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Necessary Condition for Optimal Control of Uncertain Stochastic Systems

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Abstract: This paper examines an uncertain stochastic maximum principle, aiming to establish a necessary condition for optimality in control problems through the classical variational approach. By integrating Liu's uncertainty theory with conventional stochastic optimal control theory, the study addresses a hybrid optimal control problem that merges elements from both frameworks. The analysis operates under the assumption that the associated adjoint equation, specifically the uncertain backward stochastic differential equation (UBSDE), exists uniquely. This assumption is grounded in prior research that has rigorously established the existence and uniqueness of UBSDEs.

Keywords: uncertain stochastic process, uncertain stochastic differential equation, uncertain backward stochastic differential equation

MSC: 49K45, 49K99

Abbreviation

UBSDE	Uncertain Backward Stochastic Differential Equation
USDE	Uncertain Stochastic Differential Equation

1. Introduction

Since time immemorial, Man's socio-economic pursuits have always unfolded in an environment that is characterized by at least one form of uncertainty. Indeterminacy is a term used in some academic quarters to mean generic uncertainty. Aleatory uncertainty, fuzziness and Liu's uncertainty have recently attracted the attention of prominent researchers and industry practitioners as they endeavor to model indeterminacy. For more technical discussions of these forms of uncertainty the reader is referred to [1–7]. Empirical research [8–14] has amply demonstrated that two or more forms of uncertainty normally manifest themselves simultaneously in any specified context. This research aims to investigate the interplay of aleatory uncertainty, also called randomness, and Liu's uncertainty. Liu's uncertainty is also known as epistermic uncertainty. The main contribution of this paper is the presentation of a necessary condition for optimal control under the assumption that the system and control processes are modeled by two forms of indeterminacy, randomness and

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Liu uncertainty. The study assumes that the uncertain backward stochastic differential equation exists and is unique, as proved by Fei [14].

The classical main tools for solving optimal control problems are the Bellman's [15] dynamic programming and Pontryagin's [16] maximum principle methods. The uncertain stochastic maximum principle for optimal control involves the the use of adjoint processes to solve the related adjoint equation. The adjoint equation is an uncertain backward stochastic differential equation (UBSDE). The uniqueness and existence theorem for the solutions of UBSDE was proved by Fei [14].

The maximum principle has been studied by many researchers to solve optimal control problems. Earliest versions of stochastic maximum principle were developed by Kushner [17] and Bismut [18] for the linear case. Peng [19] first proved the maximum principle for nonlinear stochastic optimal control problems in the general case. For more information on stochastic maximum principle, the reader is referred to [20–30] and some references therein.

Kushner [17] presented necessary conditions for continuous parameter stochastic optimization problems. The classical spike variation method was used by Peng [19] to prove a maximum principle for non linear stochastic optimal control problem in the general case. Tang and Li [27] presents necessary conditions for optimal control of stochastic systems with random jumps. The paper proves a maximum principle where the control is allowed to enter into both diffusion and jump terms. Meng and Tang [30] studied the general stochastic optimal control problem for stochastic systems driven by Teugel's martingales and an independent Brownian motion. Necessary and sufficient optimality conditions in the form of stochastic maximum principle were derived using the classical convex variation method.

Optimal control problems have recently been extended to Liu uncertainty framework. Some of the works on Liu uncertain optimal control are [2, 3, 31]. Zhu [3] solved an optimal control problem for Liu uncertain processes by obtaining the principle of optimality for uncertain optimal control. Zhu and Ge [32] presented a necessary condition of optimality for Liu's uncertain optimal control problem using the classical variational method.

Merton's seminal work [33], investigates a stochastic control portfolio problem, wherein stochastic analysis and control theory play a pivotal role. The primary assumption underlying this work is a two-asset financial market, wherein investors can allocate their wealth between a riskless asset and a risky asset. Randomness is employed to model the uncertainties associated with the risky asset, which is characterized by unpredictability and governed by probabilistic laws.

Probability theory provides a formal framework for studying random phenomena, and is particularly useful when large sample sizes are available to estimate probability distributions from random experiments. A paradigmatic example of a random experiment is coin tossing, wherein probability measures are used to quantify randomness.

The concept of aleatory uncertainty has given rise to stochastic analysis and its applications, which have become a distinct branch of mathematics with far-reaching implications for finance, engineering, banking, and insurance. For a comprehensive treatment of randomness, the reader is referred to [34–36], and some references therein.

In recent years, mathematicians have been preoccupied with modeling indeterminacy, which motivated Liu [6] to introduce Liu uncertainty theory. This theory addresses problems characterized by inadequate observed data generated by a system, and relies on expert opinion to evaluate personal belief degrees that each event will occur. The belief degree ranges from 0 to 1, and uncertain measures are used to quantify epistemic indeterminacy.

To illustrate this concept, consider a scenario wherein five commercial banks listed on the local stock exchange merge to form a single financial institution. The exact number and names of the banks involved are unknown, but it is known that between three and five banks have merged. The universal set Γ, comprising the number of banks that have merged, can be represented by

$$\Gamma = \{3,\ 4,\ 5\}.$$

The set $\mathscr{L}$, comprising 8 events derived from the idea of number of banks that have merged can be represented as

$$\{\phi,\ \{3\},\ \{4\},\ \{5\},\ \{3,\ 4\},\ \{3,\ 5\},\ \{4,\ 5\},\ \{3,\ 4,\ 5\}\}.$$

After assigning the set function $\mathscr{M}$ to $[0,\ 1]$, we obtain the following

$$\mathscr{M}\{3\} = 0.5,\ \mathscr{M}\{4\} = 0.4,\ \mathscr{M}\{5\} = 0.1,\ \mathscr{M}\{3,\ 4\} = 0.9,$$

$$\mathscr{M}\{3,\ 5\} = 0.7,\ \mathscr{M}\{4,\ 5\} = 0.3,\ \mathscr{M}\{3,\ 4,\ 5\} = 1,\ \mathscr{M}\{\phi\} = 0.$$

A set function $\mathscr{M}$ must satisfy the following axioms of Liu uncertainty theorem:

Axiom 1 (Normality Axiom): $\mathscr{M}\{\Gamma\} = 1$ for the universal set Γ;

Axiom 2 (Duality Axiom): Let $\wedge = \{3,\ 4\}$ and $\wedge^c = \{\phi,\ \{3\},\ \{4\},\ \{5\},\ \{3,\ 5\},\ \{4,\ 5\},\ \{3,\ 4,\ 5\}\}$. It can be easily seen that $\mathscr{M}\{\wedge\} + \mathscr{M}\{\wedge^c\} = 1$;

Axiom 3 (Subadditivity Axiom): Suppose $\wedge_1 = \phi,\ \wedge_2 = \{3\},\ \wedge_3 = \{4\},\ \cdots \wedge_8 = \{3,\ 4,\ 5\}$, $\mathscr{M}\left\{\cup_{i=1}^{8} \wedge_i\right\} = \mathscr{M}\{\phi \cup \{3\} \cup \{4\} \cdots \cup \{3,\ 4,\ 5\}\}$ should be less or equal to $0.5 + 0.4 + 0.1 \cdots + 0 = 3.9$.

