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Research Article

Peirce Decomposition of Quasi Jordan Algebras

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Abstract: Idempotents play a basic role in the study of algebras. Peirce decomposition induced by an idempotent is an important tool in the structure theory of non-associative algebras. In this note, we investigate the Peirce decomposition of a unital quasi Jordan algebra.

Keywords: quasi Jordan algebras, idempotents, Peirce decomposition

MSC: 17-XX, 17Cxx, 17C27

1. Introduction

Originally, the Peirce decomposition was introduced by Benjamin Peirce for associative algebras [1]. Peirce decomposition is a powerful tool which provides a foundation for deeper analysis of their structure. In the recent past, the study of additivity of mappings such as multiplicative maps, Jordan multiplicative maps, and derivations has been a central theme in algebra. The research in this area was initiated by Martindale in his seminal work [2]. Later, authors of [3–6] explored conditions for additivity of multiplicative isomorphisms of rings. In [3], Daif investigated conditions for the additivity of multiplicative derivations on associative rings. Afterwords, Ferreira and Nascimento [4] extended his result for alternative rings. In 2009, Ji investigated additivity of Jordan maps on Jordan algebras [5]. Very recently, Ferreira et al. studied the additivity of n-multiplicative isomorphism and derivations of Jordan rings [6].

Derivations on Jordan algebras are instrumental in constructing Lie algebra; the derivations on quasi-Jordan algebras have been discussed by Velásquez and Felipe [7, 8]. Recent advancements have employed Peirce decomposition to discuss maps and derivations; particularly, their additivity and structural behavior [9–11]. These works demonstrate the essential role of Peirce decomposition in advancing our understanding of algebraic structures and their associated functions. In this note, motivated from the above literature and Theorem 5 in [12], we begin the study of Peirce decomposition in the sitting of quasi Jordan algebras. We investigate Peirce decomposition of quasi Jordan algebras relative to idempotents of split quasi Jordan algebras (see below).

The class of quasi Jordan algebras was introduced by Velásquez and Felipe [13]: a quasi Jordan algebra is a vector space $\mathfrak{J}$ over a field of characteristic different from 2 and 3 equipped with a bilinear product, "called quasi Jordan product", $\triangleleft :\ \mathfrak{J}\times\mathfrak{J} :\to \mathfrak{J}$, satisfying $x\triangleleft(y\triangleleft z) = x\triangleleft(z\triangleleft y)$, and $(y\triangleleft x)\triangleleft x^2 = (y\triangleleft x^2)\triangleleft x$.

An element e in a quasi Jordan algebra $\mathfrak{J}$ is called a unit if $x\triangleleft e = x$ for all $x\in\mathfrak{J}$; a quasi Jordan algebra with a unit is called unital. A quasi Jordan algebra may have infinitely many units (cf. [13, 14]).

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The exploration of ideals in quasi Jordan algebras reveals a nuanced landscape of ideals, contributing to the understanding of their internal structure. The pivotal ideals $\Im_{ann} := span\{x \triangleleft y - y \triangleleft x : \ x, \ y \in \Im\}$ and $Z(\Im) := \{z \in \Im : \ x \triangleleft z = 0, \ \forall x \in \Im\}$, called the annihilator and zero part respectively, unlock deeper insights into algebras (cf. [15–18]).

In [14], Velásquez and Felipe introduced an interesting subclass of quasi Jordan: a split quasi Jordan algebras over an ideal I which satisfies $\Im_{ann} \subseteq I \subseteq Z(\Im)$ and $\Im = J \oplus I$ for some Jordan subalgebra J of $\Im$, where $J \oplus I$ denotes the direct sum of J and I. If a split quasi Jordan algebra $\Im$ is unital, then $\Im_{ann} = Z(\Im)$, and hence the split quasi algebra $\Im$ becomes $J \oplus Z(\Im)$. In such a case, each element $x \in \Im$ has a unique representation $x = x_J + x_Z$ with $x_J \in J$ and $x_Z \in Z(\Im)$, called the Jordan part and the zero part of x, respectively.

