x-ct)\sqrt{\rho_2}\right]e^{i\zeta(x+\Omega t)},\tag{27}$$

or

$$\Psi_{5.1,\,2}(x,\,t)=\mp2\sqrt{-\frac{5\mathscr{E}_2}{\rho_2\mathscr{E}_1+8\mathscr{E}_3}}\coth\left[\frac{1}{2}(x-ct)\sqrt{\rho_2}\right]e^{i\zeta(x+\Omega t)}.\tag{28}$$

Case 6 If $\rho_2=\rho_4=\rho_6=0$, the below set of solutions is evaluated:

$$\mathfrak{C}_1=0,\ \mathfrak{C}_{-1}=\frac{16}{\rho_1}\sqrt{-\frac{5\rho_0^3\mathscr{E}_2}{\rho_1^2\mathscr{E}_1+64\rho_0\mathscr{E}_3}},\ \mathfrak{C}_0=4\sqrt{-\frac{5\rho_0\mathscr{E}_2}{\rho_1^2\mathscr{E}_1+64\rho_0\mathscr{E}_3}},\ \rho_3=-\frac{\rho_1^3}{8\rho_0^2},$$

$$a=-\frac{1}{2s(\beta_2+4\gamma\zeta)}\left(\gamma\zeta(7c-\Omega)+2\beta_2c\pm\sqrt{\gamma^2\zeta^2(c+\Omega)^2-\frac{\left(16\rho_0\mathscr{E}_3-\rho_1^2\mathscr{E}_1\right)\left(\rho_1^2\mathscr{E}_1+64\rho_0\mathscr{E}_3\right)(\beta_2+4\gamma\zeta)}{19,200\gamma\rho_0^2\mathscr{E}_2}}\right),$$

$$b=\frac{1}{2s(\beta_2+4\zeta)}\left(\zeta(\Omega-7c)-2\beta_2c\pm\sqrt{\zeta^2(c+\Omega)^2-\frac{256\rho_0^2\mathscr{E}_2(\beta_2+4\zeta)\left(8\rho_0\mathscr{E}_3-3\rho_1^2\mathscr{E}_1\right)}{3\left(\rho_1^6\mathscr{E}_1+64\rho_0\rho_1^4\mathscr{E}_3\right)}}\right).$$

By inserting the above parameters for Eq. (1), get the following solution:

(6.1) If $\rho_0<0,\ \rho_1<0$ and $\mathscr{E}_2\left(\rho_1^2\mathscr{E}_1+64\rho_0\mathscr{E}_3\right)>0$, a Weierstrass elliptic doubly periodic solution is generated on the following form:

$$\Psi_{6.1}(x,\,t)=4\sqrt{-\frac{5\rho_0\mathscr{E}_2}{\rho_1^2\mathscr{E}_1+64\rho_0\mathscr{E}_3}}\left[1+\frac{4\rho_0}{\rho_1\wp\left(\frac{1}{2}\sqrt{\rho_3}(x-ct);-\frac{4\rho_1}{\rho_3},\ -\frac{4\rho_0}{\rho_3}\right)}\right]e^{i\zeta(x+\Omega t)}.\tag{29}$$

Case 7 If $\rho_0 = \rho_1 = \rho_3 = 0$, the raised sets of solutions are generated as:

$$\mathfrak{C}_0 = \mathfrak{C}_1 = \rho_6 = 0,\ \mathfrak{C}_{-1} = \pm\sqrt{-\frac{\mathscr{E}_2}{\rho_4\mathscr{E}_1}},\ \rho_2 = -\frac{\mathscr{E}_3}{\mathscr{E}_1},$$

(i) $a = -\frac{c}{s}$.
(ii) $a = \frac{\gamma\zeta(\Omega-3c)-\beta_2 c}{s(\beta_2+4\gamma\zeta)},\ b = -\frac{c}{s}$.
(iii) $a = \frac{\gamma\zeta(\Omega-3c)-\beta_2 c}{s(\beta_2+4\gamma\zeta)},\ b = -\frac{\beta_2 c+3c\zeta-\zeta\Omega}{\beta_2 s+4\zeta s}$.
By using the acquired set of solutions with Eq. (1), the below analytical types of solutions are derived:
(7.1) If $\rho_2 > 0$ and $\mathscr{E}_2\mathscr{E}_3 > 0$, a hyperbolic wave solution can be reached as:

$$\Psi_{7.1}(x,\ t) = \pm\sqrt{\frac{\mathscr{E}_2}{\mathscr{E}_3}}\sinh\left[(x-ct)\sqrt{\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{30}$$

(7.2) If $\rho_2 < 0$ and $\mathscr{E}_2\mathscr{E}_3 > 0$, the following periodic wave solution is obtained:

$$\Psi_{7.2}(x,\ t) = \pm\sqrt{\frac{\mathscr{E}_2}{\mathscr{E}_3}}\cos\left[(x-ct)\sqrt{-\rho_2}\right]\ e^{i\zeta(x+\Omega t)}. \tag{31}$$