In addition to these axioms, Liu [37] introduced the product axiom to obtain an uncertain measure of a compound event. If $\mathscr{M}$ satisfies the axioms of Liu uncertainty theorem, then $\mathscr{M}$ is a Liu uncertain measure, and the number of banks that have merged is a Liu uncertain variable.

In reality, randomness and epistemic indeterminacy often manifest simultaneously in a given process. The investigation of the interaction between randomness and Liu uncertainty has attracted considerable attention from scientists, including [8, 12–14] among others. Liu [11] was motivated to model indeterminacy using an uncertain random variable, wherein a process can display both randomness and Liu uncertainty simultaneously. For instance, suppose the price of a stock is USD 100 today. After one week, the price of the same stock is an indeterminate quantity. To estimate the new price more accurately, information on stock market experts' beliefs and experiences, as well as past data on stock prices, is required. Thus, the source of indeterminacy arises from both randomness and Liu uncertainty.

Additionally, Liu uncertain stochastic theory can be applied to solve optimal control problems in engineering. One such example is the study of inherent imperfections in integrated electronic devices [38]. The paper by Bucolo et al. [38] examines the role of inherent imperfections in integrated electronic devices, such as nonlinearities and parasitic elements introduced during manufacturing. These imperfections, though generally unintended, induce complex dynamics and can lead to chaotic behaviors in electronic circuits. The inherent imperfections in integrated devices, such as nonlinearities and parasitic elements, introduce complex dynamics and uncertainties, making Liu uncertainty theory and stochastic theory highly relevant. Liu uncertainty theory, which addresses epistemic uncertainty through uncertain variables and distributions, provides a framework for analyzing systems with ambiguous knowledge of probabilities. Conversely, stochastic theory, encompassing aleatory uncertainty, models randomness in manufacturing variations and parasitic elements. By integrating both theories, researchers can optimize control strategies for uncertain stochastic systems, leveraging imperfections for robust chaotic circuits. Applications include chaotic circuit design, exploiting nonlinearities for compactness; parameter estimation and synchronization; uncertainty quantification in manufacturing; and optimal control under combined aleatory and epistemic uncertainties. This fusion enables comprehensive analysis and control of complex electronic systems.

This paper is organized as follows: Section 2 reviews foundational concepts in probability theory and Liu uncertainty theory, while Section 3 introduces the Ito-Liu integral with an illustrative example. Section 4 presents an uncertain stochastic control problem, culminating in the paper's main result: a necessary optimality condition for uncertain stochastic optimal control. Finally, Section 5 concludes the paper, summarizing key findings and outlining future research directions.

2. Some preliminaries

The study assumes that indeterminacy in the market is modeled by a combination of randomness and Liu uncertainty. As such, in order to describe the market with uncertain stochastic systems, this section presents a review of some useful concepts and results related to chance space, the product $(\Gamma,\ \mathscr{L},\ \mathscr{M})\times(\Omega,\ \mathscr{F},\ \Pr)$ [13]. $(\Omega,\ \mathscr{F},\ \Pr)$ is a probability space where Ω is a non-empty set, and $\mathscr{F}$ is a σ-algebra over Ω. Additionally, $(\Gamma,\ \mathscr{L},\ \mathscr{M})$ is a Liu uncertainty space where Γ is a non-empty set, and $\mathscr{L}$ is a σ-algebra over Γ.

Mathematically, the study assumes a complete filtered Liu uncertainty probability space $(\Gamma\times\Omega,\ \mathscr{L}\times\mathscr{F},\ \{\mathscr{L}_t\times\mathscr{F}_t\}_{t\in[0,\ T]},\ \mathscr{M}\times\Pr)$ equipped with a filtration $\{\mathscr{L}_t\times\mathscr{F}_t\}_{t\in[0,\ T]}$, generated by a standard one-dimensional canonical Liu process $\{C_t\}_{t\in[0,\ T]}$ and a one-dimensional Brownian motion $\{W_t\}_{t\in[0,\ T]}$, which are specified in the model. We consider a hybrid space, the chance space, $(\Gamma\times\Omega,\ \mathscr{L}\times\mathscr{F},\ \mathscr{M}\times\Pr)$, where $\Gamma\times\Omega$ is the universal set, $\mathscr{L}\times\mathscr{F}$ is the product σ-algebra, and $\mathscr{M}\times\Pr$ is the product measure. The universal set $\Gamma\times\Omega$ is the set of all ordered pairs of the form $(\gamma,\ \omega)$, where $\gamma\in\Gamma$ and $\omega\in\Omega$.

Definition 1 [13]: Suppose $(\Gamma,\ \mathscr{L},\ \mathscr{M})\times(\Omega,\ \mathscr{F},\ \Pr)$. If we let $\Theta\in\mathscr{L}\times\mathscr{F}$, then the chance measure of Θ is defined as

$$Ch\{\Theta\}=\int_0^1 Pr\{\omega\in\Omega|\mathscr{M}\{\gamma\in\Gamma|\,(\gamma,\ \omega)\in\Theta\}\geq x\}dx. \tag{1}$$

$\Gamma\times\Omega$ can be understood as a rectangular coordinate system if Γ is understood as the horizontal axis and Ω as the vertical axis. The product σ-algebra $\mathscr{L}\times\mathscr{F}$ is the smallest σ-algebra containing measurable rectangles of the form $\wedge\times A$, where $\wedge\in\mathscr{L}$ and $A\in\mathscr{F}$. Each element in $\mathscr{L}\times\mathscr{F}$ is called an event in the chance space. The product measure $\mathscr{M}\times\Pr$ for an event Θ is called the chance measure and is represented by $\mathrm{Ch}\{\Theta\}$ in this study *A* chance measure satisfies the following axioms [11].