2. Idempotents in quasi Jordan algebras

Idempotents play a basic role in the structure theory of algebras. An element p in a quasi Jordan algebra $\Im$ is called an idempotent if $p^2 = p$.

Let $\Im = J \oplus Z(\Im)$ be a split quasi Jordan algebra with a unit e in J, in this case $e_Z = 0$ and $e_J = e$. If p is an idempotent in $\Im$, then,

$$p_J + p_Z = p = p^2 = p_J^2 + p_Z \triangleleft p_J.$$

This implies that $p_Z = p_Z \triangleleft p_J$ and $p_J = p_J^2$. Hence, p_J is an idempotent in J. We define $p^\perp := e - p_J$. It is clear that $p^\perp$ is an element of the Jordan part J of $\Im$ and that $e - e \triangleleft p = e - e \triangleleft (p_J + p_Z) = e - e \triangleleft p_J - e \triangleleft p_Z = e - p_J = p^\perp$. Hence, as expected, we get:

$$p \triangleleft p^\perp = p \triangleleft (e - e \triangleleft p) = p - p^2 = 0,$$

$$p^\perp \triangleleft p = (e - e \triangleleft p) \triangleleft p = p_J - p_J^2 = 0.$$

and

$$\left(p^\perp\right)^2 = (e - e \triangleleft p)^2 = (e - p_J)^2 = e^2 - 2p_J + {p_J}^2 = e - p_J = p^\perp$$

Note that, if $p^\perp = e - p$ then $p^\perp \triangleleft p = p_Z \neq 0$.

Example 1 Let $\Im$ be a split quasi Jordan algebra. For every $x \in \Im$, define $R_x : \ \Im \to \Im$ by $R_x(y) := y \triangleleft x, \ \forall y \in \Im$. From [14], we know that $R(\Im) = \{R_x : \ x \in \Im\}$, with product "$\circ$" defined by $R_x \circ R_y = R_{x \triangleleft y}$ for all $x, \ y \in \Im$, is a Jordan algebra. Moreover, if p is an idempotent, then R_p is an idempotent in $R(\Im)$. Clearly, $R_p^2 = R_p \circ R_p = R_{p \triangleleft p} = R_p$.

Alhefthi [19] recently introduced an analogue of the Jordan triple product specifically designed for quasi Jordan algebras, namely, quasi Jordan triple product; which is a natural extension of the usual Jordan triple product, broadening the scope of study within the realm of Jordan algebras. For any elements $x, \ y, \ z$ in $\Im$, define.

$$\{xyz\} = x \triangleleft (y \triangleleft z) + (x \triangleleft y) \triangleleft z - (x \triangleleft z) \triangleleft y. \tag{1}$$

If e is a unit in $\Im$ then $\{xez\} = x \triangleleft (e \triangleleft z) + (x \triangleleft e) \triangleleft z - (x \triangleleft z) \triangleleft e = x \triangleleft y$. If $\Im = J \oplus I$ is a split quasi Jordan algebra over I, then for any $x,\ y,\ z \in \Im$, the quasi Jordan triple product in the Jordan part coincides with the usual Jordan triple product (cf. [19]).

Lemma 1 Let $\Im = J \oplus Z(\Im)$ be a split quasi Jordan algebra and $x,\ y,\ z \in \Im$. Then the above triple product (1) is linear in the three variables, and

$$\{xyz\} = \{x_J yz\} + \{x_Z yz\},$$

where $\{x_J yz\} = \{x_J y_J z_J\} \in J$ and $\{x_Z yz\} \in Z(\Im)$.

