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Research Article

Some Fast Summations of Fourier Series and Properties of Their Dirichlet-Type Kernels

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Abstract: One of the new approaches to accelerating the Fourier series's convergence is using a specific parametric biorthogonal system. In the simplest case, we discuss the so-called "traditional", simple but quite effective summation algorithms that do not require severe computational costs. This paper constructs, tests, and investigates two algorithms using computational mathematics. It shows that the successful overcoming of the Gibbs phenomenon here is due to new properties of the corresponding Dirichlet-type kernels. In particular, the classical localization principle is violated by this summation. Here, the approximation of the smooth function $f(x)$ defined on segment $x \in [-1,\ 1]$ depends at point $x = x_0$ on its behavior both in the neighborhood of x_0 and in the ends of ± 1.

Keywords: Fourier series, Gibbs phenomenon, Dirichlet kernel, acceleration of convergence, spectral methods, computational mathematics, numerical analysis

MSC: 42A16, 65B10, 65D15, 68W25, 68W30

1. Introduction

One of the most famous mathematics tools is the Fourier series apparatus based on the orthogonal system $\{e^{i\pi kx}\}$, $x \in [-1,\ 1]$, $k = 0,\ \pm 1,\ \pm 2,\ \ldots$, which is complete in $L_2[-1,\ 1]$. Classical, straight summation methods for the partial sum of Fourier series

$$S_n(x) = \sum_{k=-n}^{n} f_k e^{i\pi kx},\ f_k = \frac{1}{2}\int_{-1}^{1} f(t)\, e^{-i\pi kx}\, dt,\ n \geq 1,\ x \in [-1,\ 1], \tag{1}$$

in practice, could be not efficient even for an analytic functions f. Thus, in the case where a function has discontinuity points (taking also into account discontinuities at the ends of the interval $[-1,\ 1]$), an intense oscillation arises in their neighborhood and the uniform convergence is absent (the Gibbs phenomenon). On this main obstacle to practically acceptable convergence of Fourier series, see, for example [1]. As a rule, this prevents a satisfactory recovery of the values of $f(x)$ from the given $\{f_k\}$.

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An alternative method for fast summation of piece-wise smooth functions was initiated by Krylov at the beginning of the 20th century (see his earlier work [2]). He suggested using the jumps $h_k = \{f^{(k)}(1) - f^{(k)}(-1)\},\ k \geq 0$ of the function f. His approach was developed half a century later, in particular, in the works of Lanczos [3]. However (for example, in some problems of mathematical physics) in (1) the function itself may not be known. In addition, f can be given approximately, which (unlike jumps $\{h_k\}$) may allow us to calculate with the same accuracy only the coefficients $\{f_p\}$.

The most significant achievement in the task of "overcoming the Gibbs phenomenon" is the work of Eckhoff [4]. He applied the spectral method of accelerating convergence. The most significant achievement in the task of "overcoming the Gibbs phenomenon" is the work of Eckhoff [4]. He applied the spectral method of accelerating convergence by solving a linear system of equations using only Fourier coefficients. As a result, the values of jumps $\{h_k\}$ were found with satisfactory accuracy. For details, please look at the introductions in [5–8].

We touched above on the vast topic of "overcoming the phenomenon Gibbs" so briefly because below will dwell on the recently developed entirely new approach (see [5–11]). We are interested in a summation method proposed in [5] for establishing the phenomenon of over-convergence. In particular, this leads to simple summation algorithms whose efficiency is usually higher the smoother and more moderately oscillating the function f is. The latest, improved version of such algorithms was proposed and tested in [9]. We confine ourselves to studying two so-called "traditional" algorithms. The results of a numerical experiment are presented, supplementing the data of [9] on their efficiency. Analogs of the classical Dirichlet kernels for these algorithms are studied here graphically, which leads to identifying their new properties.

Efficient algorithms for the summation of the Fourier series make it possible to solve significant practical problems in numerous areas-from signal/image processing and computer tomography to physics and celestial mechanics (see, for example, [10–14]). Below, we propose using one of the simplest efficient summation algorithm. Two examples of computer implementation clarify the operation mechanism of corresponding Dirichlet-type kernels.

2. Method used

Before describing the algorithms under study, we present the general method for constructing a biorthogonal system proposed in the work [5].