(i) Normality. $Ch\{\Lambda\times A\}=\mathscr{M}\{\Lambda\}\times Pr\{A\}$. This implies that $Ch\{\theta\}=\mathscr{M}\{\phi\}\times Pr\{\phi\}=0$ and $Ch\{\Gamma\times\Omega\}=\mathscr{M}\{\Gamma\}\times Pr\{\Omega\}=1$.

(ii) Monotonicity. $Ch\{\Theta_1\}\leq Ch\{\Theta_2\}$, for any events Θ_1 and Θ_2 with $\Theta_1\subset\Theta_2$.

(iii) Self-duality. $Ch\{\Theta\}+Ch\{\Theta^c\}=1$, for any event Θ. In order to exemplify the duality axiom, we consider the occurrence of event $\Theta\in\mathscr{L}\times\mathscr{A}$. When performing the event, the chance that event Θ occurs or does not occur must be 1.

(iv) Subadditivity. $Ch\left\{\bigcup_{i=1}^{\infty}\Theta_i\right\}\leq\sum_{i=1}^{\infty}Ch\{\Theta_i\}$, for any countable sequence of events $\Theta_1,\Theta_2,\ \cdots$.

Definition 2 (i) [13] An uncertain random variable is a function ξ from a chance space $(\Gamma,\ \mathscr{L},\ \mathscr{M})\times(\Omega,\ \mathscr{F},\ \Pr)$ to the set of real numbers such that $\{\xi\in B\}$ is an event in $\mathscr{L}\times\mathscr{F}$ for any Borel set B of real numbers.

(ii) [14] An uncertain random variable is a measurable function $\xi\in\mathbb{R}^p(resp.\ \mathbb{R}^{p\times m})$ from an uncertainty probability space $(\Gamma\times\Omega,\ \mathscr{L}\times\mathscr{F},\ \mathscr{M}\times P)$ to the set in $\mathbb{R}^p(resp.\ \mathbb{R}^{p\times m})$, that is for any Borel set A in $\mathbb{R}^p(resp.\ \mathbb{R}^{p\times m})$, the set $\{\xi\in A\}=\{(\gamma,\ \omega)\in\Gamma\times\Omega:\xi(\gamma,\ \omega)\in A\}\in\mathscr{L}\times\mathscr{F}$.

Definition 3 [13] Let ξ be an uncertain random variable, then its chance distribution of ξ is defined by

$$\Phi(x)=Ch\{\xi\leq x\},$$

for any $x\in\mathbb{R}$. The reader is referred to [13] for examples on uncertain random variable.

Definition 4 (i) [13] Let ξ be an uncertain random variable. Then its expected value is defined by

$$E_{ch}[\xi]=\int_0^{+\infty}Ch\{\xi\geq x\}dx-\int_{-\infty}^{0}Ch\{\xi\leq x\}dx, \tag{2}$$

provided that at least one of the two integrals is finite.

If the expected value of ξ exists, then by using the chance inversion theorem in Liu [11], the expected value can be denoted

$$E_{ch}[\xi] = \int_0^{+\infty} (1-\Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx. \tag{3}$$

or

$$E_{ch}[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \tag{4}$$

(ii) [14] The expected value of an uncertain random variable ξ is defined by

$$E_{ch}[\xi] = E_p\left[E_{\mathscr{M}}[\xi]\right] \triangleq \int_\Omega \left[\int_0^{+\infty} \mathscr{M}\{\xi \geq x\}dx\right] P(d\omega) - \int_\Omega \left[\int_{-\infty}^0 \mathscr{M}\{\xi \leq x\}dx\right] P(d\omega),$$

where E_p and $E_{\mathscr{M}}$ are the expected values under the Liu uncertainty and the probability space respectively.

The expected value of the uncertain random variable ξ is the probability expectation of the expected value of ξ under Liu uncertainty. In order to simplify the work and presentation, $E_{ch}(.)$ shall be denoted by $E(.)$.

Definition 5 [39] A filtration $\{\mathscr{L}_t \times \mathscr{F}_t\}_{t\in[0,\ T]}$ models the flow of information over a specific period. Given an uncertainty probability space $\{\Gamma \times \Omega,\ \mathscr{L} \times \mathscr{F},\ \mathscr{M} \times Pr\}$, a filtration $\{\mathscr{L} \times \mathscr{F}\}_{t\in[0,\ T]}$ is an increasing family of σ-algebras on $\Gamma \times \Omega$ such that, for $s \leq t$,

$$\mathscr{L}_s \times \mathscr{F}_s \subseteq \mathscr{L}_t \times \mathscr{F}_t.$$

If we suppose that $\{\Gamma \times \Omega,\ \mathscr{L} \times \mathscr{F},\ \{\mathscr{L}_t \times \mathscr{F}_t\}_{t\in[0,\ T]},\ \mathscr{M} \times Pr\}$ is a filtered uncertainty probability space satisfying the usual conditions endowed with one-dimensional Brownian motion $\{W_t\}_{t\in[0,\ T]}$ adapted to the filtration $\{\mathscr{F}_t\}_{t\in[0,\ T]}$ and a standard one-dimensional canonical Liu process $\{C_t\}_{t\in[0,\ T]}$ adapted to the filtration $\{\mathscr{L}_t\}_{t\in[0,\ T]}$. A filtration $\{\mathscr{L}_t \times \mathscr{F}_t\}_{t\in[0,\ T]}$ shows the available information at time t for the uncertain stochastic process $X_t = (W_t,\ C_t)$. For more information on canonical Liu process, the reader is referred to [13].

Definition 6 (i) [14] A hybrid process $X(t)$ is an uncertain stochastic process if $X(t)$ is an uncertain random variable for each $t \in [0,\ T]$.

An uncertain stochastic process $X(t)$ is said to be continuous if the sample paths of $X(t)$ are all continuous functions of t for almost all $(\gamma,\ \omega) \in \Gamma \times \Omega$.

(ii) [39] An uncertain stochastic process $X(t)$ is called $\mathscr{F}_t$-adapted if $X\,(t,\ \gamma)$ is $\mathscr{F}_t$-measurable for all $t \in [0,\ T]$, $\gamma \in \Gamma$. Subsequently, an uncertain stochastic process $X(t)$ is $\mathscr{L}_t \times \mathscr{F}_t$-adapted if X_t is $\mathscr{L}_t \times \mathscr{F}_t$-measurable for all $t \in [0,\ T]$. The reader is referred to [39] for more details.

3. Hybrid calculus

Before we look at uncertain stochastic differential equations, we first discuss the notion of Ito-Liu integral.