Proof. Linearity of the triple product (1) follows immediately from the linearity of the quasi Jordan product. Next, if $x,\ y,\ z \in \Im$, with decomposition $x = x_J + x_Z$, then

$$\begin{aligned}\{xyz\} =& x \triangleleft (y \triangleleft z) + (x \triangleleft y) \triangleleft z - (x \triangleleft z) \triangleleft y \\ =& x_J \triangleleft (y \triangleleft z) + (x_J \triangleleft y) \triangleleft z - (x_J \triangleleft z) \triangleleft y \\ &+ x_Z \triangleleft (y \triangleleft z) + (x_Z \triangleleft y) \triangleleft z - (x_Z \triangleleft z) \triangleleft y \\ =& \{x_J yz\} + \{x_Z yz\}.\end{aligned}$$

By using the fact that $Z(\Im)$ is an ideal and that left multiplication by any element will vanish it's zero part, we get $\{x_J yz\} = \{x_J y_J z_J\} \in J$ and $\{x_Z yz\} \in Z(\Im)$. □

We will adopt the usual notation of the quadratic operator U_v and the left multiplication operator L_v, which are defined as follows: $U_v(y) = \{vyv\}$ and $L_v y = v \triangleleft y$, for all $v,\ y \in \Im$.

Proposition 1 Let $\Im$ be a split quasi Jordan algebra with $p \in \Im$ an idempotent. Then, for all $y \in \Im$,

(i) $U_p(y) = U_{p_J}(y) + \{p_Z y_J p_J\}$.

(ii) $U_{p^\perp}(y) = U_{p^\perp}(y_J)$, $U_{p_J}(y) = U_{p_J}(y_J)$, and both belong to J.

(iii) $\{p_Z y p_J\} = \{p_Z y_J p_J\} \in Z(\Im)$.

Proof.

(i) By the definition of U_p and Lemma 1.

$$U_p(y) = \{pyp\} = \{p_J yp\} + \{p_Z yp\} = \{p_J y_J p_J\} + \{p_Z y_J p_J\} = U_{p_J}(y_J) + \{p_Z y_J p_J\}.$$

(ii) Note that $U_{p_J}(y) = \{p_J y p_J\} = p_J \triangleleft (y \triangleleft p_J) + (p_J \triangleleft y) \triangleleft p_J - (p_J \triangleleft p_J) \triangleleft y = p_J \triangleleft (y_J \triangleleft p_J) + (p_J \triangleleft y_J) \triangleleft p_J - (p_J \triangleleft p_J) \triangleleft y_J = \{p_J y_J p_J\} = U_{p_J}(y_J) \in J$, and so the triple product $\{p_J y p_J\}$ in J is precisely the usual Jordan triple product. Further, since $p^\perp = e - p_J \in J$, we get $U_{p^\perp}(y) = \left\{p^\perp y p^\perp\right\} = p^\perp \triangleleft \left(y \triangleleft p^\perp\right) + \left(p^\perp \triangleleft y\right) \triangleleft p^\perp - \left(p^\perp \triangleleft p^\perp\right) \triangleleft y = p^\perp \triangleleft \left(y_J \triangleleft p^\perp\right) + \left(p^\perp \triangleleft y_J\right) \triangleleft p^\perp - \left(p^\perp \triangleleft p^\perp\right) \triangleleft y_J = U_{p^\perp}(y_J) \in J$.

(iii) $\{p_Z y p_J\} = p_Z \triangleleft (y \triangleleft p_J) + (p_Z \triangleleft y) \triangleleft p_J - (p_Z \triangleleft p_J) \triangleleft y = p_Z \triangleleft (y_J \triangleleft p_J) + (p_Z \triangleleft y_J) \triangleleft p_J - (p_Z \triangleleft p_J) \triangleleft y_J = \{p_Z y_J p_J\} \in Z(\Im)$. □

Note that L_e is an idempotent mapping onto J, and so $(L_e)^2(x) = L_e x_J = x_J$. Hence:

$L_e U_p = U_{p_J}$;

$L_e U_p(y) = U_{p_J}(y_J) \in J$, for all $y \in \Im$;

$e_\triangleleft(x) = 0$, for all $x \in J$, and $e_\triangleleft(y) \in Z(\Im)$, for all $y \in \Im$;

$L_e(z) = 0$, for all $z \in Z(\Im)$;
$U_x(z) = 0$, for all $x \in \Im,\ z \in Z(\Im)$;
$U_x(y) \in J$, for all $x \in J$ and $y \in \Im$;
$U_{p_J} U_{p^\perp}(y) = U_{p_J} U_{p^\perp}(y_J) = 0$ [20, p. 46].