Case 8 If $\rho_1 = \rho_3 = \rho_6 = \mathfrak{C}_1 = 0$, we can deduce the below set of solutions:

$$\mathfrak{C}_{-1} = \pm 2\sqrt{\frac{5\rho_0\rho_2\mathscr{E}_2}{\rho_2^2(\rho_2\mathscr{E}_1+\mathscr{E}_3)+4\rho_0\rho_4(3\mathscr{E}_3-2\rho_2\mathscr{E}_1)}},$$

$$\beta_2 = \frac{1}{\gamma(bs+c)-(as+c)}\left(4\gamma\zeta s(a-b) - \frac{\gamma\mathscr{E}_2(\rho_2\mathscr{E}_1+\mathscr{E}_3)}{\left(\rho_2^2(\rho_2\mathscr{E}_1+\mathscr{E}_3)+4\rho_0\rho_4(3\mathscr{E}_3-2\rho_2\mathscr{E}_1)\right)(bs+c)}\right.$$
$$\left. + \frac{(\rho_2\mathscr{E}_1+6\mathscr{E}_3)\left(\rho_2^2(\rho_2\mathscr{E}_1+\mathscr{E}_3)+4\rho_0\rho_4(3\mathscr{E}_3-2\rho_2\mathscr{E}_1)\right)}{1,200\gamma\rho_2^2\mathscr{E}_2(as+c)}\right),$$

$$\Omega = \frac{1}{\gamma(bs+c)-(as+c)}\left(\gamma(4as+3c)(bs+c)-(4bs+3c)(as+c) - \frac{\mathscr{E}_2(\rho_2\mathscr{E}_1+\mathscr{E}_3)(as+c)}{\zeta\left(\rho_2^2(\rho_2\mathscr{E}_1+\mathscr{E}_3)+4\rho_0\rho_4(3\mathscr{E}_3-2\rho_2\mathscr{E}_1)\right)(bs+c)}\right.$$
$$\left. + \frac{(\rho_2\mathscr{E}_1+6\mathscr{E}_3)\left(\rho_2^2(\rho_2\mathscr{E}_1+\mathscr{E}_3)+4\rho_0\rho_4(3\mathscr{E}_3-2\rho_2\mathscr{E}_1)\right)(bs+c)}{1,200\gamma\zeta\rho_2^2\mathscr{E}_2(as+c)}\right).$$

The following outcomes resulted as Eq. (1) solutions through the above mentioned case:

(8.1) If $\rho_0 = 1,\ \rho_2 = -m^2 - 1,\ \rho_4 = m^2,\ \left(m^6 - 5m^4 - 5m^2 + 1\right)\mathscr{E}_1\mathscr{E}_2 > \left(m^4 + 14m^2 + 1\right)\mathscr{E}_3\mathscr{E}_2$ and $0 \le m \le 1$, the solutions will be as Jacobi elliptic functions (JEFs) that are displayed below:

$$\Psi_{8.1}(x,\ t) = \mp 2\sqrt{\frac{5\,(m^2+1)\,\mathscr{E}_2}{(m^6-5m^4-5m^2+1)\,\mathscr{E}_1 - (m^4+14m^2+1)\,\mathscr{E}_3}}\operatorname{ns}(x-ct)\ e^{i\zeta(x+\Omega t)}, \tag{32}$$

or

$$\Psi_{8.2}(x,\ t) = \pm 2\sqrt{\frac{5\,(m^2+1)\,\mathscr{E}_2}{(m^6-5m^4-5m^2+1)\,\mathscr{E}_1 - (m^4+14m^2+1)\,\mathscr{E}_3}}\operatorname{dc}(x-ct)\ e^{i\zeta(x+\Omega t)}. \tag{33}$$

As a special case, when either $m = 0$ or $m = 1$ for Eq. (32), either the below singular periodic or singular soliton solutions are generated:

$$\Psi_{8.3}(x,\ t) = \mp 2\sqrt{\frac{5\mathscr{E}_2}{\mathscr{E}_1 - \mathscr{E}_3}}\csc[x-ct]\ e^{i\zeta(x+\Omega t)}, \tag{34}$$

or

$$\Psi_{8.4}(x,\ t) = \mp\sqrt{-\frac{5\mathscr{E}_2}{\mathscr{E}_1 + 2\mathscr{E}_3}}\coth[x-ct]\ e^{i\zeta(x+\Omega t)}. \tag{35}$$

Special case, when $m = 0$ for Eq. (33), a singular periodic solution is obtained:

$$\Psi_{8.5}(x,\ t) = \mp 2\sqrt{\frac{5\mathscr{E}_2}{\mathscr{E}_1 - \mathscr{E}_3}}\sec[x-ct]\ e^{i\zeta(x+\Omega t)}. \tag{36}$$

(8.2) If $\rho_0 = m^2 - 1,\ \rho_2 = 2 - m^2,\ \rho_4 = -1,\ (m-1)\left(m^2-2\right)\mathscr{E}_2\left(\left(m^6+2m^4-12m^2+8\right)\mathscr{E}_1 - \left(m^4-16m^2+16\right)\mathscr{E}_3\right) > 0$ and $0 \le m < 1$, the below JEF solution is reached:

$$\Psi_{8.6}(x,\ t) = \pm 2\sqrt{\frac{5\,(m^2-1)\,(m^2-2)\,\mathscr{E}_2}{(m^6+2m^4-12m^2+8)\,\mathscr{E}_1 - (m^4-16m^2+16)\,\mathscr{E}_3}}\operatorname{nd}(x-ct)\ e^{i\zeta(x+\Omega t)}. \tag{37}$$

