Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Structural Stability of the Stokes Fluid System in a Channel

Yanping Wang, Yuanfei Li*

School of Data Science, Guangzhou Huashang College, Guangzhou, China
E-mail: liyuanfei@gdhsc.edu.cn

Received: 9 September 2024; **Revised:** 29 October 2024; **Accepted:** 11 November 2024

Abstract: This paper considers the two-dimensional Stokes system in a semi-infinite channel and is committed to deriving the structural stability of the model. Using the differential inequality technique, we obtain the expression of energy function. By making use of the earlier work, a second order differential inequality for energy function is obtained. By solving this second-order differential inequality, the continuous dependence on the coefficient of the system is established. This paper shows how to derive a priori estimates of nonlinear terms.

***Keywords*:** boussinesq equations, continuous dependence, a priori estimates

MSC: 35K05, 35K20, 35K55

1. Introduction

In this paper, we consider the following Stokes system which can be written as [1]

$$\boldsymbol{v}_t - \mu\Delta\boldsymbol{v} + \nabla p = 0, \ in\ R\times[0,\ \infty), \tag{1}$$

$$div\boldsymbol{v} = 0, \ in\ R\times[0,\ \infty). \tag{2}$$

Here $\boldsymbol{v} = (v_1,\ v_2)$ denotes the velocity of the fluid. p is the hydrostatic pressure. μ is the kinematic viscosity coefficient. The region R is defined as

$$R := \left\{(x_1,\ x_2)\middle| x_1 \geq 0,\ 0 \leq x_2 \leq h\right\},$$

where h is a fixed positive constant.

In the paper [1], the authors derived the estimates for weighted energy expression for the solution of equations (1) and (2) in R. By using these estimates, the Phragmén-Lindelöf alternative result was established. In the case of decay, the

DOI: https://doi.org/10.37256/cm.6120255683

bound of the total weighted energy was also obtained. In the present paper, we want to use the results obtained in paper [1] and study the continuous dependence on the coefficient μ. Continuous dependence questions are fundamental in that one wishes to know whether a small change in a coefficient in the equations will induce a dramatic change in the solution. In the spirit of the earlier work [1], we derive a second order partial differential inequality which leads to the continue dependence result.

In fact, a large number of articles have studied the structural stability of various types of partial differential equations. Scott [2] considered a porous mediu of Darcy type and obtained the continuous dependence on boundary reaction terms. Considering the simultaneous existence of multiple fluids in a bounded region, Li et al. [3] obtained the structural stability in resonant penetrative convection in a Brinkman-Forchheimer fluid interfacing with a Darcy fluid. Liu et al. [4] assumed that Boussinesq fluid interfaced with a Darcy fluid in a bounded region in $\mathbb{R}^2$, and they obtained the continuous dependence on the interface parameter. For more papers one can refer to [5–19].

Obviously, most of the above results usually supposed that the problems were defined in a bounded region. Meanwhile, the structural stability of solutions of partial differential equations defined in a cylinder has begun to attract attention. Using the results of [20], Li et al. [21] proved that the solutions of the nonhomogeneous Brinkman-Forchheimer equations depended on the Forchheimer coefficient continuously in a three-dimensional semi-infinite cylinder. Obviously, this type of research has not been fully carried out. As far as we know, the continuous dependence results of the solutions of partial differential equations defined on a two-dimensional semi-infinite channel have not yet appeared. Our motivation for doing this is to extend the structural stability results obtained from a three-dimensional cylinder to two-dimensional rectangular regions using the Stokes equation as an example. The argument to derive the result will be more complicated and the results are interesting. The methods which are used in our paper can be extended to other similar equations (e.g., Generalized heat equation).

2. Formulation

We also use the following notations

$$L_z := \left\{ (x_1,\, x_2) \middle| x_1 = z,\ 0 \le x_2 \le h \right\},$$

$$R_z := \left\{ (x_1,\, x_2) \middle| x_1 \ge z,\ 0 \le x_2 \le h \right\}.$$

Clearly $R_0 = R$.

Throughout this paper, the usual summation convention is employed with repeated Greek subscript summed from 1 to 2. The comma is used to indicate partial differentiation, e.g., $\varphi_{\alpha,\,\alpha} = \Sigma_{\alpha=1}^{2} \dfrac{\partial \varphi_\alpha}{\partial x_\alpha}$. Therefore, the Stokes systems (1)-(2) can be rewritten as

$$v_{\alpha,\,t} - \mu \Delta v_\alpha + p_{,\,\alpha} = 0,\ in\ R \times [0,\, \infty), \tag{3}$$

$$v_{\alpha,\,\alpha} = 0,\ in\ R \times [0,\, \infty), \tag{4}$$

with the initial-boundary conditions

$$v_\alpha(x_1,\ x_2,\ 0)=0,\ in\ R, \tag{5}$$

$$v_\alpha(x_1,\ 0,\ t)=v_\alpha(x_1,\ h,\ t)=0,\ x_1\geq 0,\ t>0, \tag{6}$$

$$v_\alpha(0,\ x_2,\ t)=f_\alpha(x_2,\ t),\ 0\leq x_2\leq h,\ t>0, \tag{7}$$

$$\boldsymbol{v},\ \nabla\boldsymbol{v},\ p,\ \nabla p=o(1),\ as\ x_1\to\infty, \tag{8}$$

for $\alpha=1,\ 2$. The functions $f_\alpha(\alpha=1,\ 2)$ satisfy the compatibility relationship $f_\alpha(0,\ t)=f_\alpha(h,\ t)=0$.

The Stokes equation describes the flow behavior of compressible fluids. It forms the foundation of theoretical mechanics and fluid mechanics, not only determining the flow behavior of mechanical flows but also playing a significant role in thermodynamics, heat transfer, and gel mechanics. The study of structural stability of the Stokes equations primarily concerns the continuous dependence and convergence of the solutions on the coefficients. By achieving structural stability of the solutions with respect to the coefficients, one can simulate physical phenomena more accurately and control errors more effectively.

Under the condition $\int_0^h f_1 dx_2=0$, it follows that at each instant of time t

$$\int_L v_1 dx_2=0.$$

Since $(v_1,\ v_2)$ is divergence-free, we have

$$\int_{L_z} v_1 dx_2=\int_{L_0} v_1 dx_2+\int_0^z\int_{L_\xi} v_{1,\ 1}dx_2 d\xi=-\int_0^z\int_{L_\xi} v_{2,\ 2}dx_2 d\xi=0.$$

Now, we list some lemmas which have been derived in [1]. These lemmas will be used in the next sections of this paper.

Combining (2.8) and (4.16) in [1], we have the following lemma.

Lemma 1 Let $v_\alpha,\ p$ are the solutions of (3)-(8) with $f_\alpha\in L^\infty$. Then for $\xi>z\geq 0$,

$$\int_0^t\int_{R_z}\Big[1+k(\xi-z)\Big]v_{\alpha,\ \beta}v_{\alpha,\ \beta}dAd\eta+\frac{1}{2}\int_{R_z}\Big[1+k(\xi-z)\Big]v_{\alpha,\ \beta}v_{\alpha,\ \beta}dA\Big|_{\eta=t}$$

$$+\frac{h^2}{\pi^2}\int_0^t\int_{R_z}\Big[1+k(\xi-z)\Big]\Big[v_{\alpha,\ \eta}v_{\alpha,\ \eta}+v_{1,\ \alpha\beta}v_{1,\ \alpha\beta}\Big]dAd\eta$$

$$+\frac{h^2}{\pi^2}\int_0^t\int_{R_z}\Big[1+k(\xi-z)\Big]\Big[v_{\alpha,\ \beta}v_{\alpha,\ \beta}+\frac{1}{2}v_{1,\ \alpha}v_{1,\ \alpha}\Big]dAd\eta$$

$$\leq 2c_2(t)e^{-kz},$$

where k is a computable positive constant and $c_2(t)$ is a positive function.

Combining (5.2) and (5.14) in [1], we have the following lemma.

Lemma 2 Let $v_\alpha,\ p$ are the solutions of (3)-(8) with $f_\alpha \in L^\infty$. Then

$$\int_0^t \int_R v_{\alpha,\ \beta} v_{\alpha,\ \beta} dA d\eta + \frac{1}{2}\int_R v_\alpha v_\alpha dA\Big|_{\eta=t} + \frac{2h^2}{\pi^2}\int_0^t \int_R v_{\alpha,\ \eta} v_{\alpha,\ \eta} dA d\eta + \frac{h^2}{\pi^2}\int_R v_{\alpha,\ \beta} v_{\alpha,\ \beta} dA\Big|_{\eta=t} \le c_1(t),$$

where $c_1(t)$ is a positive function which depends on f_α.

In view of (5.30) and (5.14) in [1], we have the following lemma.

Lemma 3 Let $v_\alpha,\ p$ are the solutions of (3)-(8) with $f_\alpha \in L^\infty$. Then

$$\int_0^t \int_R v_{1,\ \alpha\beta} v_{1,\ \alpha\beta} dA d\eta \le c_3(t),$$

where $c_3(t)$ is a positive function which depends on f_α.

In this paper, we want to establish the continuous dependence on the kinematic viscosity coefficient μ. Let $(v_\alpha,\ p)$ and $(v_\alpha^*,\ p^*)$ be solutions of (3)-(8), but with different coefficients μ and μ^*, respectively. We set

$$w_\alpha = v_\alpha - v_\alpha^*,\ \pi = q - q^*,\ \tau = \mu - \mu^*.$$

Then, it is easy to find that $(w_\alpha,\ \pi)$ satisfy

$$w_{\alpha,\ t} - \tau\Delta v_\alpha - \mu^*\Delta w_\alpha + \pi_{,\ \alpha} = 0,\ in\ R\times[0,\ \infty), \tag{9}$$

$$w_{\alpha,\ \alpha} = 0,\ in\ R\times[0,\ \infty), \tag{10}$$

with the initial-boundary conditions

$$w_\alpha(x_1,\ x_2,\ 0) = 0,\ in\ R, \tag{11}$$

$$w_\alpha(x_1,\ 0,\ t) = w_\alpha(x_1,\ h,\ t) = 0,\ x_1 \ge 0,\ t > 0, \tag{12}$$

$$w_\alpha(0,\ x_2,\ t) = 0,\ 0 \le x_2 \le h,\ t > 0, \tag{13}$$

$$\boldsymbol{w},\ \nabla\boldsymbol{w},\ \pi,\ \nabla\pi = o(1),\ as\ x_1 \to \infty. \tag{14}$$

Our main result may be written as:

Theorem 1 Assume that $(w_\alpha,\ p)$ and $(w_\alpha^*,\ p^*)$ be solutions of (9)-(14). If $f_\alpha \in L^\infty$ and $\int_0^h f_1 dx_2 = 0$, then for $\xi > z \ge 0$,

$$\mu^* \int_0^t \int_{R_z} (\xi - z) w_{\alpha,\ \beta} w_{\alpha,\ \beta} dA d\eta + \delta \int_0^t \int_{R_z} (\xi - z) w_{\alpha,\ \eta} w_{\alpha,\ \eta} dA d\eta + \delta \mu^* \int_0^t \int_{R_z} (\xi - z) w_{\alpha,\ \beta 1} w_{\alpha,\ \beta 1} dA d\eta$$

$$\leq L_1(t) \tau^2 e^{-bz} + L_2(t) \frac{1}{b-k} \tau^2 (e^{-kz} - e^{-bz}),$$

or

$$\mu^* \int_0^t \int_{R_z} (\xi - z) w_{\alpha,\ \beta} w_{\alpha,\ \beta} dA d\eta + \delta \int_0^t \int_{R_z} (\xi - z) w_{\alpha,\ \eta} w_{\alpha,\ \eta} dA d\eta + \delta \mu^* \int_0^t \int_{R_z} (\xi - z) w_{\alpha,\ \beta 1} w_{\alpha,\ \beta 1} dA d\eta$$

$$\leq L_1(t) \tau^2 e^{-bz} + L_2(t) \tau^2 z e^{-bz},$$

where $L_i(t)(i = 1,\ 2)$ are positive functions which only depend on t, b, $\delta > 0$.