Definition 7 [14] (Ito-Liu Integral) Suppose $X_t = (Y_t,\ Z_t)^T$ is an uncertain stochastic process, where $Y_t \in \mathbb{R}^{p\times m}$ and $Z_t \in \mathbb{R}^{p\times d}$. For any partition of closed interval $[a,\ b]$ with $a = t_1 < t_2 < \cdots < t_{N+1} = b$, the mesh is expressed as

$$\Delta = \max_{1 \le i \le N} |t_{i+1} - t_i|. \tag{5}$$

The Ito-Liu integral of X_t with respect to $(W_t,\ C_t)$ is defined as follows:

$$\int_a^b X_s^T d(W_s,\ C_s) = \lim_{\Delta \mapsto 0} \sum_{i=1}^{N} [Y(t_i)\left(W_{t_{i+1}} - W_{t_i}\right) + Z(t_i)\left(C_{t_{i+1}} - C_{t_i}\right)], \tag{6}$$

provided that it exists in mean square and is an uncertain random variable, where C_t and W_t are one-dimensional canonical Liu process and one-dimensional Brownian motion, respectively. When $Y_t \equiv 0,\ X_t$ is called Liu integrable.

Example 1 Consider an investor with a initial value USD $10,000$, comprising stocks modeled by the stochastic-Liu uncertain process

$$X(t) = 2 + 3t + \sigma W(t) + \theta C(t),$$

where $\sigma = 0.2$ (volatility), $\theta = 0.1$ (Liu uncertainty coefficient), $W(t)$ (Brownian motion) represents market fluctuations, $C(t)$ (Liu uncertain Canonical process represents market trends), $T = 1$ year (time horizon), $h = 0.1,\ W(0) = 0,\ W(0.1) = 0.05,\ C(0) = 0$, and $C(0.1) = 0.02$.

To quantify the accumulated impact of market fluctuations and trends on portfolio value, calculate the Ito-Liu integral by discretizing $[0,\ T]$ into small steps:

$$\begin{aligned}
\int_0^T [X(t)dW(t) + X(t)dC(t)] &= \int_0^T (2 + 3t + \sigma W(t) + \theta C(t))dW(t) + \int_0^T (2 + 3t + \sigma W(t) + \theta C(t))dC(t) \\
&\approx \sum [[(2 + 3t_i + \sigma W(t_i) + \theta C(t_i))(W(t_{i+1}) - W(t_i))] \\
&\quad + \sum [(2 + 3t_i + \sigma W(t_i) + \theta C(t_i))(C(t_{i+1}) - C(t_i))] \\
&\approx (2)(0.05) + (3)(0.05) + (0.2)(0.05) + (0.1)(0.02) + \dots
\end{aligned}$$

The Ito-Liu integral represents the accumulated impact of market fluctuations and trends on portfolio value.

Remark 1 The hybrid integral may also be written as follows

$$\int_a^b X_s^T d(W_s,\ C_s) = \int_a^b Y_t dW_t + Z_t dC_t. \tag{7}$$

Theorem 1 [14] Suppose $W = (W_t)_{0 \le t \le T}$ and $C = (C_t)_{0 \le t \le T}$ are m-dimensional standard Wiener process and d-dimensional canonical process, respectively. Assuming uncertain stochastic processes $X_1(t),\ X_2(t),\ \cdots,\ X_q(t)$ satisfy

$$dX_k(t) = u_k(t)dt + \sum_{r=1}^{n} \nu_{kr}(t)dW_t^r + \sum_{r=1}^{m} \omega_{kr}(t)dC_t^r, \quad k = 1,\ 2,\ 3\cdots,\ q, \tag{8}$$

with $u_k(t)$, $\nu_{kr}(t)$, and $\omega_{kr}(t)$ being absolute integrable, square integrable, and Liu integrable, respectively. If $\frac{\partial G}{\partial t}(t,\ x_1,\ x_2,\ \cdots x_q)$, $\frac{\partial G}{\partial x_k}(t,\ x_1,\ x_2,\ \cdots x_q)$, and $\frac{\partial^2 G}{\partial x_k x_r}(t,\ x_1,\ x_2,\ \cdots x_q)$ are continuous for $k,\ r = 1,\ 2,\ 3,\ \cdots,\ q$, then

$$\begin{aligned} dG(t,\ X_1(t),\ \cdots,\ X_q(t)) = & \frac{\partial G}{\partial t}(t,\ X_1(t),\ \cdots,\ X_q(t))dt + \sum_{k=1}^{q} \frac{\partial G}{\partial x_k}(t,\ X_1(t),\ \cdots X_q(t))dX_k(t) \\ & + \frac{1}{2}\sum_{k=1}^{q}\sum_{r=1}^{q} \frac{\partial^2 G}{\partial x_k \partial x_r}(t,\ X_1(t),\ \cdots X_q(t))dX_k(t)dX_r(t) \end{aligned} \tag{9}$$

and

$$\partial_{kr} = \begin{cases} 0, & \text{if } k \neq r, \\ 1, & \text{otherwise,} \end{cases}$$

where $dW_t^k dW_t^r = \partial_{kr}dt$ and $dW_t^k dt = dC_t^i dC_t^j = dC_t^i dt = dW_t^k dC_t^i = 0$, for $k,\ r = 1,\ \cdots,\ m$ and $i,\ j = 1,\ \cdots,\ d$.

Proof. Since $G(t,\ X_1(t),\ \cdots,\ X_q(t))$ is a continuously differentiable function, the following holds

$$\begin{aligned} \Delta G(t,\ X_1(t),\ \cdots,\ X_q(t)) = & \frac{\partial G}{\partial t}(t,\ X_1(t),\ \cdots,\ X_q(t))\Delta t + \sum_{k=1}^{q} \frac{\partial G}{\partial x_k}(t,\ X_1(t),\ \cdots X_q(t))\Delta X_k(t) \\ & + \frac{1}{2}\sum_{k=1}^{q}\sum_{r=1}^{q} \frac{\partial^2 G}{\partial x_k \partial x_r}(t,\ X_1(t),\ \cdots X_q(t))\Delta X_k(t)\Delta X_r(t) \\ & + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(t,\ X_1(t),\ \cdots X_q(t))(\Delta t)^2 + \sum_{k=1}^{q} \frac{\partial^2 G}{\partial x_k \partial t}(t,\ X_1(t),\ \cdots X_q(t))\Delta t \Delta X_r(t) \\ & + \Psi_r(\Delta t)^2 + \sum_{k=1}^{q}\sum_{r=1}^{q} \Psi_{kr}\Delta X_k(t)\Delta X_r(t) + \sum_{k=1}^{q} \Psi_k \Delta(t)\Delta X_r(t), \end{aligned} \tag{10}$$

where $\Psi_r \mapsto 0,\ \Psi_{kr} \mapsto 0,\ \Psi_k \mapsto 0$ for $k,\ r = 1,\ \cdots,\ q$ as $\Delta t \mapsto 0$. □

Since $\Delta W_k(t) \mapsto 0,\ \Delta C_r(t) \mapsto 0$,

$$\Delta X_k(t) = u_k(t)\Delta t + \sum_{r=1}^{n} \nu_{kr}\Delta W_r(t) + \sum_{r=1}^{m} \omega_{kr}\Delta C_r \mapsto 0$$

as $\Delta t \mapsto 0$.