(8.3) If $\rho_0 = -m^2\ \rho_2 = 2m^2 - 1,\ \rho_4 = 1 - m^2,\ \left(2m^2-1\right)\left(\left(-16m^4+16m^2-1\right)\mathscr{E}_3\mathscr{E}_2 + \left(8m^6-12m^4+2m^2+1\right)\right.$ $\left.\mathscr{E}_1\mathscr{E}_2 > 0\right.$ and $0 < m \le 1$, the evaluated solution is produced as JEF solution that is formed as:

$$\Psi_{8.7}(x,\ t) = \pm 2\,m\sqrt{\frac{5\,(2m^2-1)\,\mathscr{E}_2}{(8m^6-12m^4+2m^2+1)\,\mathscr{E}_1 + (-16m^4+16m^2-1)\,\mathscr{E}_3}}\operatorname{cn}(x-ct)\ e^{i\zeta(x+\Omega t)}. \tag{38}$$

Special case, by setting $m = 1$, the following bright soliton solution is produced:

$$\Psi_{8.8}(x,\ t) = \pm 2\sqrt{-\frac{5\mathscr{E}_2}{\mathscr{E}_1 + \mathscr{E}_3}}[x - ct]\ e^{i\zeta(x+\Omega t)}. \tag{39}$$

(8.4) If $\rho_0 = -1\ \rho_2 = 2 - m^2,\ \rho_4 = m^2 - 1,\ \left(\left(2 - m^2\right)\mathscr{E}_1 + 6\mathscr{E}_3\right)\left(\left(m^6 + 2m^4 - 12m^2 + 8\right)\mathscr{E}_1 - \left(m^4 - 16m^2 + 16\right)\mathscr{E}_3\right) > 0$ and $0 \le m \le 1$, the following JEF is obtained as:

$$\Psi_{8.9}(x,\ t) = \pm 2\sqrt{\frac{5\left(\left(2 - m^2\right)\mathscr{E}_1 + 6\mathscr{E}_3\right)}{\left(m^6 + 2m^4 - 12m^2 + 8\right)\mathscr{E}_1 - \left(m^4 - 16m^2 + 16\right)\mathscr{E}_3}}\mathrm{dn}(x - ct)\ e^{i\zeta(x+\Omega t)}. \tag{40}$$

Special case, by setting $m = 1$, the following bright soliton solution is reached:

$$\Psi_{8.10}(x,\ t) = \pm 2\sqrt{-\frac{5\mathscr{E}_2}{\mathscr{E}_1 + \mathscr{E}_3}}[x - ct]\ e^{i\zeta(x+\Omega t)}. \tag{41}$$

(8.5) If $\rho_0 = 1,\ \rho_2 = 2 - 4m^2,\ \rho_4 = 1,\ \left(2m^2 - 1\right)\left(\left(8m^6 - 12m^4 + 2m^2 + 1\right)\mathscr{E}_1\mathscr{E}_2 - 2\left(m^4 - m^2 + 1\right)\mathscr{E}_2\mathscr{E}_3\right) > 0$ and $0 \le m \le 1$, the following solution is determined as JEF solution:

$$\Psi_{8.11}(x,\ t) = \mp\sqrt{\frac{5\left(2m^2 - 1\right)\mathscr{E}_2}{\left(8m^6 - 12m^4 + 2m^2 + 1\right)\mathscr{E}_1 - 2\left(m^4 - m^2 + 1\right)\mathscr{E}_3}}\mathrm{nd}(x - ct)\mathrm{cn}(x - ct)\mathrm{ns}(x - ct)\ e^{i\zeta(x+\Omega t)}. \tag{42}$$

As a special case, when either $m = 0$ or $m = 1$, two types of solutions appear as the below singular periodic or singular soliton solutions:

$$\Psi_{8.12}(x,\ t) = \mp\sqrt{-\frac{5\mathscr{E}_2}{\mathscr{E}_1 - 2\mathscr{E}_3}}\cot[x - ct]\ e^{i\zeta(x+\Omega t)}, \tag{43}$$

or

$$\Psi_{8.13}(x,\ t) = \mp\sqrt{-\frac{5\mathscr{E}_2}{\mathscr{E}_1 + 2\mathscr{E}_3}}\coth[x - ct]\ e^{i\zeta(x+\Omega t)} \tag{44}$$

(8.6) If $\rho_0 = m^4 - 2m^3 + m^2,\ \rho_2 = -\dfrac{4}{m},\ \rho_4 = -m^2 + 6m - 1,\ \mathscr{E}_2((3m^9 - 24m^8 + 42m^7 - 24m^6 + 3m^5 - 4m)\mathscr{E}_3 + (8m^8 - 64m^7 + 112m^6 - 64m^5 + 8m^4 + 16)\mathscr{E}_1) > 0$ and $0 < m < 1$, the below JEF solution is reached:

$$\Psi_{8.14}(x,\ t) = \pm 2m(m - 1)$$

$$\sqrt{\frac{5\mathscr{E}_2}{(8m^8-64m^7+112m^6-64m^5+8m^4+16)\,\mathscr{E}_1+(3m^9-24m^8+42m^7-24m^6+3m^5-4m)\,\mathscr{E}_3}} \tag{45}$$

$$\times\left(\frac{1+m\mathrm{sn}^2(x-ct)}{\mathrm{cn}(x-ct)\mathrm{dn}(x-ct)}\right)\,e^{i\zeta(x+\Omega t)}.$$