Proposition 2 For any split quasi Jordan algebra $\Im$ with unit e and idempotent p, we have $(L_e U_p)^2 = L_e U_p$; $U^2_{p^\perp} = U_{p^\perp}$; $L_e U_p U_{p^\perp} = 0$; $L_e L_p U_p = L_e U_p$ and $L_e L_p U_{p^\perp} = 0$.

Proof. The operators U_{p_J} and $U_{p^\perp}$ coincide with the usual quadratic operators on the Jordan algebra J. Hence, using [20, p. 46], we get that:

$$U^2_{p_J}(x) = U_{p_J}(x);\ U^2_{p^\perp}(x) = U_{p^\perp}(x);\ U_{p_J} U_{p^\perp}(x) = U_{p^\perp} U_{p_J}(x) = 0,$$

for all $x \in J$. However, for any $y \in \Im$, $U_{p_J}(y) = U_{p_J}(y_J)$ and $U_{p^\perp}(y) = U_{p^\perp}(y_J)$. Therefore:

$$U^2_{p_J}(y) = U_{p_J}(y);\ \ U^2_{p^\perp}(y) = U_{p^\perp}(y);\ \ U_{p_J} U_{p^\perp}(y) = U_{p^\perp} U_{p_J}(y) = 0,$$

for all $y \in \Im$. This, together with the fact that $L_{p_J} = L_e L_{p_J}$, gives $(L_e U_p)^2 = U^2_{p_J}(y) = U_{p_J}(y) = L_e U_p; U^2_{p^\perp} = U_{p^\perp}$ and $L_e U_p U_{p^\perp} = U_{p^\perp} L_e U_p = U_{p^\perp} U_{p_J} = 0$.

Similarly, since $L_e L_p(y) = L_{p_J}(y) = L_{p_J}(y_J)$, for all $y \in \Im$ (cf. [20, p. 46], we get that $L_e L_p U_p = L_e U_p L_p = L_e U_p$; $L_e L_p U_{p^\perp} = L_e U_{p^\perp} L_p = U_{p^\perp} L_p = 0$. □

3. Peirce decomposition of split quasi Jordan algebras

For any idempotent p in a unital split quasi Jordan algebra $\Im$, p_J is an idempotent in the Jordan part J and since J is a unital Jordan algebra, we can apply [20, p. 45] to p_J so, $p_J \triangleleft y_J = \frac{1}{2}\left(\{p_J y_J p_J\} - \{p_J^\perp y_J p_J^\perp\} + y_J\right)$, for all $y \in \Im$. Clearly, $p_J^\perp = e - p_J = p^\perp$ and $L_e x = e \triangleleft x = x_J$, for all $x \in \Im$. So that:

$$\frac{1}{2}\left(U_{p_J}(y_J) - U_{p^\perp}(y_J) + L_e(y_J)\right) = \frac{1}{2} L_e\left(U_p(y) - U_{p^\perp}(y) + I_\Im(y)\right)$$

for all $y \in \Im$, where $I_\Im$ is the identity operator of $\Im$. Then:

$$\begin{aligned} L_e(L_p(y)) &= e \triangleleft (p_J \triangleleft y) = p_J \triangleleft y_J \\ &= L_{p_J}(y_J) = \frac{1}{2}\left(U_{p_J}(y_J) - U_{p^\perp}(y_J) + L_e(y_J)\right) \\ &= \frac{1}{2} L_e\left(U_p(y) - U_{p^\perp}(y) + I_\Im(y)\right) \end{aligned}$$

for all $y \in \Im$. Or equivalently.