Additionally, since $\Delta W_r(t) \mapsto 0,\ \Delta C_k(t) \mapsto 0,\ (\Delta W_r(t))^2 \mapsto \Delta t,\ (\Delta C_k(t))^2 \mapsto (\Delta t)^2,\ (\Delta X_k(t))^2 \mapsto \Delta t$, a chain rule is obtained.

Theorem 2 [14] (Existence and Uniqueness Theorem) Given three functions, $b(t, x, y)$, $\sigma(t, x, y)$, and $\lambda(t, x, y)$, the uncertain backward stochastic differential equation (UBSDE)

$$\begin{cases} dX_t = b(t, X(t), Y(t))dt + \sigma((t, X(t), Y(t))dW_t + \lambda((t, X(t), Y(t))dC_t, \\ X_T = \xi, \end{cases} \tag{11}$$

where $b = (b_1, b_2, ..., b_p)^T : \Gamma \times \Omega \times [0, T] \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \mapsto \mathbb{R}^p$ being $\mathscr{P} \otimes \mathscr{B}_p \otimes \mathscr{B}_{p\times m}/\mathscr{B}_p$ measurable, $\sigma = (\sigma_{kl})_{p\times m}$: $\Gamma \times \Omega \times [0, T] \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \mapsto \mathbb{R}^{p\times m}$ being $\mathscr{P} \otimes \mathscr{B}_p \otimes \mathscr{B}_{p\times m}/\mathscr{B}_{p\times m}$ measurable, $\lambda = (\lambda_{kl})_{p\times d} : \Gamma \times \Omega \times [0, T] \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \mapsto \mathbb{R}^{p\times d}$ being $\mathscr{P} \otimes \mathscr{B}_p \otimes \mathscr{B}_{p\times m}/\mathscr{B}_{p\times d}$ measurable, has a unique pair $(X, Y) \in \mathbb{M}^2(0, T; \mathbb{R}^p) \times \mathbb{M}^2(0, T; \mathbb{R}^{p\times m})$ which solves the USBDE (11), provided that $b(., 0) \in \mathbb{M}^2(0, T; \mathbb{R}^p)$, $\sigma(., 0) \in \mathbb{M}^2(0, T; \mathbb{R}^{p\times m})$ and $\lambda(., 0) \in \mathbb{M}^2(0, T; \mathbb{R}^{p\times d})$ and there exists $\phi > 0$ such that $|b(t, x_1, y_1) - b(t, x_2, y_2)| \vee |\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)| \vee |\lambda(t, x_1, y_1) - \lambda(t, x_2, y_2)| \le \phi(|x_1 - x_2| + |y_1 - y_2|)$, for all $x_1, x_2 \in \mathbb{R}^p$; $y_1, y_2 \in \mathbb{R}^{p\times m}$; (γ, ω, t)-a.e and there exists $\psi > 0$ such that

$$|\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)| \ge \psi|y_1 - y_2|$$

for all $x \in \mathbb{R}^p$; $y_1, y_2 \in \mathbb{R}^{p\times m}$, (γ, ω, t)-a.e.

Proof. The proof of the existence and uniqueness theorem for the UBSDE is found in [14].

4. Main result: The general optimal control problem

Let $(\Gamma, \mathscr{L}, \mathscr{M}) \times (\Omega, \mathscr{F}, \Pr)$ be a chance space with filtration $(\mathscr{L}_t \times \mathscr{F}_t)$ described in section 2. This type of filtration is similar to the one in [14].

The system under consideration in this section is governed by the following uncertain stochastic differential equation (USDE):

$$\begin{cases} dX_t = f(X_t, u_t, t)dt + g(X_t)dW_t + h(X_t)dC_t, \\ X_0 = x_0, \end{cases} \tag{12}$$

where X_t is an n-dimensional state vector, $u(t)$ is an n-dimensional control vector, W_t is an m-dimensional Wiener process, and C_t is a d-dimensional canonical process. In equation (12), $f : \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$, and $h : \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$ are some given functions.

Given that the objective function is an uncertain random variable for any decision, we employ the expected value-based method to optimize the uncertain stochastic objective. Let the cost functional $E[J(u)]$ be

$$E[J(u)] = E\left[\int_0^T \alpha(X_s, u(s), s)ds + \beta(X_T, T)\right]$$

where $\alpha : \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \mapsto \mathbb{R}$ and $\beta : \mathbb{R}^n \times [0, T] \mapsto \mathbb{R}$.

The objective of the hybrid optimal control problem is to minimize $E[J(u)]$, which involves both stochastic and uncertain elements, and $u^* \in U \subset \mathbb{R}^n$ is said to be optimal if

$$E[J(u^*)] = \inf_{u \in U} E[J(u)]. \tag{13}$$

Define the Hamiltonian $H : \mathbb{R}^n \times U \times \mathbb{R}^n \times [0,\ T] \mapsto \mathbb{R}$ by

$$H(X_t,\ u(t),\ p(t),\ t) = \alpha(X_t,\ u(t),\ t) + p_t^\tau f(X_t,\ u(t),\ t) \tag{14}$$

where p_t is a function of t from $[0,\ T]$ to $\mathbb{R}^n$, and p_t^τ is the transpose of the vector p_t. Let ∇_x be a gradient operator in variable x. For real-valued function α, vector-valued function $f = (f_1,\ f_2,\ f_3,\ ...,\ f_n)^\tau$, matrix-valued function $g = (g_{ij})_{n\times m}$, and matrix-valued function $h = (h_{ij})_{n\times d}$, $\nabla_x \alpha = \left(\frac{\partial \alpha}{\partial x_1},\ \frac{\partial \alpha}{\partial x_2},\ \frac{\partial \alpha}{\partial x_3},\ ...,\ \frac{\partial \alpha}{\partial x_n}\right)^\tau$; $\nabla_x f = (\nabla_x f_1,\ \nabla_x f_2,\ \nabla_x f_3,\ ...,\ \nabla_x f_n)^\tau$, $\nabla_x g = (\nabla_x g_{ij})_{n\times m}$, and $\nabla_x h = (\nabla_x h_{ij})_{n\times d}$ respectively.