(8.7) If $\rho_0=\frac{1}{4},\ \rho_2=\frac{1}{2}\left(m^2-2\right),\ \rho_4=\frac{m^4}{4},\ \mathscr{E}_2\left(\left(m^6+2m^4-12m^2+8\right)\mathscr{E}_1-8\left(m^4-m^2+1\right)\mathscr{E}_3\right)>0$ and $0\leq m\leq 1$, the solutions will be as JEFs that are given below:

$$\Psi_{8.15}(x,\ t)=\pm 2\sqrt{\frac{5\,(2-m^2)\,\mathscr{E}_2}{(m^6+2m^4-12m^2+8)\,\mathscr{E}_1-8\,(m^4-m^2+1)\,\mathscr{E}_3}}\left(\frac{\sqrt{1-m^2}+\mathrm{dn}(x-ct)}{\mathrm{cn}(x-ct)}\right)\,e^{i\zeta(x+\Omega t)}, \tag{46}$$

or

$$\Psi_{8.16}(x,\ t)=\mp 2\sqrt{\frac{5\,(2-m^2)\,\mathscr{E}_2}{(m^6+2m^4-12m^2+8)\,\mathscr{E}_1-8\,(m^4-m^2+1)\,\mathscr{E}_3}}\left(\frac{1+\mathrm{dn}(x-ct)}{\mathrm{sn}(x-ct)}\right)\,e^{i\zeta(x+\Omega t)}. \tag{47}$$

As a special case, when $m=0$ for Eq. (46), the produced solution is a singular periodic solution that is obtained in the following form:

$$\Psi_{8.17}(x,\ t)=\pm 2\sqrt{\frac{5\mathscr{E}_2}{\mathscr{E}_1-\mathscr{E}_3}}\,\sec[x-ct]\,e^{i\zeta(x+\Omega t)}. \tag{48}$$

As a special case, when either $m=0$ or $m=1$ for Eq. (47), singular periodic or singular soliton solutions are obtained as:

$$\Psi_{8.18}(x,\ t)=\mp 2\sqrt{\frac{5\mathscr{E}_2}{\mathscr{E}_1-\mathscr{E}_3}}\,\csc[x-ct]\,e^{i\zeta(x+\Omega t)}, \tag{49}$$

or

$$\Psi_{8.19}(x,\ t)=\mp 2\sqrt{-\frac{5\mathscr{E}_2}{\mathscr{E}_1+8\mathscr{E}_3}}\,\coth\left[\frac{1}{2}(x-ct)\right]\,e^{i\zeta(x+\Omega t)}. \tag{50}$$

4. Modulation instability analysis

When nonlinear and dispersive effects interact, the steady state is modulated by many nonlinear phenomena that exhibit instability. We examine modulation instability (MI) by applying the techniques of standard linear stability [31]. Assuming Eq. (1) possesses steady-state solutions as below:

$$\phi(x,\ t) = [Z(x,\ t) + \sqrt{\mathfrak{R}}]\ e^{i(t\ \mathfrak{R})}, \tag{51}$$

In this equation, $\mathfrak{R}$ symbolizes the steady-state solution for Eq. (1). In this context, $Z(x,\ t)$ represents the perturbation term.

By inserting Eq. (51) into Eq. (1) and linearizing, we can derive:

$$i\ b\ s\ Z_{xxxx} - i\ Z_{xxxt} + (\mathfrak{R} - bbeta_2\ s)\ Z_{xxx} + \beta_2\ Z_{xxt} + i\ (\beta_2\mathfrak{R} - s(s - 6a\gamma\mathfrak{R}))Z_{xx} + i\ Z_{tt} - 6i\gamma\,\mathfrak{R}\ \ Z_{xt}$$

$$2\,(3\beta_2 - 1)\ \mathfrak{R}Z_t + 6\ \mathfrak{R}\ (\gamma\,\mathfrak{R} - a\ \beta_2 s)\ \ Z_x + i\ (6\ \beta_2 - 1)\ \mathfrak{R}^2\ \ (Z^* + Z) = 0, \tag{52}$$

where Z^* denotes the conjugate of Z. Consider the solution to Eq. (52) can be stated as:

$$Z = \mathscr{F}_1\ e^{i(\mathscr{L}\ x - \omega\ t)} + \mathscr{F}_2\ e^{-i(\mathscr{L}\ x - \omega\ t)}, \tag{53}$$

where ω and $\mathscr{L}$ denotes the perturbation frequency and the normal wave number prospectively.

A linear evolution equation that has the dispersion relation $\omega = \omega(\mathscr{L})$ with constant coefficients defines the relationship between temporal oscillations $e^{i\mathscr{L}\ x}$ and spatial oscillations $e^{-i\omega\ t}$ at frequency ω. When we substitute Eq. (53) into Eq. (52), the dispersion relation is resulted as:

$$\omega = \frac{1}{12}\left[-3T_3 \pm \left(\sqrt{3T_7} + \sqrt{-6\left[-3T_3^2 + 8T_2 + \sqrt[3]{4}T_4 + \left(\frac{2\sqrt[3]{2}T_5}{T_4} \pm \frac{3\sqrt{3}T_6}{\sqrt{T_7}}\right)\right]}\right)\right], \tag{54}$$

where

$$T_2 = \mathscr{L}^2\left(-2s\left(b\mathscr{L}^2 - 6a\gamma\mathfrak{R}\right) - 2s^2 + \left(\mathscr{L}^2 - 6\gamma\mathfrak{R}\right)^2\right) + \beta_2\left(\mathscr{L}^2 - 6\mathfrak{R}\right)\left(-\beta_2\left(\mathscr{L}^2 - 6\mathfrak{R}\right) - 2\mathfrak{R}\right) - 2\mathfrak{R}^2,$$