Remark 1 Theorem 1 not only indicates that the solutions of equations (9)-(14) depend continuously on change in the coefficient μ, but also indicates that the solutions of equations (9)-(14) decay exponentially as $z \to \infty$. Continuous dependence ensures the stability of the system. When the viscosity coefficient is subjected to small disturbances, the system can recover to its original state after being disturbed, or at least remain within an acceptable range of fluctuations, which is very important for many practical applications, such as control systems and signal processing.

Remark 2 Most papers (see e.g., [1, 7, 20, 23]) obtained the spatial exponential decay results of the solutions to equations. Differently, this article demonstrates that the difference between perturbed and undisturbed solutions still decays exponentially with spatial variable.

Remark 3 From (88), we can conclude that $L_i(t)(i = 1,\ 2)$ depend on $\int_0^t \int_{R_z} v_{\alpha,\ \beta\eta} v_{\alpha,\ \beta\eta} dA d\eta$. By referring to the methods of [1], we can obtain the following theorem.

Theorem 2 Assume that $(w_\alpha,\ p)$ be solutions of (3)-(8). If $f_\alpha \in L^\infty$, then

$$\int_0^t \int_{R_z} v_{\alpha,\ \beta\eta} v_{\alpha,\ \beta\eta} dA d\eta \leq n(t),$$

where $n(t)$ is a positive function which only depends on t.

3. The definitions for energy functions

We introduce a stream function $\varphi(x_1,\ x_2,\ t)$ such that

$$w_1 = \varphi_{,\ 2},\ w_2 = -\varphi_{,\ 1}. \tag{15}$$

We can eliminate the troublesome pressure term $\pi_{,\ \alpha}$ in (9). The equations (9)-(14) may be transformed into the following form

$$(\Delta\varphi)_{,\ t} - \tau \Delta v_{1,\ 2} + \tau \Delta v_{2,\ 1} - \mu^* \Delta^2 \varphi = 0,\ in\ R \times [0,\ \infty), \tag{16}$$

with the initial-boundary conditions

$$\varphi(x_1, x_2, 0) = 0, \ in\ R, \tag{17}$$

$$\varphi(x_1, 0, t) = \varphi(x_1, h, t) = 0, \ \varphi_{,2}(x_1, 0, t) = \varphi_{,2}(x_1, h, t) = 0, \ x_1 \geq 0, \ t > 0, \tag{18}$$

$$\varphi(0, x_2, t) = \varphi_{,1}(0, x_2, t) = 0, \ 0 \leq x_2 \leq h, \ t > 0. \tag{19}$$

To deduce the continuous dependence result, the key is to set up an appropriate energy function. Now, we look for such an energy function.

Step 1: Definition for $\Phi_1(z, t)$. We start with the integral

$$\int_0^t \int_{L_z} \varphi_{,1\eta} \varphi dx_2 d\eta.$$

Making use of the divergence theorem, Eq.(16) and the initial-boundary conditions (17)-(19), we have

$$\begin{aligned}
\int_0^t \int_{L_z} \varphi_{,1\eta} \varphi dx_2 d\eta &= -\int_0^t \int_z^\infty \int_{L_\xi} (\Delta\varphi)_{,\eta} \varphi dA d\eta - \frac{1}{2}\int_z^\infty \int_{L_\xi} \varphi_{,\alpha}\varphi_{,\alpha} dA|_{\eta=t} \\
&= \int_0^t \int_{R_z} \Big[-\tau\Delta v_{1,2} + \tau\Delta v_{2,1} - \mu^*\Delta^2\varphi \Big] \varphi dA d\eta - \frac{1}{2}\int_{R_z} \varphi_{,\alpha}\varphi_{,\alpha} dA\Big|_{\eta=t} \\
&= -\frac{1}{2}\int_{R_z} \varphi_{,\alpha}\varphi_{,\alpha} dA\Big|_{\eta=t} - \mu^*\int_0^t \int_{R_z} \Delta^2\varphi\varphi dA d\eta + \int_0^t \int_{R_z} \Big[-\tau\Delta v_{1,2} + \tau\Delta v_{2,1} \Big] \varphi dA d\eta \\
&\doteq -\frac{1}{2}\int_{R_z} \varphi_{,\alpha}\varphi_{,\alpha} dA\Big|_{\eta=t} + A_1 + A_2 + A_3.
\end{aligned} \tag{20}$$

Integrating by parts, we have

$$\begin{aligned}
A_1 =& \mu^*\int_0^t \int_{L_z} \Delta\varphi_{,1}\varphi dx_2 d\eta + \mu^*\int_0^t \int_{R_z} \Delta\varphi_{,\alpha}\varphi_{,\alpha} dA d\eta \\
=& -\mu^*\frac{\partial}{\partial z}\int_0^t \int_{R_z} \Delta\varphi_{,1}\varphi dA d\eta - \mu^*\int_0^t \int_{L_z} \varphi_{,1\alpha}\varphi_{,\alpha} dx_2 d\eta - \mu^*\int_0^t \int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta} dA d\eta \\
=& \mu^*\frac{\partial}{\partial z}\Big\{ \int_0^t \int_{L_z} \varphi_{,11}\varphi dx_2 d\eta + \int_0^t \int_{R_z} \varphi_{,1\alpha}\varphi_{,\alpha} dA d\eta \Big\} \\
& -\mu^*\int_0^t \int_{L_z} \varphi_{,1\alpha}\varphi_{,\alpha} dx_2 d\eta - \mu^*\int_0^t \int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta} dA d\eta
\end{aligned}$$

$$=\mu^*\frac{\partial}{\partial z}\Big\{\int_0^t\int_{L_z}\varphi_{,\,11}\varphi dx_2d\eta+\int_0^t\int_{L_z}\varphi_{,\,\alpha}\varphi_{,\,\alpha}dx_2d\eta\Big\}$$

$$-\mu^*\int_0^t\int_{R_z}\varphi_{,\,\alpha\beta}\varphi_{,\,\alpha\beta}dAd\eta, \tag{21}$$

and

$$A_2+A_3=\tau\int_0^t\int_{L_z}v_{1,\,12}\varphi dx_2d\eta+\tau\int_0^t\int_{R_z}v_{1,\,2\alpha}\varphi_{,\,\alpha}dAd\eta-\tau\int_0^t\int_{L_z}v_{2,\,11}\varphi dx_2d\eta-\tau\int_0^t\int_{R_z}v_{2,\,1\alpha}\varphi_{,\,\alpha}dAd\eta$$

$$=\tau\int_0^t\int_{L_z}v_{1,\,12}\varphi dx_2d\eta+\tau\int_0^t\int_{R_z}v_{1,\,2\alpha}\varphi_{,\,\alpha}dAd\eta-\tau\frac{\partial}{\partial z}\Big\{\int_0^t\int_{L_z}v_{2,\,1}\varphi dx_2d\eta\Big\}+\tau\int_0^t\int_{L_z}v_{2,\,1}\varphi_{,\,1}dx_2d\eta$$

$$+\tau\int_0^t\int_{L_z}v_{2,\,\alpha}\varphi_{,\,\alpha}dx_2d\eta+\tau\int_0^t\int_{R_z}v_{2,\,\alpha}\varphi_{,\,1\alpha}dAd\eta. \tag{22}$$

If we define

$$\Phi_1(z,\,t)=\mu^*\int_0^t\int_{R_z}\varphi_{,\,\alpha\beta}\varphi_{,\,\alpha\beta}dAd\eta,$$

then inserting (21) and (22) into (20), we have

$$\Phi_1(z,\,t)+\frac{1}{2}\int_{R_z}\varphi_{,\,\alpha}\varphi_{,\,\alpha}dA\Big|_{\eta=t}=y_{11}(z,\,t)+y_{12}(z,\,t)+y_{13}(z,\,t), \tag{23}$$

where

$$y_{11}(z,\,t)=\frac{\partial}{\partial z}\int_0^t\int_{L_z}\Big[\mu^*\varphi_{,\,11}\varphi-\mu^*\varphi_{,\,\alpha}\varphi_{,\,\alpha}-\tau v_{2,\,1}\varphi\Big]dx_2d\eta, \tag{24}$$

$$y_{12}(z,\,t)=\int_0^t\int_{L_z}\Big[\tau v_{1,\,12}\varphi_{,\,1}+\tau v_{2,\,1}\varphi_{,\,1}-\varphi_{,\,1\eta}\varphi+\tau v_{2,\,\alpha}\varphi_{,\,\alpha}\Big]dx_2d\eta, \tag{25}$$

$$y_{13}(z,\,t)=\int_0^t\int_{R_z}\Big[\tau v_{1,\,2\alpha}\varphi_{,\,\alpha}+\tau v_{2,\,\alpha}\varphi_{,\,1\alpha}\Big]dAd\eta. \tag{26}$$

Step 2: Definition of $\Phi_2(z,\,t)$. We consider the integral

$$\int_0^t \int_{R_z} \varphi_{,\alpha\eta} \varphi_{,\alpha\eta} dA d\eta.$$

By using the divergence theorem and the initial-boundary conditions (17)-(19), we have

$$\begin{aligned}
\int_0^t \int_{R_z} \varphi_{,\alpha\eta} \varphi_{,\alpha\eta} dA d\eta &= -\int_0^t \int_{L_z} \varphi_{,1\eta} \varphi_{,\eta} dx_2 d\eta - \int_0^t \int_{R_z} \varphi_{,\eta} (\Delta\varphi)_{,\eta} dA d\eta \\
&= -\frac{1}{2}\frac{\partial}{\partial z} \int_0^t \int_{L_z} \varphi_{,\eta}^2 dx_2 d\eta - \mu^* \int_0^t \int_{R_z} \Delta^2 \varphi \varphi_{,\eta} dA d\eta \\
&\quad - \int_0^t \int_{R_z} \left[\tau \Delta v_{1,2} - \tau \Delta v_{2,1} \right] \varphi_{,\eta} dA d\eta \\
&\doteq -\frac{1}{2}\frac{\partial}{\partial z} \int_0^t \int_{L_z} \varphi_{,\eta}^2 dx_2 d\eta + B_1 + B_2 + B_3. \qquad (27)
\end{aligned}$$