The adjoint equation in p_t is the Uncertain Backward Stochastic Differential Equation (UBSDE)

$$\begin{cases} -dp_t = \nabla_x H(X_t^*,\ u^*(t),\ p_t,\ t)dt + p_t^\tau(\nabla_x g(X_t^*))dW_t + p_t^\tau(\nabla_x g(X_t^*))dC_t, \\ p_T = \nabla_x \beta(X_T^*,\ T). \end{cases} \tag{15}$$

Theorem 3 (A necessary condition of optimality) Let $\nabla_u \alpha(X_t^*,\ u^*(t),\ t)$, $\nabla_x f(X_t^*,\ u^*(t),\ t)$, $\nabla_x g X_t^*$, and $\nabla_x h(X_t^*)$ be bounded, and suppose that (13) admits a control $u^*(t)$ with corresponding solution X_t^* satisfying (12). Additionally, if

$$H(X_t^*,\ u^*(t),\ p_t,\ t) = \alpha(X_t^*,\ u^*(t),\ t) + p_t^\tau f(X_t^*,\ u^*(t),\ t),$$

then

$$\nabla_u H(X_t^*,\ u^*(t),\ p_t,\ t) = 0 \quad almost \quad surely, \tag{16}$$

and p_t satisfies (15).

Proof. Consider a small perturbation $\delta u(t)$ to the optimal control $u^*(t)$. Let $u^\varepsilon(t) = u^*(t) + \varepsilon \delta u(t)$, where $t \in [0,\ T]$, $0 < \varepsilon < 1$, and $\delta u(t),\ u^\varepsilon(t) \in U$. The trajectory corresponding to $u^\varepsilon(t)$ is denoted $X_t^\varepsilon = X_t^* + \varepsilon \delta x(t)$, and the variation in J can be expressed as

$$\begin{aligned} J(u^\varepsilon) - J(u^*) &= \left[\left(\int_0^T \alpha(X_t^\varepsilon,\ u^\varepsilon(t),\ t)dt + \beta(X_T^\varepsilon,\ T)\right) - \left(\int_0^T \alpha(X_t^*,\ u^*(t),\ t)dt + \beta(X_T^*,\ T)\right) \right] \\ &= \beta(X_T^\varepsilon,\ T) - \beta(X_T^*,\ T) + \int_0^T (\alpha(X_t^\varepsilon,\ u^\varepsilon(t),\ t) - \alpha(X_t^*,\ u^*(t),\ t))\,dt. \end{aligned} \tag{17}$$

From (17),

$$\lim_{\varepsilon\mapsto 0}\frac{E[J(u^{\varepsilon})-J(u^{*})]}{\varepsilon}=\lim_{\varepsilon\mapsto 0}\frac{1}{\varepsilon}E\left[\beta(X_T^{\varepsilon},\ T)-\beta(X_T^{*},\ T)+\int_0^T(\alpha(X_t^{\varepsilon},\ u^{\varepsilon}(t),\ t)-\alpha(X_t^{*},\ u^{*}(t),\ t))\,dt\right]. \tag{18}$$

Since $J(u^{\varepsilon})=J(u^{*}+\varepsilon\delta u)$, after applying the Taylor series expansion we obtain

$$J(u^{\varepsilon})-J(u^{*})=\nabla_u J(u^{*})^{\tau}\varepsilon\delta u(t)+o(\varepsilon). \tag{19}$$

The boundedness of $\nabla_u\alpha(X_t^{*},\ u^{*}(t),\ t)$ implies the boundedness of $\nabla_u J(u^{*})$. Consequently, (19) can be rewritten as

$$J(u^{\varepsilon})-J(u^{*})=O(\varepsilon)+o(\varepsilon), \tag{20}$$

given that $\nabla_u J(u^{*})^{\tau}\varepsilon\delta u(t)=O(\varepsilon)$.

To achieve the optimal outcome,

$$\lim_{\varepsilon\mapsto 0}\frac{E[J(u^{\varepsilon})]-E[J(u^{*})]}{\varepsilon}=0. \tag{21}$$

As $\varepsilon\mapsto 0,\ E[J(u^{\varepsilon})]-E[J(u^{*})]=E[J(u^{*})]+O(\varepsilon)-E[J(u^{*})]=O(\varepsilon)+o(\varepsilon)$.
Utilizing (21), we obtain $\lim_{\varepsilon\mapsto 0}\dfrac{O(\varepsilon)}{\varepsilon}=0$, leading to $\nabla_u J(u^{*})^{\tau}\varepsilon\delta u(t)=o(\varepsilon)$.
Therefore,

$$\lim_{\varepsilon\mapsto 0}\frac{E[J(u^{\varepsilon})]-E[J(u^{*})]}{\varepsilon}=\lim_{\varepsilon\mapsto 0}\frac{E[\nabla_u J(u^{*})^{\tau}\varepsilon\delta u(t)]}{\varepsilon}=\lim_{\varepsilon\mapsto 0}\frac{O(\varepsilon)}{\varepsilon}=\lim_{\varepsilon\mapsto 0}\frac{o(\varepsilon)}{\varepsilon}=0.$$

Equation (18) is equivalently rewritten as

$$\lim_{\varepsilon\mapsto 0}\frac{E[J(u^{\varepsilon})-J(u^{*})]}{\varepsilon}=\lim_{\varepsilon\mapsto 0}\frac{1}{\varepsilon}E\left[L(u^{\varepsilon})-L(u^{*})\right], \tag{22}$$

utilizing $L(u^{\varepsilon})$ and $L(u^{*})$ as the Lagrange functions.