$$T_3 = 12\gamma\mathscr{L}\mathfrak{R} - 2\mathscr{L}^3,$$

$$T_4 = (2T_2^3 - 72T_0T_2\mathscr{L}^2 + 108T_1^2\mathscr{L}^2 + 27T_0T_3^2\mathscr{L}^2 - 18T_1T_3T_2\mathscr{L}$$

$$+\sqrt{(2T_2^3 + 108T_1^2\mathscr{L}^2 - 9T_0(8T_2 - 3T_3^2)\mathscr{L}^2 - 18T_1T_3T_2\mathscr{L})^2 - 4(T_2^2 + 12T_0\mathscr{L}^2 - 6T_1T_3\mathscr{L})^3})^{\frac{1}{3}},$$

$$T_5 = T_2^2 + 12T_0\mathscr{L}^2 - 6T_1T_3\mathscr{L}, \quad T_6 = T_3^3 - 4T_2T_3 + 16T_1\mathscr{L}, \quad T_7 = 3T_3^2 - 8T_2 + 2\sqrt[3]{4}T_4 + \frac{4\sqrt[3]{2}T_5}{T_4}.$$

The steady-state's linear stability analysis is given by equation (54). The steady-state stability is indicated by a real value of ω. Conversely, instability in the steady-state solution is indicated by an imaginary ω, defined by the exponential growth of the disturbance. Therefore, the gain spectrum of modulation instability is determined as follows:

$$G(\mathfrak{R}) = 2\ \mathrm{Im}\left(\frac{1}{12}\left[-3T_3 \pm \left(\sqrt{3T_7} + \sqrt{-6\left[-3T_3^2 + 8T_2 + \sqrt[3]{4}T_4 + \left(\frac{2\sqrt[3]{2}T_5}{T_4} \pm \frac{3\sqrt{3}T_6}{\sqrt{T_7}}\right)\right]}\right)\right]\right). \tag{55}$$

Increasing b typically causes the wave to become more stable by dispersing its energy, as seen in graphs with larger dispersion values and a slower rate of MI increase. On the other hand, graphs where wave amplitudes develop quickly and soliton production occurs to show that greater a enhances MI. Higher levels of phase velocity (s) bring the waves closer together and increase their interaction, which, depending on the configuration, can either stabilize or destabilize the system. s also impacts the spatial dynamics of the wave. Plotting these changes in three dimensions over time highlights how careful parameter tuning is essential to managing wave stability by showing the transition from stable continuous waves to soliton structures as parameters vary.

5. Visual explanations and graphical presentations of some resulted solutions

Different solution sets were generated for Eq. (1) by varying the values of the parameters that are located in the model used in this study. Consequently, this scheme has produced some new results that were not published or attained previously in the mentioned literature. Plots for different particular solutions in two and three dimensions are presented to illustrate the physical properties of the obtained solutions.

Figure 1 shows 3D graphical depictions for dual-mode waves for the bright soliton solution of Eq. (12) when selecting parameters as $c = 1.1,\ s = 0.8,\ \zeta = -0.7,\ \beta_2 = 0.8,\ \gamma = 0.5,\ \Omega = -0.86,\ \rho_2 = 1,$ and x from -10 to 10. And all its 2D depictions are shown in Figure 2. Figure 3 clarifies 3D graphical depictions for dual-mode waves for the singular periodic solution of Eq. (13) with choosing the parameters' values as $c = 1.1,\ s = 0.8,\ \zeta = -0.7,\ \beta_2 = 0.8,\ \gamma = -0.5,\ \Omega = 0.86,\ \rho_2 = -1$ and x from -10 to 10. Besides, the 2D depictions that represent Eq. (13) are drawn in Figure 4. Figure 5 shows 3D graphical depictions for dual-mode waves for the dark soliton solution of Eq. (19) when giving the parameters the next values as $c = -1.1,\ \beta_2 = -0.8,\ s = 0.8,\ \zeta = 0.7,\ \gamma = 0.5,\ \Omega = 0.86,\ \rho_2 = -1$ and $-10 \leq x \leq 10$. Furthermore, Figure 6 displays the 2D graphical depictions that represent this dark soliton providing the previously mentioned restrictions. Figure 7 shows 3D graphical depictions of dual-mode waves for the singular soliton solution of Eq. (15) when applying the parameters' values as $c = -1.3,\ s = 0.7,\ \zeta = 0.8,\ \beta_2 = -0.95,\ \gamma = 0.7,\ \Omega = 0.8,\ \rho_2 = -1$ and $-10 \leq x \leq 10$. Furthermore, Figure 8 clarifies the 2D graphical depictions that represent this singular soliton by obeying the same restrictions. Figure 9 displays a three-dimensional plot illustrating the L, R, and $L-R$ waves forming regions of the modulation instability gain spectrum described by Eq. (55). The parameters used in these graphs are $s = 1,\ \beta_2 = 0.8$ and $\gamma = 0.5$.