By the divergence theorem, we have

$$\begin{aligned}
B_1 &= \mu^* \int_0^t \int_{L_z} \Delta\varphi_{,1} \varphi_{,\eta} dx_2 d\eta + \mu^* \int_0^t \int_{R_z} \Delta\varphi_{,\alpha} \varphi_{,\alpha\eta} dx_2 d\eta \\
&= \frac{\partial}{\partial z}\Big\{ -\mu^* \int_0^t \int_{R_z} \Delta\varphi_{,1} \varphi_{,\eta} dA d\eta \Big\} - \mu^* \int_0^t \int_{L_z} \varphi_{,1\alpha} \varphi_{,\alpha\eta} dx_2 d\eta - \frac{1}{2}\mu^* \int_{R_z} \varphi_{,\alpha\beta} \varphi_{,\alpha\beta} dA \Big|_{\eta=t} \\
&= \frac{\partial}{\partial z}\mu^* \int_0^t \int_{L_z} \varphi_{,11} \varphi_{,\eta} dx_2 d\eta - 2\mu^* \int_0^t \int_{L_z} \varphi_{,1\alpha} \varphi_{,\alpha\eta} dx_2 d\eta - \frac{1}{2}\mu^* \int_{R_z} \varphi_{,\alpha\beta} \varphi_{,\alpha\beta} dA \Big|_{\eta=t}, \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
B_2 + B_3 &= \tau \int_0^t \int_{L_z} v_{1,12} \varphi_{,\eta} dx_2 d\eta + \tau \int_0^t \int_{R_z} v_{1,2\alpha} \varphi_{,\alpha\eta} dA d\eta \\
&\quad - \tau \int_0^t \int_{L_z} v_{2,11} \varphi_{,\eta} dx_2 d\eta - \tau \int_0^t \int_{R_z} v_{2,1\alpha} \varphi_{,\alpha\eta} dA d\eta \\
&= \frac{\partial}{\partial z}\Big\{ \tau \int_0^t \int_{L_z} v_{1,2} \varphi_{,\eta} dx_2 d\eta - \tau \int_0^t \int_{L_z} v_{2,1} \varphi_{,\eta} dx_2 d\eta \Big\} \\
&\quad - \tau \int_0^t \int_{L_z} v_{1,2} \varphi_{,1\eta} dx_2 d\eta + \tau \int_0^t \int_{L_z} v_{2,1} \varphi_{,1\eta} dx_2 d\eta
\end{aligned}$$

$$+\tau\int_0^t\int_{L_z} v_{2,\,\alpha}\varphi_{,\,\alpha\eta}dx_2d\eta+\tau\int_0^t\int_{R_z} v_{2,\,\alpha}\varphi_{,\,1\alpha\eta}dAd\eta$$

$$=\frac{\partial}{\partial z}\Big\{\tau\int_0^t\int_{L_z} v_{1,\,2}\varphi_{,\,\eta}dx_2d\eta-\tau\int_0^t\int_{L_z} v_{2,\,1}\varphi_{,\,\eta}dx_2d\eta\Big\}$$

$$-\tau\int_0^t\int_{L_z} v_{1,\,2}\varphi_{,\,1\eta}dx_2d\eta+\tau\int_0^t\int_{L_z} v_{2,\,1}\varphi_{,\,1\eta}dx_2d\eta$$

$$-\tau\int_0^t\int_{L_z} v_{2,\,\alpha}\varphi_{,\,\alpha\eta}dx_2d\eta-\tau\int_{R_z} v_{2,\,\alpha}\varphi_{,\,1\alpha}dA\Big|_{\eta=t}$$

$$-\tau\int_0^t\int_{L_z} v_{2,\,\eta}\varphi_{,\,11}dAd\eta-\tau\int_0^t\int_{R_z} v_{2,\,\eta}\varphi_{,\,1\alpha\alpha}dAd\eta. \tag{29}$$

If we define

$$\Phi_2(z,\,t)=\int_0^t\int_{R_z}\varphi_{,\,\alpha\eta}\varphi_{,\,\alpha\eta}dAd\eta,$$

then combining (27)-(29) we conclude that

$$\Phi_2(z,\,t)+\frac{1}{2}\mu^*\int_{R_z}\varphi_{,\,\alpha\beta}\varphi_{,\,\alpha\beta}dA\Big|_{\eta=t}=y_{21}(z,\,t)+y_{22}(z,\,t)+y_{23}(z,\,t), \tag{30}$$

where

$$y_{21}(z,\,t)=\frac{\partial}{\partial z}\int_0^t\int_{L_z}\Big[-\frac{1}{2}\varphi_{,\,\eta}^2+\mu^*\varphi_{,\,11}\varphi_{,\,\eta}+\tau v_{1,\,2}\varphi_{,\,\eta}-\tau v_{2,\,1}\varphi_{,\,\eta}\Big]dx_2d\eta, \tag{31}$$

$$y_{22}(z,\,t)=\int_0^t\int_{L_z}\Big[-2\mu^*\varphi_{,\,1\alpha}\varphi_{,\,\alpha\eta}-\tau v_{1,\,2}\varphi_{,\,1\eta}+\tau v_{2,\,1}\varphi_{,\,1\eta}-\tau v_{2,\,\alpha}\varphi_{,\,\alpha\eta}-\tau v_{2,\,\eta}\varphi_{,\,11}\Big]dx_2d\eta, \tag{32}$$

$$y_{23}(z,\,t)=-\tau\int_{R_z} v_{2,\,\alpha}\varphi_{,\,1\alpha}dA\Big|_{\eta=t}-\tau\int_0^t\int_{R_z} v_{2,\,\eta}\varphi_{,\,1\alpha\alpha}dAd\eta. \tag{33}$$

Step 3: Definition of $\Phi_3(z,\,t)$. We define

$$\Phi_3(z,\,t)=\mu^*\int_0^t\int_{R_z}\varphi_{,\,1\alpha\beta}\varphi_{,\,1\alpha\beta}dAd\eta.$$

Integrating by parts and using (16) and the initial-boundary conditions (17)-(19), we have

$$\Phi_3(z,\, t) = -\mu^* \int_0^t \int_{L_z} \varphi_{,\,1\alpha}\varphi_{,\,11\alpha} dx_2 d\eta - \mu^* \int_0^t \int_{R_z} \varphi_{,\,1\alpha}\varphi_{,\,1\alpha\beta\beta} dA d\eta$$

$$= -\mu^* \int_0^t \int_{L_z} \varphi_{,\,1\alpha}\varphi_{,\,11\alpha} dx_2 d\eta + \mu^* \int_0^t \int_{L_z} \varphi_{,\,1}\varphi_{,\,11\beta\beta} dx_2 d\eta + \mu^* \int_0^t \int_{R_z} \varphi_{,\,1}\Delta^2\varphi_{,\,1} dA d\eta$$

$$= \frac{\partial}{\partial z}\Big\{ -\frac{\mu^*}{2} \int_0^t \int_{L_z} \varphi_{,\,1\alpha}\varphi_{,\,1\alpha} dx_2 d\eta + \mu^* \int_0^t \int_{L_z} \varphi_{,\,1}\varphi_{,\,1\beta\beta} dx_2 d\eta \Big\}$$

$$-\mu^* \int_0^t \int_{L_z} \varphi_{,\,11}\varphi_{,\,1\beta\beta} dx_2 d\eta + \int_0^t \int_{R_z} \varphi_{,\,1}\Delta\varphi_{,\,1\eta} dA d\eta + \int_0^t \int_{R_z} \varphi_{,\,1}\Big[-\tau v_{1,\,12} + \tau v_{2,\,11}\Big] dA d\eta. \tag{34}$$

Since

$$\int_0^t \int_{R_z} \varphi_{,\,1}\Delta\varphi_{,\,1\eta} dA d\eta = -\int_0^t \int_{L_z} \varphi_{,\,1}\varphi_{,\,11\eta} dx_2 d\eta - \int_0^t \int_{R_z} \varphi_{,\,1\alpha}\varphi_{,\,1\alpha\eta} dA d\eta$$

$$= -\frac{\partial}{\partial z} \int_0^t \int_{L_z} \varphi_{,\,1}\varphi_{,\,1\eta} dx_2 d\eta + \int_0^t \int_{L_z} \varphi_{,\,11}\varphi_{,\,1\eta} dx_2 d\eta - \frac{1}{2} \int_{R_z} \varphi_{,\,1\alpha}\varphi_{,\,1\alpha} dA \Big|_{\eta=t}$$

and

$$\int_0^t \int_{R_z} \varphi_{,\,1}\Big[-\tau v_{1,\,12} + \tau v_{2,\,11}\Big] dA d\eta = \tau \int_0^t \int_{R_z} \varphi_{,\,12} v_{1,\,1} dA d\eta - \tau \int_0^t \int_{L_z} v_{2,\,1}\varphi_{,\,1} dx_2 d\eta - \tau \int_0^t \int_{R_z} \varphi_{,\,11} v_{2,\,1} dA d\eta,$$

we conclude that

$$\Phi_3(z,\, t) + \frac{1}{2} \int_{R_z} \varphi_{,\,1\alpha}\varphi_{,\,1\alpha} dA \Big|_{\eta=t} = y_{31}(z,\, t) + y_{32}(z,\, t) + y_{33}(z,\, t), \tag{35}$$

where

$$y_{31}(z,\, t) = \frac{\partial}{\partial z} \int_0^t \int_{L_z} \Big[-\frac{\mu^*}{2}\varphi_{,\,1\alpha}\varphi_{,\,1\alpha} + \mu^*\varphi_{,\,1}\varphi_{,\,1\beta\beta} - \varphi_{,\,1}\varphi_{,\,1\eta} \Big] dx_2 d\eta, \tag{36}$$

$$y_{32}(z,\, t) = \int_0^t \int_{L_z} \Big[-\mu_1^*\varphi_{,\,11}\varphi_{,\,1\beta\beta} - \tau v_{2,\,1}\varphi_{,\,1} + \varphi_{,\,11}\varphi_{,\,1\eta} \Big] dx_2 d\eta, \tag{37}$$

$$y_{33}(z,\, t) = \int_0^t \int_{R_z} \Big[\tau\varphi_{,\,12} v_{1,\,1} - \tau\varphi_{,\,11} v_{2,\,1} \Big] dA d\eta. \tag{38}$$

Now, we define

$$\Phi(z,\, t)=\mu^*\int_0^t\int_{R_z}\varphi_{,\,\alpha\beta}\varphi_{,\,\alpha\beta}dAd\eta+\delta\int_0^t\int_{R_z}\varphi_{,\,\alpha\eta}\varphi_{,\,\alpha\eta}dAd\eta+\delta\mu^*\int_0^t\int_{R_z}\varphi_{,\,1\alpha\beta}\varphi_{,\,1\alpha\beta}dAd\eta, \tag{39}$$

where $\delta>0$. Then, we let

$$\begin{aligned}\Psi(z,\, t)=&\int_z^\infty\Phi(\xi,\, t)d\xi\\ =&\mu^*\int_0^t\int_{R_z}(\xi-z)\varphi_{,\,\alpha\beta}\varphi_{,\,\alpha\beta}dAd\eta+\delta\int_0^t\int_{R_z}(\xi-z)\varphi_{,\,\alpha\eta}\varphi_{,\,\alpha\eta}dAd\eta\\ &+\delta\mu^*\int_0^t\int_{R_z}(\xi-z)\varphi_{,\,1\alpha\beta}\varphi_{,\,1\alpha\beta}dAd\eta.\end{aligned} \tag{40}$$