$$L_eL_p = L_eU_p + \frac{1}{2}\left(L_e - L_eU_p - U_{p^\perp}\right) + 0\cdot\left(U_{p^\perp} - e_\triangleleft\right), \tag{2}$$

where $-e_\triangleleft(y) := y - e\triangleleft y = y_Z$ and $(-e_\triangleleft)^2(y) = -e_\triangleleft(y_Z) = y_Z$. Thus, $-e_\triangleleft$ is an idempotent mapping onto $Z(\mathfrak{I})$.

Proposition 3 Let $\mathfrak{I} = J\oplus Z(\mathfrak{I})$ be a split quasi Jordan algebra with a unit e included in J and p an idempotent in $\mathfrak{I}$. Then the operators L_eU_p, $L_e - L_eU_p - U_{p^\perp}$, $U_{p^\perp} - e_\triangleleft$ are orthogonal to each other. Further, each of these operators is an idempotent mapping such that $L_eU_p + \left(L_e - L_eU_p - U_{p^\perp}\right) + \left(U_{p^\perp} - e_\triangleleft\right) = I_{\mathfrak{I}}$.

Proof. For all $y\in\mathfrak{I}$,

$$(L_eU_p)\left(L_e - L_eU_p - U_{p^\perp}\right)(y) = U_{p_J}\left(L_e(y) - U_{p_J}(y) - U_{p^\perp}(y)\right) = U_{p_J}L_e(y) - U_{p_J}U_{p_J}(y) - U_{p_J}U_{p^\perp}(y) = 0;$$

$$(L_eU_p)\left(U_{p^\perp} - e_\triangleleft\right)(y) = U_{p_J}U_{p^\perp}(y) + 0 = 0;$$

$$\left(L_e - L_eU_p - U_{p^\perp}\right)(L_eU_p)(y) = \left(L_e - U_{p_J} - U_{p^\perp}\right)U_{p_J}(y) =$$

$$L_e\left(U_{p_J}(y_J)\right) - U_{p_J}^2(y_J) - U_{p^\perp}\left(U_{p_J}(y_J)\right) = U_{p_J}(y) - U_{p_J}(y) - 0 = 0;$$

$$\left(L_e - L_eU_p - U_{p^\perp}\right)\left(U_{p^\perp} - e_\triangleleft\right)(y) = \left(L_e - U_{p_J} - U_{p^\perp}\right)U_{p^\perp}(y) + 0 =$$

$$L_eU_{p^\perp}(y_J) - U_{p_J}U_{p^\perp}(y_J) - U_{p^\perp}^2(y_J) = U_{p^\perp}(y_J) - 0 - U_{p^\perp}(y_J) = 0;$$

$$\left(U_{p^\perp} - e_\triangleleft\right)\left(L_e - L_eU_p - U_{p^\perp}\right)(y) = U_{p^\perp}\left(L_e - U_{p_J} - U_{p^\perp}\right)(y) - e_\triangleleft\left(L_e - U_{p_J} - U_{p^\perp}\right)(y) =$$

$$U_{p^\perp}L_e(y) - U_{p^\perp}U_{p_J}(y_J) - U_{p^\perp}^2(y) - 0 = U_{p^\perp}(y_J) - 0 - U_{p^\perp}(y_J) = 0$$

and

$$\left(U_{p^\perp} - e_\triangleleft\right)L_eU_p(y) = U_{p^\perp}U_{p_J}(y_J) - e_\triangleleft\left(U_{p_J}(y_J)\right) = 0.$$