From (12), namely $dX_t-f(X_t,\ u_t,\ t)dt-g(X_t)dW_t-h(X_t)dC_t=0$, we apply integration by parts to derive

$$
\begin{aligned}
L(u^{\varepsilon}) = & \beta(X_T^{\varepsilon},\, T) + \int_0^T \alpha(X_t^{\varepsilon},\, u^{\varepsilon}(t),\, t)dt - \int_0^T p_t^{\tau} dX_t^{\varepsilon} + \int_0^T p_t^{\tau} f(X_t^{\varepsilon},\, u^{\varepsilon}(t),\, t)dt \\
& + \int_0^T p_t^{\tau} g(X_t^{\varepsilon})dW_t + \int_0^T p_t^{\tau} h(X_t^{\varepsilon})dC_t \\
= & \beta(X_T^{\varepsilon},\, T) - [p_t^{\tau} X_t^{\varepsilon}] \,|_0^T + \int_0^T X_t^{\varepsilon} d p_t^{\tau} + \int_0^T H(X_t^{\varepsilon},\, u^{\varepsilon}(t),\, p_t,\, t)dt \\
& + \int_0^T p_t^{\tau} g(X_t^{\varepsilon})dW_t + \int_0^T p_t^{\tau} h(X_t^{\varepsilon})dC_t
\end{aligned}
$$

and

$$
\begin{aligned}
L(u^{*}) = & \beta(X_T^{*},\, T) + \int_0^T \alpha(X_t^{*},\, u^{*}(t),\, t)dt - \int_0^T p_t^{\tau} dX_t^{*} + \int_0^T p_t^{\tau} f(X_t^{*},\, u^{*}(t),\, t)dt \\
& + \int_0^T p_t^{\tau} g(X_t^{*})dW_t + \int_0^T p_t^{\tau} h(X_t^{*})dC_t \\
= & \beta(X_T^{*},\, T) - [p_t^{\tau} X_t^{*}] \,|_0^T + \int_0^T X_t^{*} d p_t^{\tau} + \int_0^T H(X_t^{*},\, u^{*}(t),\, p_t,\, t)dt \\
& + \int_0^T p_t^{\tau} g(X_t^{*})dW_t + \int_0^T p_t^{\tau} h(X_t^{*})dC_t,
\end{aligned}
$$

where p_t serves as the Lagrange multiplier function.

With

$$
\begin{aligned}
0 = & \lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon} [L(u^{\varepsilon}) - L(u^{*})] \\
= & \lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon} \{ [\beta(X_T^{\varepsilon},\, T) - \beta(X_T^{*},\, T)] - [p_t^{\tau} X_t^{\varepsilon} - p_t^{\tau} X_t^{*}] \\
& + \int_0^T [X_t^{\varepsilon} - X_t^{*}] d p_t + \int_0^T [H(X_t^{\varepsilon},\, u^{\varepsilon}(t),\, p_t,\, t) - H(X_t^{*},\, u^{*}(t),\, p_t,\, t)]dt \\
& + \int_0^T [p_t^{\tau} g(X_t^{\varepsilon}) - p_t^{\tau} g(X_t^{*})]dW_t + \int_0^T [p_t^{\tau} h(X_t^{\varepsilon}) - p_t^{\tau} h(X_t^{*})]dC_t \},
\end{aligned} \tag{23}
$$

let us define

$$M_1 = [\beta(X_T^{\varepsilon},\ T) - \beta(X_T^{*},\ T)],$$

$$M_2 = [p_T^{\tau} X_T^{\varepsilon} - p_T^{\tau} X_T^{*}],$$

$$M_3 = [X_t^{\varepsilon} - X_t^{*}],$$

$$M_4 = [H(X_t^{\varepsilon},\ u^{\varepsilon}(t),\ p_t,\ t) - H(X_t^{*},\ u^{*}(t),\ p_t,\ t)],$$

$$M_5 = [p_t^{\tau} g(X_t^{\varepsilon}) - p_t^{\tau} g(X_t^{*})], \quad \text{and}$$

$$M_6 = [p_t^{\tau} h(X_t^{\varepsilon}) - p_t^{\tau} h(X_t^{*})].$$

Applying Taylor series expansion to $M_1,\ M_2,\ M_3,\ M_4,\ M_5$, and M_6, yields:

$$M_1 = \beta(X_T^{\varepsilon},\ T) - \beta(X_T^{*},\ T) = \nabla_x \beta(X_T^{*},\ T)^{\tau} \varepsilon \delta x(T) + o(\varepsilon),$$

$$M_2 = p_T^{\tau} X_T^{\varepsilon} - p_T^{\tau} X_T^{*} = p_T^{\tau} \varepsilon \delta x(T),$$

$$M_3 = X_t^{\varepsilon} - X_t^{*} = \varepsilon \delta x(t)^{\tau},$$

$$M_4 = H(X_t^{\varepsilon},\ u^{\varepsilon}(t),\ p_t,\ t) - H(X_t^{*},\ u^{*}(t),\ p_t,\ t)$$

$$= \varepsilon \delta x(t)^{\tau} \nabla_x H(X_t^{*},\ u^{*}(t),\ p_t,\ t) + \varepsilon \delta u(t)^{\tau} \nabla_u H(X_t^{*},\ u^{*}(t),\ p_t,\ t) + o(\varepsilon),$$

$$M_5 = p_t^{\tau} g(X_t^{\varepsilon}) - p_t^{\tau} g(X_t^{*})$$

$$= \varepsilon \delta x(t)^{\tau} \nabla_x (p_t^{\tau} g(X_t^{*})) + o(\varepsilon)$$

$$= \varepsilon \delta x(t)^{\tau} p_t^{\tau} \nabla_x g(X_t^{*}) + o(\varepsilon), \ \text{and}$$

$$M_6 = p_t^{\tau} h(X_t^{\varepsilon}) - p_t^{\tau} h(X_t^{*})$$

$$= \varepsilon \delta x(t)^{\tau} \nabla_x (p_t^{\tau} h(X_t^{*})) + o(\varepsilon)$$

$$= \varepsilon \delta x(t)^{\tau} p_t^{\tau} \nabla_x h(X_t^{*}) + o(\varepsilon).$$

Substituting the Taylor series expansion of $M_1,\ M_2,\ M_3,\ M_4,\ M_5$, and M_6 into (23) yields:

$$0 = E[(-p_T + \nabla_x \beta(X_T^*, T))^\tau \delta x(T)$$

$$+ \int_0^T \delta x(t)^\tau \{dp_t + \nabla_x H(X_t^*, u^*(t), p_t, t)dt + p_t^\tau \nabla_x g(X_t^*)dW_t + p_t^\tau \nabla_x h(X_t^*)dC_t\}$$

$$+ \int_0^T \delta u(t)^\tau H(X_t^*, u^*(t), p_t, t)dt].$$

The UBSDE (15) has solution p_t. Equation (16) holds, given that $\delta x(t)$ and $\delta u(t)$ represent arbitralily small perturbations.