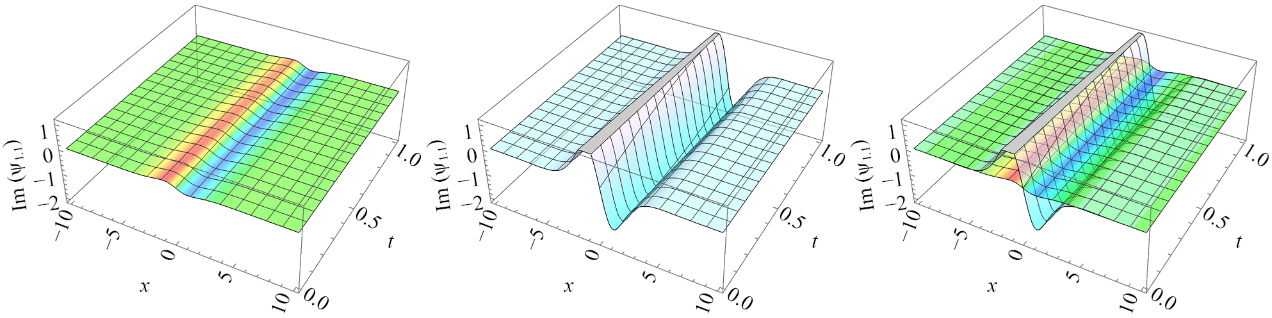

Figure 1. 3D graphical depictions for dual-mode waves for the solution of Eq. (12)

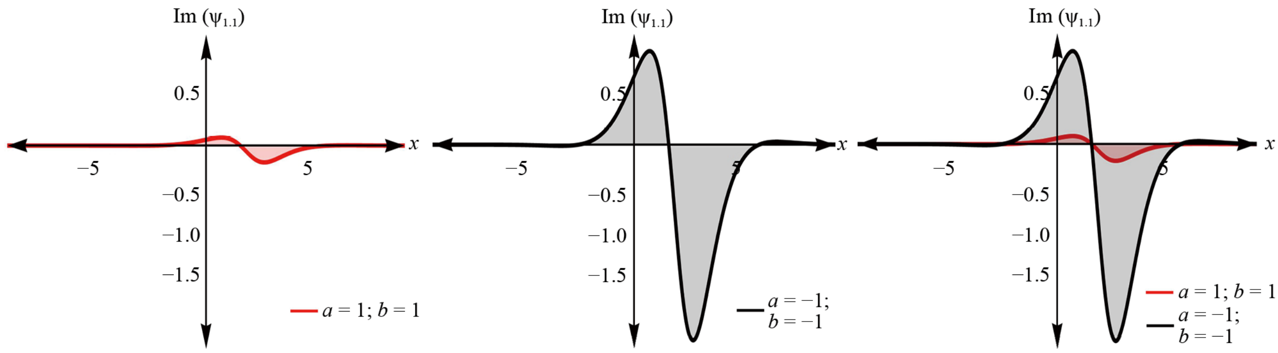

Figure 2. 2D graphical depictions for the dual-modes waves for the solution of Eq. (12)

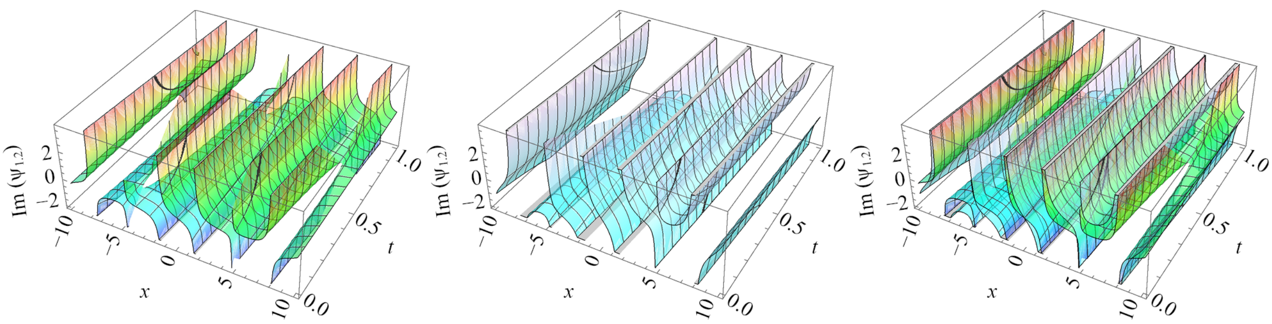

Figure 3. 3D graphical depictions for dual-mode waves for the solution of Eq. (13)

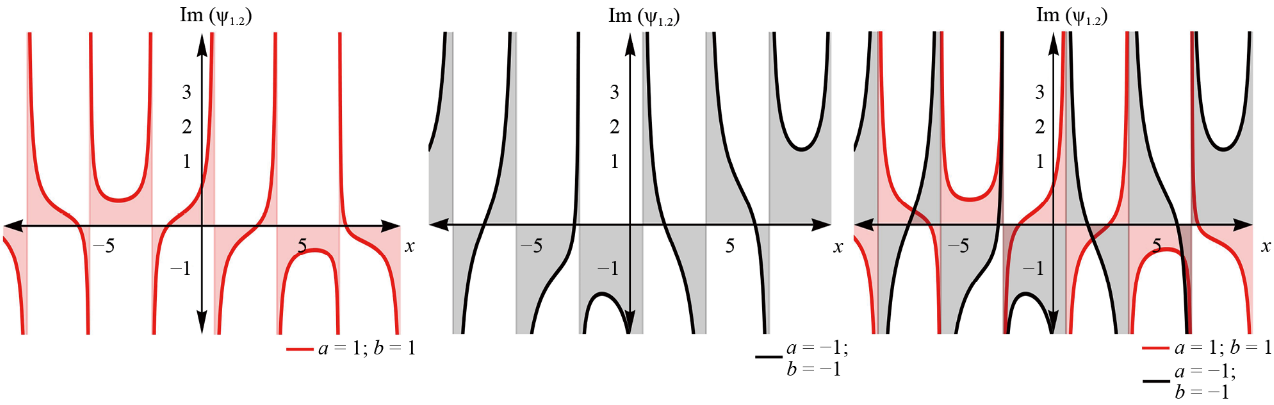

Figure 4. 2D graphical depictions for the dual-modes waves for the solution of Eq. (13)