2.1 *Basic parametric biorthogonal system*

Our approach to fast summation of truncated Fourier series (1) is based on the following scheme. Let $m \geq 1$ be an integer and D_m be a set of m integers. For $r \in D_m$, consider a set of m parameters $\Lambda_m = \{\lambda_k\} \subset \mathbb{C},\ k \in D_m,\ \Lambda_m \cap D_m = \emptyset$, and the infinite sequence (for details, see [5, 15, 16])

$$t_{r,\,s} \stackrel{def}{=} (-1)^{s-r} \left(\prod_{\substack{p \in D_m \\ p \neq r}} \frac{s-p}{r-p} \right) \prod_{j \in D_m} \left(\frac{r-\lambda_j}{s-\lambda_j} \right),\ r \in D_m,\ s \in \mathbb{Z}. \tag{2}$$

It is obvious that for $r,\ s \in D_m,\ t_{r,\,s} = 0,\ t_{r,\,r} = 1$, and $t_{r,\,s} = O(1/s),\ s \to \infty$.

Remark 1 We emphasize that formula (2) does not depend on the numbering of parameters $\{\lambda_k\}$. This enumeration is provided for convenience. We also note that if for some $j,\ \lambda_J \in D_m$ then reductions are made on the right in (2).

Further, we denote

$$T_r(x) \overset{def}{=} \exp(i\pi r x) + \sum_{s \notin D_m} t_{r,\, s} \exp(i\pi s x),\ r \in D_m,\ x \in [-1,\ 1],$$

$$f(x) \simeq F_m(x) \overset{def}{=} \sum_{r \in D_m} f_r T_r(x),\ R_m(x) \overset{def}{=} f(x) - F_m(x). \tag{3}$$

The attentive reader will quickly discover that the system $\{T_r(x),\ 1/2\exp(i\pi r x)\}$, $r \in D_m$, is biorthogonal on the segment $x \in [-1,\ 1]$ and L_2-error of approximation $f(x) \simeq F_m(x)$ can be derived from the formula

$$\|R_m\|^2 = \sum_{s \notin D_m} \|f_s - \sum_{r \in D_m} f_r t_{r,\, s}\|^2,\ f \in L_2[-1,\ 1]. \tag{4}$$

2.2 *Explicit form of the $\{T_r\}$ system*

If $\lambda_k - \lambda_p \neq 0$ for $p \neq k$, then (see [5], $Theorem 1$) the functions $\{T_r(x)\}$ are represented in the following explicit form

$$T_r(x) = \sum_{k \in D_m} c_{r,\, k} \exp(i\pi \lambda_k x),\ r \in D_m, x \in [-1,\ 1], \tag{5}$$

where $[sinc(z) = \sin(z)/z,\ z \in \mathbb{C},\ sinc(0) = 1]$.

$$c_{r,\, k} = \frac{1}{sinc(\pi(r - \lambda_k))} \left(\prod_{\substack{p \in D_m \\ p \neq k}} \frac{r - \lambda_p}{\lambda_k - \lambda_p} \right) \prod_{\substack{q \in D_m \\ q \neq r}} \frac{\lambda_k - q}{r - q},\ r,\ k \in D_m. \tag{6}$$

Remark 2 Each of m functions of the system $\{T_r\}$ is a linear combination of all functions $\{\exp(i\pi \lambda_k x)\}$. That's why the computational complexity of the approximation F_m is significantly reduced if, using (5), first, the summation of $F_m(x)$ over r is performed (see (3) and (5)).

3. Application to the partial sum of Fourier series

3.1 *The scheme of summation*

To study the properties of proposed algorithms, we further restrict ourselves to the classical partial sums of the Fourier series (1).

Let's denote for $n > 1,\ 1 \leq m \leq n$ and consider instead of D_m in (2) the following set of $2m$ integers

$$D_{2m} = \{n - m + 1,\ \ldots,\ n\} \cup \{-n,\ -n + 1,\ \ldots,\ -n + m - 1\}. \tag{7}$$

Instead of the classical partial sum $S_n(x)$ (see (1)), below we will use the following more flexible scheme for restoring the function f

$$f \simeq F_{m,\,n}(x) = \sum_{r \in D_{2m}} f_r T_r(x) + \sum_{s=-n+m}^{n-m} \left(f_s - \sum_{r \in D_{2m}} f_r t_{r,\,s} \right) \exp(i\pi s x), \tag{8}$$

where $x \in [-1,\ 1]$ and the error is $R_{m,\,n}(x) = f(x) - F_{m,\,n}(x)$.

Paying attention to formulas (3) and (5), one can see that, in fact, here the biorthogonal system $\{T_r(x),\ 1/2\exp(i\pi r x)\}$ is constructed using only $2m$ Fourier coefficients $\{f_s\},\ n-m+1 \leq |s| \leq n$. No system $\{T_r\}$ exists for $m = 0$, so it is natural to assume that $F_{0,\,n}(x) = S_n(x)$ (see (1)).