5. Conclusion

This paper has significantly advanced the field of optimal control in stochastic-uncertain environments, developing innovative strategies to mitigate indeterminacy and fostering a deeper understanding of complex systems. By establishing necessary optimality conditions and illustrating the application of the Ito-Liu integral in portfolio selection, our research has far-reaching implications for finance, engineering, economics, and beyond, enhancing risk management, decision-making, and computational methods for solving stochastic-uncertain optimal control problems.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Kao EPC. *An Introduction to Stochastic Processes*. Wadsworth Publishing Company; 1997.

[2] Zhu Y. *Uncertain Optimal Control*. Singapore: Springer Nature; 2019.

[3] Zhu Y. Uncertain optimal control with application to a portfolio selection model. *Cybernetics and Systems*. 2010; 41(7): 535-547.

[4] Liu YK, Liu B. Fuzzy random variables: A scalar expected operator. *Fuzzy Optimization and Decision Making*. 2003; 1(2): 143-160.

[5] Liu B. Why is there a need for uncertainty theory. *Journal of Uncertainty Systems*. 2011; 6(1): 3-10.

[6] Liu B. *Uncertainty Theory*. 2nd ed. Beijing: Uncertainty Theory Laboratories; 2007.

[7] Hester P. Epistemic uncertainty analysis: An approach using expert judgement and evidencial credibility. *Journal of Quality and Reliability Engineering*. 2012; 2012(1): 617481.

[8] Gao J, Yao K. Some concepts and theorems of uncertainty and randomness. *Soft Computing*. 2015; 17(4): 625-634.

[9] Hou Y, Peng W. Distance between uncertainty and random variables. *Mathematical Modelling of Engineering Problems*. 2014; 1(1): 15-20.

[10] Liu B. Fuzzy process, hybrid process and uncertain process. *Journal of Uncertainty Systems*. 2008; 2(1): 3-16.

[11] Liu Y. Uncertain random variables: A mixture of uncertainty and randomness. *Soft Computing*. 2013; 17(4): 625-634.

[12] Liu B. Uncertain random programming with applications. *Fuzzy Optimization and Decision Making*. 2013; 12(2): 153-169.

[13] Liu B. *Uncertainty Theory*. 5th ed. Beijing: Uncertainty Theory Laboratories; 2019.

[14] Fei W. On existence and uniqueness of solutions to uncertain backwards stochastic differential equations. *A Journal of Chinese Universities*. 2014; 29(1): 53-66.
[15] Bellman R. *Dynamic Programming*. USA: Princeton University Press; 1967.
[16] Pontryagin LS, Boltyanskii VG, Gamkrelidze RV, Mishchenko EF. The mathematical theory of optimal processes. In: Gamkrelidze RV. (ed.) *Interscience*. 1st ed. U.S. Hoboken: John Wiley Sons; 1962. p.149-170.
[17] Kusher HJ. Necessary conditions for continuous parameter stochastic optimization problems. *SIAM Journal on Control*. 1972; 10(3): 550-565.
[18] Bismut JM. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*. 1973; 44(2): 384-404.
[19] Peng S. A general stochastic maximum principle for optimal control problems. *SIAM Journal on Control and Optimization*. 1990; 28(4): 966-979.
[20] Bensoussan A. Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions. *Stochastics*. 1983; 9(3): 169-222.
[21] Haussmann UG. *A Stochastic Maximum Principle for Optimal Control of Diffusions*. Harlow: Longman Scientific and Technical; 1986.
[22] Peng S. Backward stochastic differential equation and application to optimal control. *Applied Mathematics and Optimization*. 1993; 27(2): 125-144.
[23] Cadenillas A, Haussmann UG. The stochastic maximum principle for a singular control problem. *Stochastics and Stochastic Reports*. 1994; 49(3-4): 211-237.
[24] Mao X. Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients. *Stochastic Processes and Their Applications*. 1995; 58(2): 281-292.
[25] Xu W. Stochastic maximum principle for optimal control problem of forward and backward system. *Journal of Australian Mathematical Society Series B*. 1995; 37(2): 172-185.
[26] Kohlman M. Optimality conditions in optimal control of jump processes-extended abstract. In: Brockhoff K, Dinkelbach W, Kall P, Pressmar DB, Spicher K. (eds.) *Proceedings in Operations Research 7*. 2nd ed. Wurzburg, Deutsh: Springer Nature; 1977. p.48-57.
[27] Tang SJ, Li XJ. Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM Journal on Control and Optimization*. 1994; 32: 1447-1475.
[28] Kabanov YM. On the Pontryagin maximum principle for SDEs with a Poisson-type driving noise. In: Kabanov YM. (ed.) *Statistics and Control of Stochastic Processes*. River Edge, NJ: World Scientific; 1997. p.173-190.
[29] Framstad NC, Oksendal B, Sulem A. A sufficient stochastic maximum principle for optimal control of jump diffusions and applications to finance. *Pure Mathematics*. 2001; 22: 1-23.
[30] Meng Q, Tang M. Necessary and sufficient conditions for optimal control of stochastic systems associated with Lévy processes. *Science in China Series F: Information Science*. 2009; 52(11): 1982-1992.
[31] Zhu Q. *Continuous Portfolio Models Under Uncertain Circumstances*. World Academic Union; 2015.
[32] Ge X, Zhu Y. A necessary condition of optimality for uncertain optimal control problem. *Fuzzy Optim Decision Making*. 2013; 12(1): 41-51.
[33] Merton RC. Optimal consumption and portfolio rules in a continuous time model. *Journal of Econom Theory*. 1971; 3(4): 373-413.
[34] Oksendal B. *Stochastic Differential Equations: An Introduction With Applications*. 5th ed. New York: Springer-Verlag Heidelberg; 2000.
[35] Ikeda N, Wanabe S. *Stochastic Differential Equations and Diffusion Processes*. Amsterdam: North Holland; 1981.
[36] Liu P. Portfolio selection in stochastic environments. *The Review of Financial Studies*. 2007; 20(1): 1-39.
[37] Liu B. Some research problems in uncertainty theory. *Journal of Uncertainty Systems*. 2010; 3(1): 3-10.
[38] Bucolo M, Buscarino A, Famoso C, Fortuna L, Gagliano S. Imperfections in integrated devices allow the emergence of unexpected strange attractors in electronic circuits. *IEEE Access*. 2020; 9: 29573-29583.
[39] Chirima J, Chikodza E, Hove-Musekwa SD. Uncertain stochastic option pricing in the presence of uncertain jumps. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*. 2019; 27(4): 613-635.