Clearly, we find

$$-\frac{\partial}{\partial z}\Psi(z,\, t)=\Phi(z,\, t), \tag{41}$$

and

$$\frac{\partial^2}{\partial z^2}\Psi(z,\, t)=\mu^*\int_0^t\int_{L_z}\varphi_{,\,\alpha\beta}\varphi_{,\,\alpha\beta}dx_2d\eta+\delta\int_0^t\int_{L_z}\varphi_{,\,\alpha\eta}\varphi_{,\,\alpha\eta}dx_2d\eta+\mu^*\delta\int_0^t\int_{L_z}\varphi_{,\,1\alpha\beta}\varphi_{,\,1\alpha\beta}dx_2d\eta. \tag{42}$$

A combination of (23), (30) and (35) leads to

$$\Psi(z,\, t)+\frac{1}{2}\int_{R_z}(\xi-z)\Big[\delta\varphi_{,\,1\alpha}\varphi_{,\,1\alpha}+\mu^*\delta\varphi_{,\,\alpha\beta}\varphi_{,\,\alpha\beta}+\varphi_{,\,\alpha}\varphi_{,\,\alpha}\Big]dA\Big|_{\eta=t}=J_1+J_2+J_3, \tag{43}$$

where

$$\begin{aligned}J_1=&-\int_0^t\int_{L_z}\Big[\mu^*\varphi_{,\,11}\varphi-\mu^*\varphi_{,\,\alpha}\varphi_{,\,\alpha}-\tau v_{2,\,1}\varphi-\frac{1}{2}\delta\varphi_{,\,\eta}^2+\mu^*\delta\varphi_{,\,11}\varphi_{,\,\eta}\\ &+\delta\tau v_{1,\,2}\varphi_{,\,\eta}-\delta\tau v_{2,\,1}\varphi_{,\,\eta}-\frac{\mu^*}{2}\delta\varphi_{,\,1\alpha}\varphi_{,\,1\alpha}+\mu^*\delta\varphi_{,\,1}\varphi_{,\,1\beta\beta}-\delta\varphi_{,\,1}\varphi_{,\,1\eta}\Big]dx_2d\eta,\end{aligned} \tag{44}$$

$$J_2=\int_z^\infty\Big[y_{12}(\xi,\, t)+\delta y_{22}(\xi,\, t)+\delta y_{32}(\xi,\, t)\Big]d\xi, \tag{45}$$

$$J_3=\int_z^\infty\Big[y_{13}(\xi,\, t)+\delta y_{23}(\xi,\, t)+\delta y_{33}(\xi,\, t)\Big]d\xi. \tag{46}$$

4. Some useful lemmas

To derive continuous dependence result we will use the basic results which have been derived in section 3. In addition, we will also frequently use the following well-known inequalities:

(1). If $\omega(x_2) \in C^1(0,\ h)$ and $\omega(0) = \omega(h) = 0$, then

$$\int_{L_z} \omega^2 dx_2 \le \frac{h^2}{\pi^2} \int_{L_z} \omega_{,2}^2 dx_2. \tag{47}$$

(2). If $\omega(x_2) \in C^2(0,\ h)$ and $\omega(0) = \omega_{,2}(0) = \omega(h) = \omega_{,2}(h) = 0$, then

$$\int_{L_z} \omega_{,2}^2 dx_2 \le \frac{h^2}{4\pi^2} \int_{L_z} \omega_{,22}^2 dx_2. \tag{48}$$

(3). If $\omega(x_2) \in C^2(0,\ h)$ and $\omega(0) = \omega_{,2}(0) = \omega(h) = \omega_{,2}(h) = 0$, then

$$\int_{L_z} \omega^2 dx_2 \le \left(\frac{2}{3}\right)^4 \frac{h^4}{\pi^4} \int_{L_z} \omega_{,22}^2 dx_2. \tag{49}$$

These inequalities can be found in Ref [22]. In addition to (15)-(17), we also use some Sobolev inequality in $R_z \times [0,\ T]$ which have been widely used in the study of Navier-Stokes equations or Boussinesq equations (see e.g. Refs.[24–28]).

Next, we derive upper bounds for J_1, J_2 and J_3. We have the following lemmas.

Lemma 4 For J_1 which defined in (44), we have the following inequality

$$J_1 \le k_1(t)\frac{\partial^2}{\partial z^2}\Psi(z,\ t) + \frac{11h^2}{9\pi^2}\tau^2 \int_0^t \int_{L_z} v_{\alpha,\ \beta} v_{\alpha,\ \beta} dx_2 d\eta,$$

where $k_1(t)$ is a positive function which only depends on t.

Proof. Using the Schwarz inequality and (49), we have

$$\begin{aligned} -\mu^* \int_0^t \int_{L_z} \varphi_{,11}\varphi dx_2 d\eta &\le \mu^* \left[\int_0^t \int_{L_z} \varphi_{,11}^2 dx_2 d\eta \int_0^t \int_{L_z} \varphi^2 dx_2 d\eta\right]^{\frac{1}{2}} \\ &\le \frac{4h^2}{9\pi^2}\mu^* \left[\int_0^t \int_{L_z} \varphi_{,11}^2 dx_2 d\eta \int_0^t \int_{L_z} \varphi_{,22}^2 dx_2 d\eta\right]^{\frac{1}{2}} \\ &\le \frac{2h^2}{9\pi^2}\frac{\partial^2}{\partial z^2}\Psi(z,\ t). \end{aligned} \tag{50}$$

Using the Schwarz inequality and (47), we have

$$\int_0^t \int_{L_z} \left[\mu^* \varphi_{,\,\alpha} \varphi_{,\,\alpha} + \frac{1}{2} \delta \varphi_{,\,\eta}^2 - \mu^* \delta \varphi_{,\,11} \varphi_{,\,\eta} - \mu^* \delta \varphi_{,\,1} \varphi_{,\,1\beta\beta} - \varphi_{,\,1} \varphi_{,\,1\eta}\right] dx_2 d\eta$$

$$\leq \frac{h^2}{2\pi^2} \mu^* \int_0^t \int_{L_z} \varphi_{,\,\alpha 2} \varphi_{,\,\alpha 2} dx_2 d\eta + \frac{h^2}{2\pi^2} \delta \int_0^t \int_{L_z} \varphi_{,\,2\eta}^2 dx_2 d\eta$$

$$+ \frac{h}{2\pi} \sqrt{\delta \mu^*} \left[\mu^* \int_0^t \int_{L_z} \varphi_{,\,11}^2 dx_2 d\eta + \delta \int_0^t \int_{L_z} \varphi_{,\,2\eta}^2 dx_2 d\eta\right]$$

$$+ \frac{h}{2\pi} \sqrt{\delta} \left[\mu^* \int_0^t \int_{L_z} \varphi_{,\,11}^2 dx_2 d\eta + \delta \mu^* \int_0^t \int_{L_z} \varphi_{,\,1\beta\beta}^2 dx_2 d\eta\right]$$

$$+ \frac{h\sqrt{\delta}}{2\sqrt{\mu^*}\pi} \left[\mu^* \int_0^t \int_{L_z} \varphi_{,\,12}^2 dx_2 d\eta + \delta \int_0^t \int_{L_z} \varphi_{,\,1\eta}^2 dx_2 d\eta\right]$$

$$\leq \max\left\{\frac{h^2}{\pi^2},\ \frac{h}{\pi}\sqrt{\delta\mu^*} + \frac{h}{\pi}\sqrt{\delta},\ \frac{h\sqrt{\delta}}{2\sqrt{\mu^*}\pi}\right\} \frac{\partial^2}{\partial z^2} \Psi(z,\ t), \tag{51}$$

and

$$\int_0^t \int_{L_z} \left[\tau v_{2,\,1} \varphi - \delta \tau v_{1,\,2} \varphi_{,\,\eta} + \delta \tau v_{2,\,1} \varphi_{,\,\eta}\right] dx_2 d\eta \leq \frac{2h^2}{9\pi^2} \left[\int_0^t \int_{L_z} \tau^2 v_{2,\,1}^2 dx_2 d\eta + \int_0^t \int_{L_z} \varphi_{,\,22}^2 dx_2 d\eta\right]$$

$$+ \left[\frac{h^2}{2\pi^2} \int_0^t \int_{L_z} \tau^2 v_{1,\,2}^2 dx_2 d\eta + \frac{1}{2} \delta^2 \int_0^t \int_{L_z} \varphi_{,\,2\eta}^2 dx_2 d\eta\right]$$

$$+ \left[\frac{h^2}{2\pi^2} \int_0^t \int_{L_z} \tau^2 v_{2,\,1}^2 dx_2 d\eta + \frac{1}{2} \delta^2 \int_0^t \int_{L_z} \varphi_{,\,2\eta}^2 dx_2 d\eta\right]$$

$$\leq \frac{11h^2}{9\pi^2} \tau^2 \int_0^t \int_{L_z} v_{\alpha,\,\beta} v_{\alpha,\,\beta} dx_2 d\eta$$

$$+ \max\left\{\delta,\ \frac{2h^2}{9\pi^2\mu^*}\right\} \frac{\partial^2}{\partial z^2} \Psi(z,\ t). \tag{52}$$

Choosing $k_1(t) = \frac{1}{2}\delta + \frac{2h^2}{9\pi^2} + \max\left\{\frac{h^2}{\pi^2},\ \frac{h}{\pi}\sqrt{\delta\mu^*} + \frac{h}{\pi}\sqrt{\delta}\right\} + \max\left\{\delta,\ \frac{2h^2}{9\pi^2\mu^*}\right\}$ and combining (50)-(52) and (44), we can get Lemma 4.

Lemma 5 For J_2 which defined in (45), we have the following inequality

$$J_2 \le k_2(t)\Big[-\frac{\partial}{\partial z}\Psi(z,\, t)\Big] + \Big[\frac{2h^2}{\pi^2} + 4 + \frac{h^2}{\pi^2\mu^*\sqrt{\delta}}\Big]c_2(t)\tau^2 e^{-kz},$$

where $k_2(t)$ is a positive function.