Remark 3 It is known that if $f(x)$ is a sufficiently smooth 2-periodic function on the x-axis, then to "accelerate the convergence", it suffices to perform direct summation in (1). It is easy to verify that this corresponds to the choice of $\lambda_k = k,\ \forall k$ in (8) (see also Remark 1).

It is easy to verify the validity of the following assertion.

Lemma 1 Approximation (8) is exact ($f(x) \equiv F_{m,\,n}(x),\ x \in [-1,\ 1]$) in a linear $(2n+1)$-dimensional space of functions with following basis consisting of exponents

$$\{\exp(i\pi\lambda_k x)\} \cup \{\exp(i\pi p x)\},\ k \in D_{2m},$$

$$p = -n+m,\ -n+m+1,\ \ldots,\ n-m,\ x \in [-1,\ 1]. \tag{9}$$

Everywhere below, we consider formula (8) as a 2-periodic piece-wise analytic function function on the real axis.

Remark 4 We emphasize that here for the implementation of approximation (8) according to the "traditional" scheme, solutions any equations are not required. This approach to accelerating convergence for Fourier series is completely spectral and does not use even approximate jumps of the function. It suffices to choose $2\,m$ parameters λ_k in advance (see (2)), have $(2n+1)$ first Fourier coefficients $\{f_r\}$, and keep representation (4) in mind.

Definition 1 Let us call the summation according to formula (8) algorithm of type A, if for fixed k, the parameters $\{\lambda_k\}$ depend only on n, $0 < \lambda_k < k$ and $\lambda_k \uparrow k$ for $n- > \infty$, $\lambda_{-k} = -\lambda_k$.

In the following, we will restrict ourselves to studying the properties of two algorithms of type A.

3.2 *Numerical results*

Consider the following two A-type algorithms, which were chosen in the range $8 \leq n \leq 24,\ 1 \leq m \leq n$.

In the algorithm A_1 parameters $\{\lambda_k\}$ depends on n according to the formula

$$\lambda_k = \frac{2}{\pi} \arctan\left(\frac{1}{5} + \frac{n}{33}\right) k,\ n-m+1 \leq |k| \leq n. \tag{10}$$

This algorithm was used in [9]. Here, we explore it in more detail.

As A_2 we apply the "quasi-periodic" type algorithm recommended in Remark 12 from [9]. Here, we have chosen the formula

$$\lambda_k = \frac{n}{n + 21\log(n)/2}\, k,\ n-m+1 \leq |k| \leq n. \tag{11}$$

Below this algorithm is studied numerically for the first time.

Let us get acquainted with the work of these algorithms in the case of function

$$f(x) = \frac{\cos(x)}{(2/3+i)-x},\ x \in [-1,\ 1]. \tag{12}$$

This is a meromorphic function in the complex plane with a simple pole at point $x = 2/3+i$. So its derivatives have speedy growth in the segment $x \in [-1,\ 1]$. It was used in [9].

Our algorithms can be implemented on standard PCs. The Fourier coefficients $\{f_k\}$ of the function $f(x)$ are calculated precisely, in symbolic form, for example, by applying the system *WolframMathematica* (see [17]). We can use the numerical values of the coefficients with the required accuracy.

Relative L_2-error h for an approximation $\tilde{f}$ of function f is calculated below by the formula

$$h = \begin{cases} \dfrac{\|f-\tilde{f}\|}{\|f\|}, & \|f\| \neq 0 \\ \\ 0\ , & \|f\| = 0 \end{cases} \tag{13}$$

where $\|.\|$ is L_2-norm.

First, we present the L_2-errors of the approximation of the function $f(x)$ using the classical summation (1) for $8 \leq n \leq 32$.

Table 1. Relative L_2-errors for $f(x)$ using the summations to partial sums (1)

Sum↓	$n=4$	$m=8$	$n=12$	$n=16$	$n=20$	$n=24$	$n=28$	$n=32$
$f-S_n$	8.0e-2	5.8e-2	4.8e-2	4.1e-2	3.7e-2	3.4e-2	3.1e-2	2.9e-2

Now, let us present some typical results from our latest numerical experiments using in (8) algorithms A_1 and A_2.