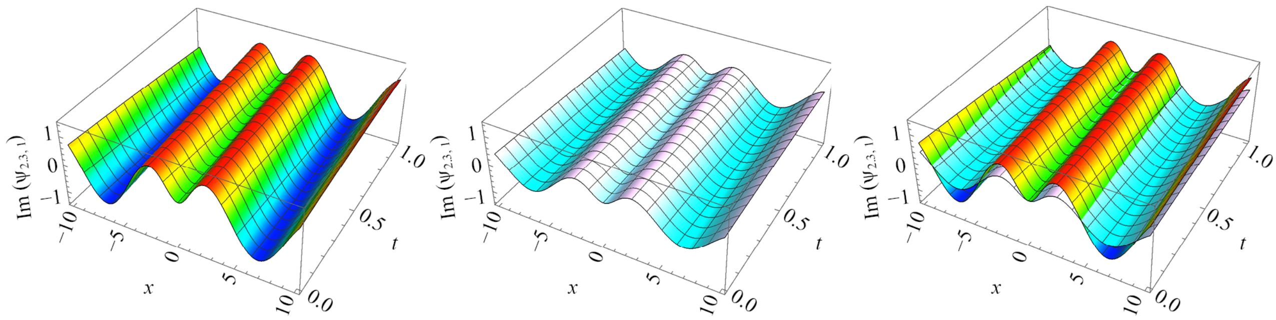

Figure 5. 3D graphical depictions for dual-mode waves for the solution of Eq. (19)

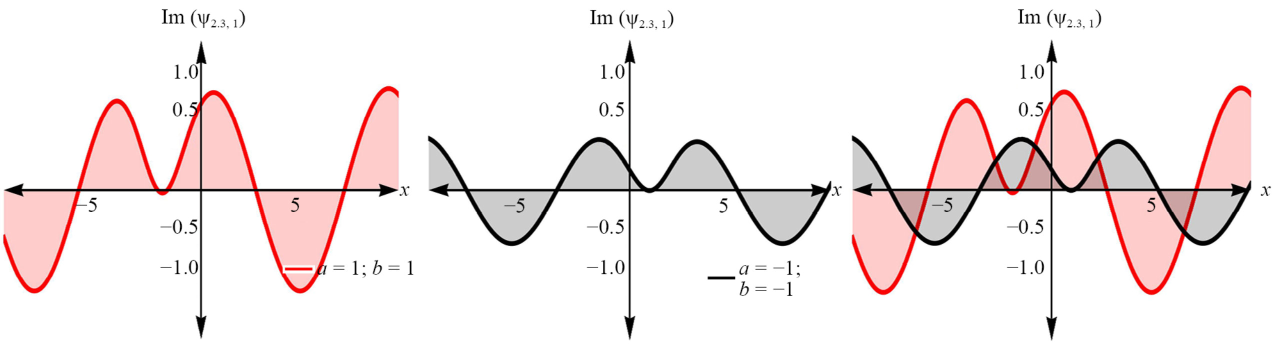

Figure 6. 2D graphical depictions for the dual-modes waves for the solution of Eq. (19)

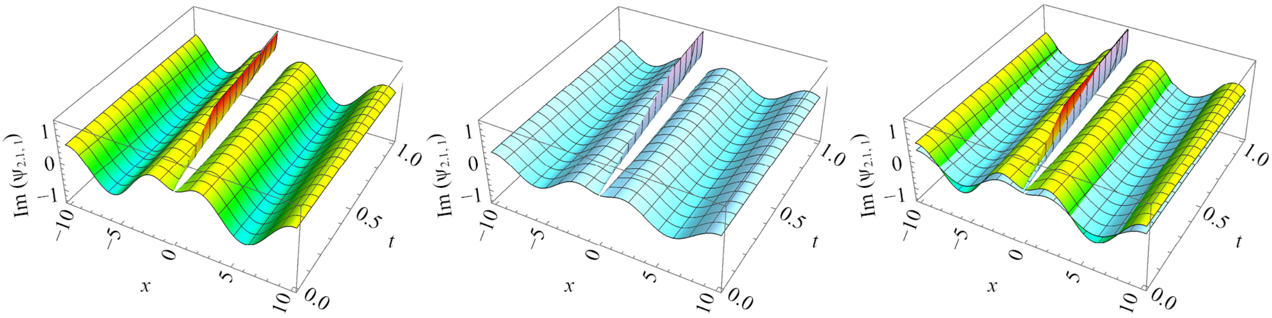

Figure 7. 3D graphical depictions for dual-mode waves for the solution of Eq. (15)