Proof. From (45), (46), (25), (32) and (37), we have

$$\int_z^\infty y_{12}(\xi,\, t)d\xi = \int_0^t\int_{R_z}\Big[\tau v_{1,\,12}\varphi_{,\,1} + \tau v_{2,\,1}\varphi_{,\,1} - \varphi_{,\,1\eta}\varphi + \tau v_{2,\,\alpha}\varphi_{,\,\alpha}\Big]dxd\eta, \tag{53}$$

$$\delta\int_z^\infty y_{22}(\xi,\, t)d\xi = \delta\int_0^t\int_{R_z}\Big[-2\mu^*\varphi_{,\,1\alpha}\varphi_{,\,\alpha\eta} - \tau v_{1,\,2}\varphi_{,\,1\eta} + \tau v_{2,\,1}\varphi_{,\,1\eta} - \tau v_{2,\,\alpha}\varphi_{,\,\alpha\eta} - \tau v_{2,\,\eta}\varphi_{,\,11}\Big]dxd\eta, \tag{54}$$

$$\delta\int_z^\infty y_{32}(\xi,\, t)d\xi = \int_0^t\int_{R_z}\Big[-\mu^*\varphi_{,\,11}\varphi_{,\,1\beta\beta} - \tau v_{2,\,1}\varphi_{,\,1} + \varphi_{,\,11}\varphi_{,\,1\eta}\Big]dxd\eta. \tag{55}$$

Using the Schwarz inequality, Lemma 1 and (47), we have

$$\begin{aligned}\int_0^t\int_{R_z}\Big[\tau v_{1,\,12}\varphi_{,\,1} + \tau v_{2,\,1}\varphi_{,\,1} + \tau v_{2,\,\alpha}\varphi_{,\,\alpha}\Big]dxd\eta \le& \Big[\frac{h^2}{2\pi^2}\delta\int_0^t\int_{R_z}\tau^2 v_{1,\,12}^2 dxd\eta + \frac{1}{2\delta}\int_0^t\int_{R_z}\varphi_{,\,12}^2 dxd\eta\Big]\\ &+ \Big[\frac{h^2}{2\pi^2}\int_0^t\int_{R_z}\tau^2 v_{2,\,1}^2 dxd\eta + \frac{1}{2}\int_0^t\int_{R_z}\varphi_{,\,12}^2 dxd\eta\Big]\\ &+ \Big[\frac{h^2}{2\pi^2}\int_0^t\int_{R_z}\tau^2 v_{2,\,\alpha}v_{2,\,\alpha}dxd\eta + \frac{1}{2}\int_0^t\int_{R_z}\varphi_{,\,\alpha2}\varphi_{,\,\alpha2}dxd\eta\Big]\\ \le& \frac{2h^2}{\pi^2}\tau^2 c_2(t)e^{-kz} + \Big[\frac{1}{2\delta} + 1\Big]\Big[-\frac{\partial}{\partial z}\Psi(z,\, t)\Big].\end{aligned} \tag{56}$$

Using (49), we have

$$\begin{aligned}-\int_0^t\int_{R_z}\varphi_{,\,1\eta}\varphi dxd\eta \le& \frac{2h^2}{9\pi^2\sqrt{\mu^*\delta}}\Big[\delta\int_0^t\int_{R_z}\varphi_{,\,1\eta}^2 dxd\eta + \mu^*\int_0^t\int_{R_z}\varphi_{,\,22}^2 dxd\eta\Big]\\ \le& \frac{2h^2}{9\pi^2\sqrt{\mu^*\delta}}\Big[-\frac{\partial}{\partial z}\Psi(z,\, t)\Big].\end{aligned} \tag{57}$$

Inserting (56) and (57) into (53), we have

$$\int_z^\infty y_{12}(\xi,\, t)d\xi \le \frac{2h^2}{\pi^2}\tau^2 c_2(t)e^{-kz} + \Big[\frac{1}{2\delta} + 1 + \frac{2h^2}{9\pi^2\sqrt{\mu^*\delta}}\Big]\Big[-\frac{\partial}{\partial z}\Psi(z,\, t)\Big]. \tag{58}$$

Using the Hölder inequality, we obtain

$$-2\mu^*\delta\int_0^t\int_{R_z}\varphi_{,\,1\alpha}\varphi_{,\,\alpha\eta}dxd\eta \le \sqrt{\mu^*\delta}\Big[\mu^*\int_0^t\int_{R_z}\varphi_{,\,1\alpha}\varphi_{,\,1\alpha}dxd\eta+\delta\int_0^t\int_{R_z}\varphi_{,\,\alpha\eta}\varphi_{,\,\alpha\eta}dxd\eta\Big]$$

$$\le \sqrt{\mu^*\delta}\Big[-\frac{\partial}{\partial z}\Psi(z,\,t)\Big]. \tag{59}$$

Using the Höldrer inequality and Lemma 1, we obtain

$$\delta\int_0^t\int_{R_z}\Big[-\tau v_{1,\,2}\varphi_{,\,1\eta}+\tau v_{2,\,1}\varphi_{,\,1\eta}-\tau v_{2,\,\alpha}\varphi_{,\,\alpha\eta}-\tau v_{2,\,\eta}\varphi_{,\,11}\Big]dxd\eta$$

$$\le\Big[\int_0^t\int_{R_z}\tau^2v_{1,\,2}^2dxd\eta+\frac{1}{4}\delta^2\int_0^t\int_{R_z}\varphi_{,\,1\eta}^2dxd\eta\Big]$$

$$+\Big[\int_0^t\int_{R_z}\tau^2v_{2,\,1}^2dxd\eta+\frac{1}{4}\delta^2\int_0^t\int_{R_z}\varphi_{,\,1\eta}^2dxd\eta\Big]$$

$$+\Big[\int_0^t\int_{R_z}\tau^2v_{2,\,\alpha}v_{2,\,\alpha}dxd\eta+\frac{1}{4}\delta^2\int_0^t\int_{R_z}\varphi_{,\,\alpha\eta}\varphi_{,\,\alpha\eta}dxd\eta\Big]$$

$$+\Big[\int_0^t\int_{R_z}\tau^2v_{2,\,\eta}^2dxd\eta+\frac{1}{4}\delta^2\int_0^t\int_{R_z}\varphi_{,\,11}^2dxd\eta\Big]$$

$$\le 4\tau^2c_2(t)e^{-kz}+\Big[\frac{1}{2}\delta+\frac{1}{4\mu^*}\delta^2\Big]\Big[-\frac{\partial}{\partial z}\Psi(z,\,t)\Big]. \tag{60}$$

Inserting (59) and (60) into (54), we have

$$\int_z^\infty y_{22}(\xi,\,t)d\xi\le 4\tau^2c_2(t)e^{-kz}+\Big[\sqrt{\mu^*\delta}+\frac{1}{2}\delta+\frac{1}{4\mu^*}\delta^2\Big]\Big[-\frac{\partial}{\partial z}\Psi(z,\,t)\Big]. \tag{61}$$

Similar, we have

$$\delta\int_z^\infty y_{32}(\xi,\ t)d\xi \le \frac{\sqrt{\delta}}{2}\Big[\mu^*\int_0^t\int_{R_z}\varphi_{,11}^2 dxd\eta + \delta\mu^*\int_0^t\int_{R_z}\varphi_{,1\beta\beta}^2 dxd\eta\Big]$$

$$+\frac{h^2\tau^2}{2\pi^2\mu^*\sqrt{\delta}}\int_0^t\int_{R_z} v_{2,\ 1}^2 dxd\eta + \frac{\sqrt{\delta}}{2}\mu^*\int_0^t\int_{R_z}\varphi_{,12}^2 dxd\eta$$

$$+\sqrt{\frac{\delta}{\mu^*}}\Big[\mu^*\int_0^t\int_{R_z}\varphi_{,11}^2 dxd\eta + \delta\int_0^t\int_{R_z}\varphi_{,1\eta}^2 dxd\eta\Big]$$

$$\le \max\Big\{\frac{\sqrt{\delta}}{2},\ \sqrt{\frac{\delta}{\mu^*}}\Big\}\Big[-\frac{\partial}{\partial z}\Psi(z,\ t)\Big] + \frac{h^2\tau^2}{\pi^2\mu^*\sqrt{\delta}}c_2(t)e^{-kz}. \tag{62}$$

Inserting (58), (61) and (62) into (45) and choosing $k_2(t) = \max\Big\{\frac{\sqrt{\delta}}{2},\ \sqrt{\frac{\delta}{\mu^*}}\Big\} + \frac{1}{4\mu^*}\delta^2 + \frac{1}{2\delta} + 1 + \frac{2h^2}{9\pi^2\sqrt{\mu^*\delta}} + \frac{\sqrt{\delta}}{2}$, we can obtain Lemma 5.

Lemma 6 For J_3 which defined in (46), we have the following inequality

$$J_3 \le k_3(t)\tau^2 e^{-kz} + \frac{1}{2}\int_{R_z}(\xi - z)\varphi_{,\ \alpha 1}\varphi_{,\ \alpha 1}dx\Big|_{\eta=t} + \frac{1}{2}\Psi(z,\ t).$$

where $k_3(t) = \frac{2}{k}\max\Big\{\frac{1}{\mu^*},\ \frac{h^2}{\delta\mu^*\pi^2} + \frac{1}{\mu^*}\delta^2,\ \frac{\pi^2}{h^2\mu^*\delta},\ \delta^2\Big\}c_2(t)$.

Proof. Inserting (26), (33) and (38) into (46), we have

$$J_3 = \int_0^t\int_{R_z}(\xi - z)\Big[\tau v_{1,\ 2\alpha}\varphi_{,\ \alpha} + \tau v_{2,\ \alpha}\varphi_{,\ 1\alpha} - \tau\delta v_{2,\ \eta}\varphi_{,\ 1\alpha\alpha}\Big]dAd\eta$$

$$+\int_0^t\int_{R_z}(\xi - z)\Big[\tau\delta\varphi_{,\ 12}v_{1,\ 1} - \tau\delta\varphi_{,\ 11}v_{2,\ 1}\Big]dAd\eta - \tau\delta\int_{R_z}(\xi - z)v_{2,\ \alpha}\varphi_{,\ 1\alpha}dA\Big|_{\eta=t}. \tag{63}$$