Further, each of the above three operators is an idempotent mapping since $(L_eU_p)(L_eU_p) = U_{p_J}U_{p_J} = U_{p_J} = L_eU_p$;

$$\left(L_e - L_eU_p - U_{p^\perp}\right)\left(L_e - L_eU_p - U_{p^\perp}\right)$$

$$=L_e\left(L_e - U_{p_J} - U_{p^\perp}\right) - U_{p_J}\left(L_e - U_{p_J} - U_{p^\perp}\right) - U_{p^\perp}\left(L_e - U_{p_J} - U_{p^\perp}\right)$$

$$=\left(L_e - U_{p_J} - U_{p^\perp}\right) - \left(U_{p_J} - U_{p_J} - 0\right) - \left(U_{p^\perp} - 0 - U_{p^\perp}\right) = L_e - L_eU_p - U_{p^\perp},$$

and $\left(U_{p^\perp} - e_\triangleleft\right)(U_{p^\perp} - e_\triangleleft) = U^2_{p^\perp} - U_{p^\perp}e_\triangleleft + e_\triangleleft U_{p^\perp} + (-e_\triangleleft)^2 = U_{p^\perp} + 0 + 0 - e_\triangleleft = U_{p^\perp} - e_\triangleleft$.

Finally, $L_eU_p + \left(L_e - L_eU_p - U_{p^\perp}\right) + \left(U_{p^\perp} - e_\triangleleft\right) = L_e - e_\triangleleft = I_\Im$. □

From (2) and Proposition 3, we see that the operator L_eL_p has eigenvalues 0, $\frac{1}{2}$, 1, and we obtain the following vector space decomposition:

$$\Im = \Im_1 \oplus \Im_{1/2} \oplus \Im_0,$$

where $\Im_\lambda$ is the eigen subspace of $\Im$ defined by $\Im_\lambda = \{x \in \Im : \ L_eL_p(x) = \lambda x\}$ corresponding to the eigenvalue $\lambda \in \{1, \ \frac{1}{2}, \ 0\}$. This decomposition is called the Peirce decomposition of $\Im$ induced by the idempotent p, and each eigensubspace $\Im_\lambda$ is called a Peirce space.

Proposition 4 Let $\Im = J \oplus Z(\Im)$ be a split quasi Jordan algebra with unit $e \in J$, and p an idempotent in $\Im$. Then L_eU_p, $2U_{p^\perp,\,p}$, $U_{p^\perp} - e_\triangleleft$ are idempotent mappings onto $\Im_1$, $\Im_{1/2}$, and $\Im_0$ of $\Im$, respectively, correspond to the Peirce decomposition.

Proof. Recall that $L_eU_p = U_{p_J}$. Also note that $U_{p^\perp,\,p}(y) = \{p^\perp y p\} = (p^\perp \triangleleft y_J) \triangleleft p_J + p^\perp \triangleleft (y_J \triangleleft p_J) - (p^\perp \triangleleft p_J) \triangleleft y_J = U_{p^\perp,\,p_J}(y_J)$, for all $y \in \Im$. So, both operators U_{p_J} and $U_{p^\perp,\,p_J}$, when restricted to the Jordan algebra J, coincide with the usual quadratic Jordan operator on J (a unital Jordan algebra). Hence, by [20, p. 47], $U_{p^\perp,\,p_J}(y_J) = U_{p_J,\ p^\perp}(y_J) = \frac{1}{2}\left(L_e - L_eU_p - U_{p^\perp}\right)(y)$. Thus, by Proposition 3, L_eU_p, $2U_{p^\perp,\,p}$, $U_{p^\perp} - e_\triangleleft$ are idempotent mappings onto $\Im_1$, $\Im_{1/2}$, $\Im_0$, respectively, corresponding to the Peirce decomposition. □

Remark 1

(i) Since the Jordan part p_J of the idempotent p is an idempotent itself in the Jordan algebra J, a natural construction of the Peirce decomposition of this Jordan algebra appears with respect to p_J, namely, $J = J_1 \oplus J_{1/2} \oplus J_0$, which coincides with the restriction of the Peirce decomposition of $\Im$ with respect to p.That is, $J_1 = \Im_1$, $J_{1/2} = \Im_{1/2}$ and $J_0 = \Im_0 \cap J$.