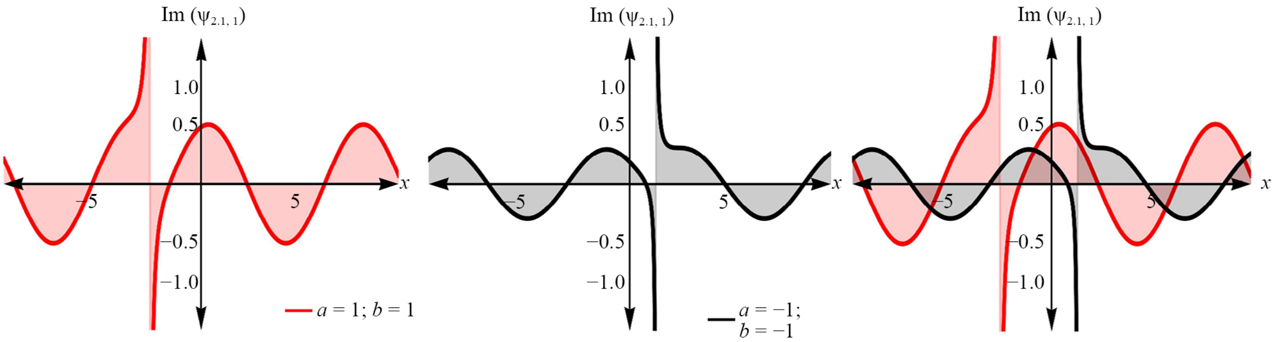

Figure 8. 2D graphical depictions for the dual-modes waves for the solution of Eq. (15)

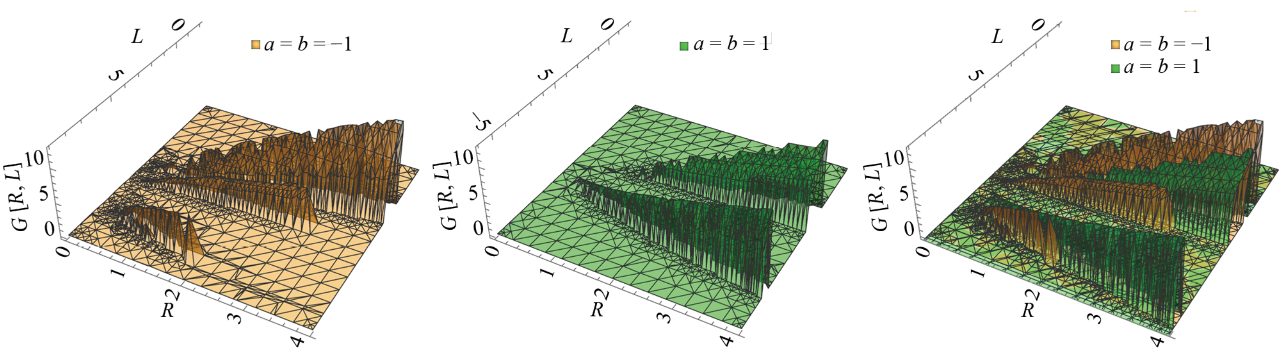

Figure 9. MI gain spectrum regions of Eq. (55) for different values of a, b, and other model parameters

6. Physical interpretations of the obtained solutions

In this part, we aim to provide a brief summary of physical interpretations of the obtained solutions. Solitons with characteristics such as bright, dark, and singular can characterize localized energy disturbances and maintain their shape as they propagate. The lone waves are likened to a weak, black soliton in comparison to the background. It has been demonstrated that dark solitons are more challenging to handle than regular solitons, but they are also more stable and resistant to losses. Singular solitons are a different kind of solitary waves that have singularities, usually endless discontinuities. When solitary solitons have an imagined center, they might be compared to single waves. This solution is a rare instance in nonlinear physics with its point of singularity or intensity divergence. The abrupt shift at the point is illustrated, and details on the interplay between nonlinearity and dispersion that produce anomalous solitary waves are given. Therefore, a discussion of lone solutions is necessary. This kind of solution may provide a description of the creation of rogue waves since it has spikes in them. Peakons and compactons are examples of such solitary waves, with peaks that have a discontinuous first derivative. Compaction on compactons has limited compact support. The number of cycles per second is known as the frequency, and the length of time needed for a waveform cycle to complete is known as the period. Periodic wave solutions characterize waves with a continuous, repeated pattern that determines their wavelength and frequency.

Concerning the double periodic, the complex periodic pattern known as Weierstrass elliptic solutions may find use in fields such as crystal lattices. They show cyclic activity in both space and time.

7. Conclusive remarks

In this work, we have created a revolutionary two-mode NLSE. Finding totally traveling wave solutions was our goal, and we used mathematical analysis to look at the physical characteristics of these extracted solutions. We used the MEDAM, which is a reputable and reliable approach. Among the retrieved solutions, we got (dark, singular, bright) soliton solutions, singular periodic, rational wave, a periodic wave, JEF, hyperbolic, and Weierstrass elliptic double periodic solutions. Some of the solutions acquired for the analyzed model were visually shown via the 3D and 2D displays. Through non-linear dynamical systems, our findings provided additional insight into the breadth of space-time and spatial patterns of solitons by generalizing most of the solutions and expanding certain previously retrieved results.

The given methodology performed better than others for solitons with the senses of controlled parameters and transient stability, as confirmed in section 4 by the simulation of the results using Mathematica software. This provided a simulation example that illustrated the efficacy of the suggested scheme in this research, together with its stability analysis. Therefore, it can be said that the system in use is functional. Moreover, the discussion of the system's stability confirmed the viability of the suggested approach. Additionally, our accomplishments achieved valuable knowledge and information for the community of nonlinear scientists. For example, when we compared our findings with those published through [32], We highlighted their uniqueness, originality, and noteworthy contribution to the current field knowledge and understanding. The efficiency of our approach suggested that it can be applied to numerous nonlinear issues across multiple fields, including soliton theory. The work's findings might have an effect on how integrated telecommunication systems for data transfer develop. In particular, the dual-wave doubling phenomenon may function as a carrier wave to facilitate the multi-way transmission of specific types of data.

Conflict of interest

The authors declare no competing financial interest.

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