Using the Hölder inequality, Young's inequality and Lemma 1, we have

$$\int_0^t \int_{R_z} (\xi - z)\Big[\tau v_{1,\, 2\alpha}\varphi_{,\, \alpha} + \tau v_{2,\, \alpha}\varphi_{,\, 1\alpha} - \tau\delta v_{2,\, \eta}\varphi_{,\, 1\alpha\alpha}\Big] dAd\eta$$

$$\leq \frac{h}{\pi}\Big[\int_0^t \int_{R_z} (\xi - z)\tau^2 v_{1,\, 2\alpha} v_{1,\, 2\alpha} dxd\eta\Big]^{\frac{1}{2}} \Big[\int_0^t \int_{R_z} (\xi - z)\varphi_{,\, 2\alpha}\varphi_{,\, 2\alpha} dxd\eta\Big]^{\frac{1}{2}}$$

$$+ \frac{h}{\pi}\Big[\int_0^t \int_{R_z} (\xi - z)\tau^2 v_{2,\, \alpha} v_{2,\, \alpha} dxd\eta\Big]^{\frac{1}{2}} \Big[\int_0^t \int_{R_z} (\xi - z)\varphi_{,\, 12\alpha}\varphi_{,\, 12\alpha} dxd\eta\Big]^{\frac{1}{2}}$$

$$+ \delta\Big[\int_0^t \int_{R_z} (\xi - z)\tau^2 v_{2,\, \eta}^2 dxd\eta\Big]^{\frac{1}{2}} \Big[\int_0^t \int_{R_z} (\xi - z)\varphi_{,\, 1\alpha\alpha}^2 dxd\eta\Big]^{\frac{1}{2}}$$

$$\leq \frac{h^2\tau^2}{\mu^*\pi^2}\int_0^t \int_{R_z} (\xi - z) v_{1,\, 2\alpha} v_{1,\, 2\alpha} dxd\eta + \frac{1}{4}\mu^* \int_0^t \int_{R_z} (\xi - z)\varphi_{,\, 2\alpha}\varphi_{,\, 2\alpha} dxd\eta$$

$$+ \frac{h^2\tau^2}{\delta\mu^*\pi^2}\int_0^t \int_{R_z} (\xi - z) v_{2,\, \alpha} v_{2,\, \alpha} dxd\eta + \frac{1}{4}\mu^*\delta \int_0^t \int_{R_z} (\xi - z)\varphi_{,\, 12\alpha}\varphi_{,\, 12\alpha} dxd\eta$$

$$+ \frac{\tau^2\delta}{\mu^*}\int_0^t \int_{R_z} (\xi - z) v_{2,\, \eta}^2 dxd\eta + \frac{1}{4}\mu^*\delta \int_0^t \int_{R_z} (\xi - z)\varphi_{,\, 1\alpha\alpha}^2 dxd\eta, \tag{64}$$

$$\int_0^t \int_{R_z} (\xi - z)\Big[\tau\delta\varphi_{,\, 12} v_{1,\, 1} - \tau\delta\varphi_{,\, 11} v_{2,\, 1}\Big] dAd\eta$$

$$\leq \frac{1}{\mu^*}\tau^2\delta^2 \int_0^t \int_{R_z} (\xi - z) v_{1,\, 1}^2 dxd\eta + \frac{1}{4}\mu^* \int_0^t \int_{R_z} (\xi - z)\varphi_{,\, 12}^2 dxd\eta$$

$$+ \frac{1}{\mu^*}\tau^2\delta^2 \int_0^t \int_{R_z} (\xi - z) v_{2,\, 1}^2 dxd\eta + \frac{1}{4}\mu^* \int_0^t \int_{R_z} (\xi - z)\varphi_{,\, 11}^2 dxd\eta$$

$$\leq \frac{1}{\mu^*}\tau^2\delta^2 \int_0^t \int_{R_z} (\xi - z) v_{\alpha,\, 1} v_{\alpha,\, 1} dxd\eta + \frac{1}{4}\mu^* \int_0^t \int_{R_z} (\xi - z)\varphi_{,\, \alpha 1}\varphi_{,\, \alpha 1} dxd\eta \tag{65}$$

and

$$-\tau\delta \int_{R_z} (\xi - z) v_{2,\, \alpha}\varphi_{,\, 1\alpha} dA\Big|_{\eta = t} \leq \frac{1}{2}\tau^2\delta^2 \int_{R_z} (\xi - z) v_{2,\, \alpha} v_{2,\, \alpha} dx\Big|_{\eta = t} + \frac{1}{2}\int_{R_z} (\xi - z)\varphi_{,\, \alpha 1}\varphi_{,\, \alpha 1} dx\Big|_{\eta = t}. \tag{66}$$

Inserting (64)-(66) into (63), we can have

$$J_3 \leq \frac{h^2\tau^2}{\mu^*\pi^2}\int_0^t\int_{R_z}(\xi-z)v_{1,\,2\alpha}v_{1,\,2\alpha}dxd\eta+\Big[\frac{h^2}{\delta\mu^*\pi^2}+\frac{1}{\mu^*}\delta^2\Big]\tau^2\int_0^t\int_{R_z}(\xi-z)v_{\alpha,\,\beta}v_{\alpha,\,\beta}dxd\eta$$

$$+\frac{\tau^2\delta}{\mu^*}\int_0^t\int_{R_z}(\xi-z)v_{2,\,\eta}^2dxd\eta+\frac{1}{2}\tau^2\delta^2\int_{R_z}(\xi-z)v_{2,\,\alpha}v_{2,\,\alpha}dx\Big|_{\eta=t}$$

$$+\frac{1}{2}\int_{R_z}(\xi-z)\varphi_{,\,\alpha 1}\varphi_{,\,\alpha 1}dx\Big|_{\eta=t}+\frac{1}{2}\Psi(z,\,t). \tag{67}$$

In (67) using Lemma 1 we can obtain Lemma 6.

In next section, we use Lemmas 4, 5 and 6 to proof our main results.

5. The proof of Theorem 1

Combining Lemmas 4-6 and (43), we have

$$\Psi(z,\,t)\leq 2k_1(t)\frac{\partial^2}{\partial z^2}\Psi(z,\,t)+2k_2(t)\Big[-\frac{\partial}{\partial z}\Psi(z,\,t)\Big]+\frac{22h^2}{9\pi^2}\tau^2\int_0^t\int_{L_z}v_{\alpha,\,\beta}v_{\alpha,\,\beta}dx_2d\eta+\tau^2k_4(t)e^{-kz}, \tag{68}$$

where $k_4(t)=2\Big[k_3(t)+\frac{2h^2}{\pi^2}+4+\frac{h^2}{\pi^2\mu^*\sqrt{\delta}}\Big]c_2(t)$. From (68), we obtain an inequality of the form

$$\frac{\partial^2}{\partial z^2}\Psi(z,\,t)-\widetilde{k}_1\frac{\partial}{\partial z}\Psi(z,\,t)-\widetilde{k}_2\Psi(z,\,t)\geq-\tau^2\widetilde{k}_3(t)e^{-kz}-\tau^2\widetilde{k}_4(t)\int_0^t\int_{L_z}v_{\alpha,\,\beta}v_{\alpha,\,\beta}dx_2d\eta, \tag{69}$$

where $\widetilde{k}_1=\frac{k_2(t)}{k_1(t)}$, $\widetilde{k}_2=\frac{1}{2k_1(t)}$, $\widetilde{k}_3=\frac{k_4(t)}{2k_1(t)}$, $\widetilde{k}_4(t)=\frac{11h^2}{9k_1(t)\pi^2}$. From (69) it follows that

$$\Big[\frac{\partial}{\partial z}-a\Big]\Big[\frac{\partial}{\partial z}\Psi(z,\,t)+b\Psi(z,\,t)\Big]\geq-\tau^2\widetilde{k}_3e^{-kz}-\tau^2\widetilde{k}_4(t)\int_0^t\int_{L_z}v_{\alpha,\,\beta}v_{\alpha,\,\beta}dx_2d\eta,$$

or

$$\frac{\partial}{\partial z}\Big\{e^{-az}\Big[\frac{\partial}{\partial z}\Psi(z,\,t)+b\Psi(z,\,t)\Big]\Big\}\geq-\tau^2\widetilde{k}_3(t)e^{-(a+k)z}-\tau^2\widetilde{k}_4(t)e^{-az}\int_0^t\int_{L_z}v_{\alpha,\,\beta}v_{\alpha,\,\beta}dx_2d\eta, \tag{70}$$

where

$$a=\frac{\widetilde{k}_1+\sqrt{\widetilde{k}_1^2+4\widetilde{k}_2}}{2},\ b=\frac{-\widetilde{k}_1+\sqrt{\widetilde{k}_1^2+4\widetilde{k}_2}}{2}.$$

An integration of (70) from z to ∞ leads to

$$\frac{\partial}{\partial z}\Psi(z,\ t)+b\Psi(z,\ t)\leq\tau^2\frac{1}{a+k}\widetilde{k}_3(t)e^{-kz}+\tau^2\widetilde{k}_4(t)e^{az}\int_z^\infty e^{-a\xi}\Big(\int_0^t\int_{L_\xi}v_{\alpha,\ \beta}v_{\alpha,\ \beta}dx_2d\eta\Big)d\xi. \tag{71}$$

Since

$$e^{-a\xi}\leq e^{-az},\ \xi\geq z,$$

we have by using Lemma 1

$$\int_z^\infty e^{-a\xi}\Big(\int_0^t\int_{L_\xi}v_{\alpha,\ \beta}v_{\alpha,\ \beta}dx_2d\eta\Big)d\xi\leq e^{-az}\int_0^t\int_{R_z}v_{\alpha,\ \beta}v_{\alpha,\ \beta}dAd\eta\leq 2c_2(t)e^{-(a+k)z}. \tag{72}$$

Inserting (72) into (71), we have

$$\frac{\partial}{\partial z}\Psi(z,\ t)+b\Psi(z,\ t)\leq\tau^2\widetilde{k}_5(t)e^{-kz}, \tag{73}$$

where $\widetilde{k}_5(t)=\dfrac{1}{a+k}\widetilde{k}_3+2c_2(t)\widetilde{k}_4(t)$. The inequality (73) can be rewritten as

$$\frac{\partial}{\partial z}\Big[\Psi(z,\ t)e^{bz}\Big]\leq\tau^2\widetilde{k}_5(t)e^{(b-k)z}. \tag{74}$$

(A) If $b\neq k$, then integrating (74) from 0 to z, we have

$$\Psi(z,\ t)\leq\Psi(0,\ t)e^{-bz}+\tau^2\frac{1}{b-k}\widetilde{k}_5(t)(e^{-kz}-e^{-bz}). \tag{75}$$

(B) If $b=k$, then integrating (74) from 0 to z, we have

$$\Psi(z,\ t)\leq\Psi(0,\ t)e^{-bz}+\tau^2\widetilde{k}_5(t)ze^{-bz}. \tag{76}$$

To establish continuous dependence on the coefficients μ, we have to give the upper bound for $\Psi(0,\ t)$. From (73), we have

$$b\Psi(0,\ t)\leq-\frac{\partial}{\partial z}\Psi(0,\ t)+\tau^2\widetilde{k}_5(t). \tag{77}$$

So, we only need to bound $-\dfrac{\partial}{\partial z}\Psi(0,\ t)$. We choose $z=0$ in (39) and combine (15) to have

$$-\frac{\partial}{\partial z}\Psi(0,\, t) = \mu^* \int_0^t \int_{R_0} \varphi_{,\,\alpha\beta}\varphi_{,\,\alpha\beta} dAd\eta + \delta \int_0^t \int_{R_0} \varphi_{,\,\alpha\eta}\varphi_{,\,\alpha\eta} dAd\eta + \delta\mu^* \int_0^t \int_{R_0} \varphi_{,\,1\alpha\beta}\varphi_{,\,1\alpha\beta} dAd\eta$$

$$= \mu^* \int_0^t \int_{R_0} w_{\alpha,\,\beta} w_{\alpha,\,\beta} dAd\eta + \delta \int_0^t \int_{R_0} w_{\alpha,\,\eta} w_{\alpha,\,\eta} dAd\eta + \delta\mu^* \int_0^t \int_{R_0} w_{\alpha,\,1\beta} w_{\alpha,\,1\beta} dAd\eta. \tag{78}$$