(ii) Since R_p is an idempotent in the Jordan algebra $R(\Im)$ whenever p is an idempotent, we can obtain a Peirce decomposition in $R(\Im)$ of R_p. One can easily see that the Peirce decomposition, $R(\Im)_1 \oplus R(\Im)_{1/2} \oplus R(\Im)_0$ coincides with $R(\Im_1) \oplus R(\Im_{1/2}) \oplus R(\Im_0)$.

(iii) An other decomposition of $\Im$ is $\Im_1 \oplus \Im_{1/2} \oplus \Im_0 \oplus Z(\Im)$, with idempotent mappings L_eU_p, $2U_{p^\perp,\,p}$, $U_{p^\perp}$, $-e_\triangleleft$ onto $\Im_1$, $\Im_{1/2}$, $\Im_0$, $Z(\Im)$, respectively.

The following result gives some properties of the Peirce spaces.

Proposition 5 Let $\Im$ be a unital split quasi Jordan algebra and p an idempotent in $\Im$. Let $\Im = \Im_1 \oplus \Im_{1/2} \oplus \Im_0$ be the corresponding Peirce decomposition. Then we have the following multiplication rules:

(i) $\Im_1 \triangleleft \Im_1 \subseteq \Im_1$;

(ii) $\Im_0 \triangleleft \Im_0 \subseteq \Im_0$;

(iii) $\Im_1 \triangleleft \Im_0 = 0$;

(iv) $\Im_0 \triangleleft \Im_1 \subseteq Z(\Im)$;

(v) $\Im_{1/2} \triangleleft (\Im_0 \oplus \Im_1) \subseteq \Im_{1/2}$;
(vi) $\Im_{1/2} \triangleleft \Im_{1/2} \subseteq \Im_0 \oplus \Im_1$.

Proof.

(i) Let $\Im$ be a unital split quasi algebra, $\Im = J \oplus Z(\Im)$, and let $p = p_J + p_Z$ be an idempotent in $\Im$. Since the Jordan part J is a unital Jordan algebra and p_J is an idempotent in J, the Jordan algebra J has the usual Peirce decomposition, namely, $J = J_1 \oplus J_{1/2} \oplus J_0$ induced by the idempotent p_J. By the construction of the corresponding idempotent of J_j and $\Im_j$, for $j \in \{0, \frac{1}{2}, 1\}$, we get $\Im_1 = J_1$, $\Im_{1/2} = J_{1/2}$ and $\Im_0 = J_0 \oplus Z(\Im)$. Then, using [20, Lemma 2.6.3], we get $J_1 \triangleleft J_1 \subseteq J_1$, and hence $\Im_1 \triangleleft \Im_1 \subseteq \Im_1$.

(ii) Since $J_0 \triangleleft J_0 \subseteq J_0 \subseteq \Im_0$ and using the fact that $Z(\Im)$ is an ideal, we get, $Z(\Im) \triangleleft Z(\Im)$, $Z(\Im) \triangleleft J_0$ and $J_0 \triangleleft Z(\Im)$ are all subspaces of $Z(\Im) \subseteq \Im_0$. Hence, $\Im_0 \triangleleft \Im_0 \subseteq \Im_0$.

(iii) Keeping in view the fact that right multiplication of any element in $\Im$ with a zero element is zero and left multiplication with a zero element falls in $Z(\Im)$, it becomes clear that $J_0 \triangleleft J_1 = 0 = J_1 \triangleleft J_0$. Therefore, $\Im_1 \triangleleft \Im_0 = J_1 \triangleleft (J_0 \oplus Z(\Im)) = J_1 \triangleleft J_0 \oplus J_1 \triangleleft Z(\Im) = 0$.