Multiplying (9) with w_α and integrating in $R \times (0,\, t)$, we have

$$\int_0^t \int_R \left[w_{\alpha,\,\eta} - \tau\Delta v_\alpha - \mu^*\Delta w_\alpha - \pi_{,\,\alpha} \right] w_\alpha dAd\eta = 0. \tag{79}$$

Integrating (79) by parts, we have

$$\frac{1}{2}\int_R w_\alpha w_\alpha dA|_{\eta=t} + \mu^* \int_0^t \int_R w_{\alpha,\,\beta} w_{\alpha,\,\beta} dAd\eta = \tau \int_0^t \int_R v_{\alpha,\,\beta} w_{\alpha,\,\beta} dAd\eta$$

$$\leq \frac{\tau^2}{2\mu^*} \int_0^t \int_R v_{\alpha,\,\beta} v_{\alpha,\,\beta} dAd\eta + \frac{1}{2}\mu^* \int_0^t \int_R w_{\alpha,\,\beta} w_{\alpha,\,\beta} dAd\eta. \tag{80}$$

Using Lemma 2, we have

$$\frac{1}{2}\int_R w_\alpha w_\alpha dA|_{\eta=t} + \mu^* \int_0^t \int_R w_{\alpha,\,\beta} w_{\alpha,\,\beta} dAd\eta \leq \frac{\tau^2}{\mu^*} c_1(t). \tag{81}$$

Multiplying (9) with $w_{\alpha,\,\eta}$ and integrating in $R \times (0,\, t)$, we have

$$\int_0^t \int_R \left[w_{\alpha,\,\eta} - \tau\Delta v_\alpha - \mu^*\Delta w_\alpha - \pi_{,\,\alpha} \right] w_{\alpha,\,\eta} dAd\eta = 0. \tag{82}$$

Integrating (82) by parts, we have

$$\frac{1}{2}\mu^*\int_R w_{\alpha,\ \beta}w_{\alpha,\ \beta}dA\Big|_{\eta=t}+\int_0^t\int_R w_{\alpha,\ \eta}w_{\alpha,\ \eta}dAd\eta$$

$$=\tau\int_0^t\int_R v_{\alpha,\ \beta}w_{\alpha,\ \beta\eta}dAd\eta$$

$$=\tau\int_R v_{\alpha,\ \beta}w_{\alpha,\ \beta}dA\Big|_{\eta=t}-\tau\int_0^t\int_R v_{\alpha,\ \beta\eta}w_{\alpha,\ \beta}dAd\eta$$

$$\le\frac{\tau^2}{2\mu^*}\int_R v_{\alpha,\ \beta}v_{\alpha,\ \beta}dA\Big|_{\eta=t}+\frac{1}{2}\mu^*\int_R w_{\alpha,\ \beta}w_{\alpha,\ \beta}dA\Big|_{\eta=t}$$

$$+\frac{\tau^2}{2\mu^*}\int_0^t\int_R v_{\alpha,\ \beta\eta}v_{\alpha,\ \beta\eta}dAd\eta+\frac{1}{2}\mu^*\int_0^t\int_R w_{\alpha,\ \beta}w_{\alpha,\ \beta}dAd\eta. \tag{83}$$

Using (81) and Lemma 2, we have from (83)

$$\int_0^t\int_R w_{\alpha,\ \eta}w_{\alpha,\ \eta}dAd\eta\le\frac{\tau^2\pi^2}{2h^2\mu^*}c_1(t)+\frac{\tau^2}{2\mu^*}\int_0^t\int_R v_{\alpha,\ \beta\eta}v_{\alpha,\ \beta\eta}dAd\eta+\frac{\tau^2}{2\mu^*}c_1(t). \tag{84}$$

To bound $\int_0^t\int_{R_0}w_{\alpha,\ 1\beta}w_{\alpha,\ 1\beta}dAd\eta$, we consider the following identity

$$\int_0^t\int_R\Big[w_{\alpha,\ 1\eta}-\tau\Delta v_{\alpha,\ 1}-\mu^*\Delta w_{\alpha,\ 1}-\pi_{,\ \alpha 1}\Big]w_{\alpha,\ 1}dAd\eta=0.$$

Therefore, using Lemma 3 we have

$$\frac{1}{2}\int_R w_{\alpha,\ 1}w_{\alpha,\ 1}dA|_{\eta=t}+\mu^*\int_0^t\int_R w_{\alpha,\ \beta 1}w_{\alpha,\ \beta 1}dAd\eta$$

$$\le\Big[\int_0^t\int_R\tau^2 v_{\alpha,\ \beta 1}v_{\alpha,\ \beta 1}dAd\eta\int_0^t\int_R w_{\alpha,\ \beta 1}w_{\alpha,\ \beta 1}dAd\eta\Big]^{\frac{1}{2}}$$

$$\le\frac{1}{2\mu^*}\tau^2c_3(t)+\frac{1}{2}\mu^*\int_0^t\int_R w_{\alpha,\ \beta 1}w_{\alpha,\ \beta 1}dAd\eta. \tag{85}$$

From (85), we have

$$\mu^*\int_0^t\int_R w_{\alpha,\ \beta 1}w_{\alpha,\ \beta 1}dAd\eta\le\frac{1}{\mu^*}\tau^2c_3(t). \tag{86}$$

Combining (81), (84), (86) and (78), we have

$$-\frac{\partial}{\partial z}\Psi(0,\, t) \leq \widetilde{k}_6(t)\tau^2, \tag{87}$$

where

$$\widetilde{k}_6(t) = \frac{1}{\mu^*}c_1(t) + \frac{\pi^2}{2h^2\mu^*}\delta c_1(t) + \frac{1}{2\mu^*}\delta\int_0^t\int_R v_{\alpha,\,\beta\eta}v_{\alpha,\,\beta\eta}dAd\eta + \frac{1}{2\mu^*}\delta c_1(t) + \frac{1}{\mu^*}\delta c_3(t) \tag{88}$$

Inserting (87) into (77), we have

$$\Psi(0,\, t) \leq \frac{1}{b}\widetilde{k}_6(t)\tau^2 + \tau^2\frac{1}{(a+k)b}\widetilde{k}_5(t). \tag{89}$$

Combining (75), (76) and (89), we can obtain

$$\Psi(z,\, t) \leq \left[\frac{1}{b}\widetilde{k}_6(t) + \frac{1}{(a+k)b}\widetilde{k}_5(t)\right]\tau^2 e^{-bz} + \tau^2\frac{1}{(a+k)(b-k)}\widetilde{k}_5(t)(e^{-kz} - e^{-bz}),\ b \neq k,$$

$$\Psi(z,\, t) \leq \left[\frac{1}{b}\widetilde{k}_6(t) + \frac{1}{(a+k)b}\widetilde{k}_5(t)\right]\tau^2 e^{-bz} + \tau^2\frac{1}{a+k}\widetilde{k}_5(t)ze^{-bz},\ b = k.$$

Choosing that

$$L_1(t) = \frac{1}{b}\widetilde{k}_6(t) + \frac{1}{(a+k)b}\widetilde{k}_5(t), L_2(t) = \frac{1}{a+k}\widetilde{k}_5(t)$$

and combining (40) and (15), we can complete the proof of Theorem 1.

6. The proof of Theorem 2

In this section, we seek bound for $\int_0^t\int_{R_z} v_{\alpha,\,\beta\eta}v_{\alpha,\,\beta\eta}dAd\eta$. To do this, we introduce another stream function $u(x_1,\, x_2,\, t)$ such that

$$v_1 = u_{,\,2},\ v_2 = -u_{,\,1}. \tag{90}$$

The initial-boundary problem (3)-(8) can be rewritten as

$$\mu\Delta^2 u = (\Delta u)_t,\ in\ R\times[0,\, \infty)$$

$$u(x_1,\, 0,\, t) = u_n(x_1,\, 0,\, t) = 0,\ x_1 > 0,\ t > 0,$$

$$u(x_1,\ h,\ t) = u_n(x_1,\ h,\ t) = 0,\ x_1 > 0,\ t > 0,$$

$$u(0,\ x_2,\ t) = g_1(x_2,\ t) = \int_0^{x_2} f_1(s,\ t)ds,\ 0 \le x_2 \le h,\ t > 0,$$

$$u_{,\,1}(0,\ x_2,\ t) = -f_2(x_2,\ t),\ 0 \le x_2 \le h,\ t > 0,$$

$$u_{,\,\alpha}(x_1,\ x_2,\ 0) = 0,\ in\ R.$$

Now, we introduce an auxiliary function

$$\widehat{u}(x_1,\ x_2,\ t) = \left[\int_0^{x_2} \frac{\partial g_1}{\partial \xi}(\xi,\ t)d\xi + x_1\left(g_1(x_2,\ t) + \int_0^{x_2} \frac{\partial g_1}{\partial \xi}(\xi,\ t)d\xi\right)\right]e^{-x_1}.$$

Obviously, $\widehat{u}(x_1,\ x_2,\ t)$ has the same initial-boundary conditions as u. We compute

$$\begin{aligned}
\mu\int_0^t\int_R v_{\alpha,\,\beta\eta}\widehat{u}_{,\,\alpha\beta\eta}dAd\eta =& \mu\int_0^t\int_R u_{,\,\alpha\beta\eta}\widehat{u}_{,\,\alpha\beta\eta}dAd\eta \\
=& \mu\int_0^t\int_R u_{,\,\alpha\beta\eta}u_{,\,\alpha\beta\eta}dAd\eta + \mu\int_0^t\int_R u_{,\,\alpha\beta\eta}\left[\widehat{u}_{,\,\alpha\beta\eta} - u_{,\,\alpha\beta\eta}\right]dAd\eta \\
=& \mu\int_0^t\int_R u_{,\,\alpha\beta\eta}u_{,\,\alpha\beta\eta}dAd\eta - \mu\int_0^t\int_R u_{,\,\alpha\beta\beta\eta}\left[\widehat{u}_{,\,\alpha\eta} - u_{,\,\alpha\eta}\right]dAd\eta \\
=& \mu\int_0^t\int_R u_{,\,\alpha\beta\eta}u_{,\,\alpha\beta\eta}dAd\eta + \mu\int_0^t\int_R \Delta^2 u_\eta\left[\widehat{u}_\eta - u_\eta\right]dAd\eta \\
=& \mu\int_0^t\int_R u_{,\,\alpha\beta\eta}u_{,\,\alpha\beta\eta}dAd\eta + \int_0^t\int_R \Delta u_{\eta\eta}\left[\widehat{u}_\eta - u_\eta\right]dAd\eta \\
=& \mu\int_0^t\int_R u_{,\,\alpha\beta\eta}u_{,\,\alpha\beta\eta}dAd\eta - \int_0^t\int_R u_{,\,\alpha\eta\eta}\left[\widehat{u}_{,\,\alpha\eta} - u_{,\,\alpha\eta}\right]dAd\eta \\
=& \mu\int_0^t\int_R u_{,\,\alpha\beta\eta}u_{,\,\alpha\beta\eta}dAd\eta - \int_R u_{,\,\alpha t}\widehat{u}_{,\,\alpha t}dA\Big|_{\eta=t} \\
& + \int_0^t\int_R u_{,\,\alpha\eta}\widehat{u}_{,\,\alpha\eta\eta}dAd\eta + \frac{1}{2}\int_R u_{,\,\alpha t}u_{,\,\alpha t}dA\Big|_{\eta=t}.
\end{aligned} \tag{91}$$

Using the Schwarz inequality and (47), we have

$$-\int_R u_{,\alpha t}\widehat{u}_{,\alpha t}dA\Big|_{\eta=t} \geq -\left[\int_R u_{,\alpha t}u_{,\alpha t}dA\Big|_{\eta=t}\right]^{\frac{1}{2}}\left[\int_R \widehat{u}_{,\alpha t}\widehat{u}_{,\alpha t}dA\Big|_{\eta=t}\right]^{\frac{1}{2}}$$

$$\geq -\frac{1}{2}\int_R u_{,\alpha t}u_{,\alpha t}dA\Big|_{\eta=t} - \frac{1}{2}\int_R \widehat{u}_{,\alpha t}\widehat{u}_{,\alpha t}dA\Big|_{\eta=t}, \tag{92}$$

and

$$\int_0^t\int_R u_{,\alpha\eta}\widehat{u}_{,\alpha\eta\eta}dAd\eta \geq -\frac{h}{\pi}\left[\int_0^t\int_R u_{,\alpha 2\eta}u_{,\alpha 2\eta}dAd\eta\right]^{\frac{1}{2}}\left[\int_0^t\int_R \widehat{u}_{,\alpha\eta\eta}\widehat{u}_{,\alpha\eta\eta}dAd\eta\right]^{\frac{1}{2}}$$

$$\geq -\frac{1}{4}\mu\int_0^t\int_R u_{,\alpha 2\eta}u_{,\alpha 2\eta}dAd\eta - \frac{h^2}{\mu\pi^2}\int_0^t\int_R \widehat{u}_{,\alpha\eta\eta}\widehat{u}_{,\alpha\eta\eta}dAd\eta. \tag{93}$$

Inserting (92) and (93) into (91), we have

$$\mu\int_0^t\int_R u_{,\alpha\beta\eta}\widehat{u}_{,\alpha\beta\eta}dAd\eta \geq \frac{3}{4}\mu\int_0^t\int_R u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}dAd\eta$$

$$-\frac{1}{2}\int_R \widehat{u}_{,\alpha t}\widehat{u}_{,\alpha t}dA\Big|_{\eta=t} - \frac{h^2}{\mu\pi^2}\int_0^t\int_R \widehat{u}_{,\alpha\eta\eta}\widehat{u}_{,\alpha\eta\eta}dAd\eta. \tag{94}$$

On the other hand, we have

$$\mu\int_0^t\int_R u_{,\alpha\beta\eta}\widehat{u}_{,\alpha\beta\eta}dAd\eta \leq \mu\left[\int_0^t\int_R u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}dAd\eta\int_0^t\int_R \widehat{u}_{\alpha,\beta\eta}\widehat{u}_{,\alpha\beta\eta}dAd\eta\right]^{\frac{1}{2}}. \tag{95}$$

If we define

$$\mathscr{F}(u) = \mu\int_0^t\int_R u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}dAd\eta,\ \ \mathscr{F}(\widehat{u}) = \mu\int_0^t\int_R \widehat{u}_{\alpha,\beta\eta}\widehat{u}_{,\alpha\beta\eta}dAd\eta, \tag{96}$$

and

$$Q(t) = \frac{1}{2}\int_R \widehat{u}_{,\alpha t}\widehat{u}_{,\alpha t}dA\Big|_{\eta=t} + \frac{h^2}{\mu\pi^2}\int_0^t\int_R \widehat{u}_{,\alpha\eta\eta}\widehat{u}_{,\alpha\eta\eta}dAd\eta,$$

then we have from (94) and (95)

$$\frac{3}{4}\mathscr{F}(u) - Q(t) \leq \sqrt{\mathscr{F}(u)\mathscr{F}(\widehat{u})}$$

or

$$\left[\sqrt{\mathscr{F}(u)}-\frac{2}{3}\sqrt{\mathscr{F}(\widehat{u})}\right]^2 \leq \frac{4}{3}Q(t)+\frac{4}{9}\mathscr{F}(\widehat{u}). \tag{97}$$

From (97) it follows that

$$\mathscr{F}(u) \leq \left[\frac{2}{3}\sqrt{\mathscr{F}(\widehat{u})}+\sqrt{\frac{4}{3}Q(t)+\frac{4}{9}\mathscr{F}(\widehat{u})}\right]^2. \tag{98}$$

Combining (90), (96) and (98) and choosing $n(t)=\left[\frac{2}{3}\sqrt{\mathscr{F}(\widehat{u})}+\sqrt{\frac{4}{3}Q(t)+\frac{4}{9}\mathscr{F}(\widehat{u})}\right]^2$, we can obtain Theorem 2.2.

7. Conclusion

This article establishes the stability of Stokes equation for disturbances in viscosity coefficients. This indicates that small perturbations in the viscosity coefficient will not have a significant impact on the solution. When the viscosity coefficient of the Stokes equation is slightly disturbed by external factors, the properties of the partial differential equation solution (such as existence, uniqueness, and stability) will not undergo significant changes. The topic of this article can be further discussed in depth. For example, considering the structural stability of nonlinear models (e.g., Brinkman equations) on two-dimensional rectangular regions remains an open topic. How to construct prior estimates for nonlinear terms will be a challenge.

Acknowledgement

This work is partial supported by the Guangzhou Huashang College Tutorial Scientific Research Project (2023HSDS21)and the National Natural Science Foundation of China (11371175).

Conflict of interest

The authors declare no competing financial interest.

References

[1] Liu Y, Lin CH. Phragmén-Lindeöf type alternative results for the Stokes flow equation. *Mathematical Inequalities and Applications*. 2006; 9(4): 671-694.

[2] Scott NL. Continuous dependence on boundary reaction terms in a porous medium of Darcy type. *Journal of Mathematical Analysis and Applications*. 2013; 399: 667-675.

[3] Li YF, Chen XJ, Shi JC. Structural stability in resonant penetrative convection in a Brinkman-Forchheimer fluid interfacing with a Darcy fluid. *Applied Mathematics and Optimization*. 2021; 84(1): 979-999.

[4] Liu Y, Qin XL, Shi JC, Zhi WJ. Structural stability of the Boussinesq fluid interfacing with a Darcy fluid in a bounded region in $\mathbb{R}^2$. *Applied Mathematics and Computation*. 2021; 411: 126488.

[5] Scott NL, Straughan B. Continuous dependence on the reaction terms in porous convection with surface reactions. *Quarterly of Applied Mathematics*. 2013; 71: 501-508.

[6] Gentile M, Straughan B. Structural stability in resonant penetrative convection in a Forchheimer porous material. *Nonlinear Analysis: Real World Applications*. 2013; 14(1): 397-401.
[7] Quintanilla R, Racke R. Spatial behavior in phase-lag heat conduction. *Differential and Integral Equations*. 2015; 28: 291-308.
[8] Li YF, Xiao SZ, Zeng P. The applications of some basic mathematical inequalities on the convergence of the primitive equations of moist atmosphere. *Journal of Mathematical Inequalities*. 2021; 15(1): 293-304.
[9] Liu Y, Xiao SZ, Lin YW. Continuous dependence for the Brinkman-Forchheimer fluid interfacing with a Darcy fluid in a bounded domain. *Mathematics and Computers in Simulation*. 2018; 150: 66-82.
[10] Liu Y. Continuous dependence for a thermal convection model with temperature-dependent solubility. *Applied Mathematics and Computation*. 2017; 308: 18-30.
[11] Chen W, Ikehata R. The Cauchy problem for the Moore-Gibson-Thompson equation in the dissipative case. *Journal of Differential Equations*. 2021; 292: 176-219.
[12] Quintanilla R. Damping of end effects in a thermoelastic theory. *Applied Mathematics Letters*. 2001; 14: 137-141.
[13] Shi JC, Liu Y. Structural stability for the Forchheimer equations interfacing with a Darcy fluid in a bounded region in $\mathbb{R}^3$. *Boundary Value Problems*. 2021; 2021: 46.
[14] Chen W. Cauchy problem for thermoelastic plate equations with different damping mechanisms. *Communications in Mathematical Sciences*. 2020; 18(2): 429-457.
[15] Ciarletta M, Straughan B, Tibullo V. Structural stability for a thermal convection model with temperature-dependent solubility. *Nonlinear Analysis: Real World Applications*. 2015; 22: 34-43.
[16] Straughan B. Continuous dependence on the heat source in resonant porous penetrative convection. *Studies in Applied Mathematics*. 2011; 127: 302-314.
[17] Dhayal R, Malik M. Stability analysis of damped fractional stochastic differential systems with Poisson jumps: a successive approximation approach. *International Journal of Systems Science*. 2024; 8: 1-13.
[18] Dhayal R, Malik M, Abbas S. Existence, stability and controllability results of stochastic differential equations with non-instantaneous impulses. *International Journal of Control*. 2021; 95(7): 1719-1730.
[19] Dhayal R, Malik M, Abbas S, Debbouche A. Optimal controls for second-order stochastic differential equations driven by mixed-fractional Brownian motion with impulses. *Mathematical Methods in the Applied Sciences*. 2020; 43(7): 4107-4124.
[20] Li YF, Liu Y, Luo SG, Lin CH. Decay estimates for the Brinkman-Forchheimer equations in a semi-infinite pipe. *Zeitschrift für Angewandte Mathematik und Mechanik [Journal of Applied Mathematics and Mechanics]*. 2012; 92(2): 160-176.
[21] Li YF, Lin C. Continuous dependence for the nonhomogeneous Brinkman-Forchheimer equations in a semi-infinite pipe. *Applied Mathematics and Computation*. 2014; 244: 201-208.
[22] Payne LE, Schaefer PW. Some Phragmén-Lindeöf type results for the biharmonic equation. *Zeitschrift für Angewandte Mathematik und Physik [Journal of Applied Mathematics and Physics]*. 1994; 45: 414-432.
[23] Li YF. Structural stability for double-diffusion Brinkman fluid with Soret effect. *Boundary Value Problems*. 2024; 2024: 135.
[24] Payne LE. *Uniqueness Criteria for Steady State Solutions of the Navier-Stokes Equation*. Edizioni Cremonse, Rome; 1965.
[25] Payne LE. Uniqueness and continuous dependence criteria for the Navier-Stokes equations. *Rocky Mountain Journal of Mathematics*. 1971; 2: 641-660.
[26] Payne LE, Philippin GA. Pointwise bounds and spatial decay estimates in heat conduction problems. *Mathematical Models and Methods in Applied Sciences*. 1995; 5: 755-775.
[27] Payne LE, Schaefer PW, Song JC. Growth and decay results in heat conduction problems with nonlinear boundary conditions. *Nonlinear Analysis*. 1999; 33: 269-286.
[28] Quintanilla R. Some remarks on the fast spatial growth/decay in exterior regions. *Zeitschrift für Angewandte Mathematik und Physik [Journal of Applied Mathematics and Physics]*. 2019; 70: 83.