(iv) Since $\Im_1 = J_1$ and $\Im_0 = J_0 \oplus Z(\Im)$, $\Im_0 \triangleleft \Im_1 = (J_0 \oplus Z(\Im)) \triangleleft J_1 = J_0 \triangleleft J_1 \oplus Z(\Im) \triangleleft J_1 = 0 \oplus Z(\Im) \subseteq Z(\Im)$.

(v) $\Im_{1/2} \triangleleft (\Im_0 \oplus \Im_1) = J_{1/2} \triangleleft (J_0 \oplus Z(\Im) \oplus J_1) = J_{1/2} \triangleleft Z(\Im) \oplus J_{1/2} \triangleleft (J_0 \oplus J_1) = J_{1/2} \triangleleft (J_0 \oplus J_1) \subseteq J_{1/2} = \Im_{1/2}$.

(vi) $\Im_{1/2} \triangleleft \Im_{1/2} = J_{1/2} \triangleleft J_{1/2} \subseteq J_0 \oplus J_1 \subseteq J_0 \oplus Z(\Im) \oplus J_1 = \Im_0 \oplus \Im_1$. □

4. Conclusion

In the recent past, Velásquez and Felipe introduced the class of quasi Jordan algebras [13] and then after a couple of years they introduced its subclass, namely, the split quasi Jordan algebras [8], which is a noncommutative generalization of Jordan algebras. It is well known that idempotents (also known as tripotents in Jordan triple systems) dominates the structure of Jordan algebras and Peirce decomposition induced by an idempotent is an important tool in the structure theory of Jordan algebras.

As a sequent of our research papers [17, 19] extending Jordan algebras results to the setting of quasi Jordan algebras or split quasi Jordan algebras, we in this article observed some properties of idempotents and by exploiting the quasi Jordan triple product $\{xyz\} = x \triangleleft (y \triangleleft z) + (x \triangleleft y) \triangleleft z - (x \triangleleft z) \triangleleft y$ (introduced in [19]), discussed the possible Peirce decomposition relative to idempotents of unital split quasi algebras. We proved that, for any idempotent p in a split quasi Jordan algebra $\Im$ with a unit e included in J, $L_eL_p = L_eU_p + \frac{1}{2}(L_e - LeU_p - U_{p^\perp}) + 0 \cdot (U_{p^\perp} - e_\triangleleft)$; further that L_eU_p, $L_e - L_eU_p - U_{p^\perp}$, $U_{p^\perp} - e_\triangleleft$ are orthogonal idempotents satisfying that $L_eU_p + L_e - L_eU_p - Up^\perp + U_{p^\perp} - e_\triangleleft = I_\Im$, where $I_\Im$ is the identity operator of $\Im$. This led us to the Peirce decomposition: $\Im = \Im_1 \oplus \Im_{\frac{1}{2}} \oplus \Im_0$, where $\Im_\lambda$ is the eigen subspace of $\Im$ defined by $\Im_\lambda = \{x \in \Im : L_eL_p(x) = \lambda x\}$ corresponding to the eigenvalue $\lambda \in \{1, \frac{1}{2}, 0\}$ (see Propositions 1 and 2).

This study paves the way for several promising directions for future research. A key area of interest is the detailed investigation of the structure of Peirce subspaces $\Im_\lambda$ relative to an idempotent in a unital split algebra $\Im$; particularly, when $\Im$ is equipped with additional structures such as a complete norm [17] or an involution [18]. Another intriguing question is how different types of idempotents influence these decompositions, offering insights into their algebraic and geometric properties. Beyond this, exploring the feasibility of Peirce decomposition in broader settings, such as and general algebras, presents an exciting challenge.

An additional open problem is to obtain some appropriate analogue of Theorem 5 of [12] in the sitting of quasi Jordan algebras. Furthermore, Peirce decomposition serves as a critical tool for studying the characterization and additivity of functions, as highlighted in [9–11]. This raises another open problem: under what conditions can we study the additivity and characterization of certain functions using Peirce decomposition of quasi Jordan algebras?

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Conflict of interest

The authors declare no competing financial interest.

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