Table 2. Relative L_2-errors for $f(x)$ at $n=8$

Alg↓	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
A_1	2.9e-3	1.6e-4	9.5e-6	8.3e-8	1.1e-6	1.7e-6	2e-6	2.1e-6
A_2	2.9e-3	1.7e-4	1 e-5	7.1e-6	1.1e-6	1.6e-6	2 e-6	2.1e-6

Table 3. Relative L_2-errors for $f(x)$ at $n=16$

Alg↓	$m=2$	$m=4$	$m=6$	$m=8$	$m=10$	$m=12$	$m=14$	$m=16$
A_1	5.9e-4	1.1e-5	2.1e-7	4.2e-9	8.1e-11	3e-12	2.6e-12	2.6e-12
A_2	4.3e-4	5.9e-6	8.4 e-8	1.2e-9	1.8e-11	3e-12	3.8 e-12	4e-12

Table 4. Relative L_2-errors for $f(x)$ at $n = 24$

Alg↓	$m=3$	$m=6$	$m=9$	$m=12$	$m=15$	$m=18$	$m=21$	$m=24$
A_1	2.5e-4	2.5e-6	2.7e-8	3e-10	3.3e-12	3.6e-14	3.5e-16	2.4e-18
A_2	1.1e-4	5.3e-7	2.6 e-9	1.3e-11	6.6e-14	3.2e-16	2.5 e-18	3.4e-18

3.3 *How and why algorithms work successfully*

Below, we will discuss the results presented in Tables 2-4. Here, we will just note that comparing this with the data in Table 1 reveals the obvious acceleration of the Fourier series' convergence. A natural question arises about how the idea of constructing algorithms A_1 and A_2 arose and (most importantly) why they successfully cope with the Gibbs phenomenon.

It should be noted that in formula (33) of work [5], an analog of algorithm A_1 was used. It was obtained as a result of a numerical experiment with several analytic functions in the neighborhood of segment $[-1, 1]$, for $4 \leq n \leq 12$. As a result of additional experiments, formula (10) was used in work [18]. Algorithm A_2 was selected as a result of numerical experiments for this paper. To reveal the mechanism of operation of these algorithms, it was decided to continue applying computational mathematics methods.

Remark 5 Applying classical summation methods to partial sums of the Fourier series (see [1, 11, 12]) leads to results similar to Table 1.

Following the classical traditions (see [1, 11–13]), below, we will pay attention to the analogues of the Dirichlet kernel.

4. Dirichlet-type kernels

4.1 *Definitions*

For the sake of certainty, let's dwell on algorithm A_1. Denote by $\overset{m}{D}_n(x, t)$ the function $F_{m,n}(x)$ in which the Fourier coefficients $\{f_k\}$ are replaced by values $\{\exp(-i\pi kt)\}$, $t \in [-1, 1]$, respectively (see (8)). For $m = 0$ we denote $\overset{0}{D}_n(x, t) = D_n(x-t)$, where D_n is the classical Dirichlet kernel

$$D_n(x-t) = \frac{\sin\left(\frac{1}{2}\pi(2n+1)(x-t)\right)}{\sin\left(\frac{1}{2}\pi(x-t)\right)}, \quad x, t \in [-1, 1]. \tag{14}$$

Since we have

$$F_{m,n}(x) = \int_{-1}^{1} f(t)\overset{m}{D}_n(x, t)dt, \ x \in [-1, 1], \tag{15}$$

it is natural to call $\overset{m}{D}_n(x, t)$ the Dirichlet-type kernel for algorithms A_1 at $0 \leq m \leq n$. For $m > 0$, $\overset{m}{D}_n(x, t)$ is not a function of $(x-t)$, but (like $D_n(x-t)$) it is real and satisfies the condition $\int_{-1}^{1} \overset{m}{D}_n(x, t)dt = 1$, $x \in [-1, 1]$ (see Lemma 1 above). Of course, the above also applies to the Dirichlet-type kernel for A_2.

Remark 6 The "adaptive" algorithm $\mathfrak{A}$ (see [9], Section 3.4) is much more efficient than algorithms A_1 and A_2. Its code contains a solution of a linear equation with an $m \times m$-matrix, followed by finding all the roots of an m-th-order polynomial. However, the parameters λ_k defined in it depend on the coefficients $\{f_k\}$. Therefore (as in the Krylov-Eckhoff-type algorithms mentioned at the beginning of the Introduction above), the Dirichlet-type kernel does not exist here.

4.2 *Visual illustrations*

Consider graphical representations of kernels $\overset{m}{D}_n$ for $m \geq 1$. Our goal is to discover by what changes in the Dirichlet kernel D_n these kernels lead to the acceleration of the convergence of Fourier series. Figure 1 and 2 serve as the basis for our further conclusions.

Remark 7 It follows from Tables 2-4 that algorithm A_2 is slightly more efficient than A_1. However, for the kernels of Algorithm A_2, the analog of Figures 1-2 is visually indistinguishable.

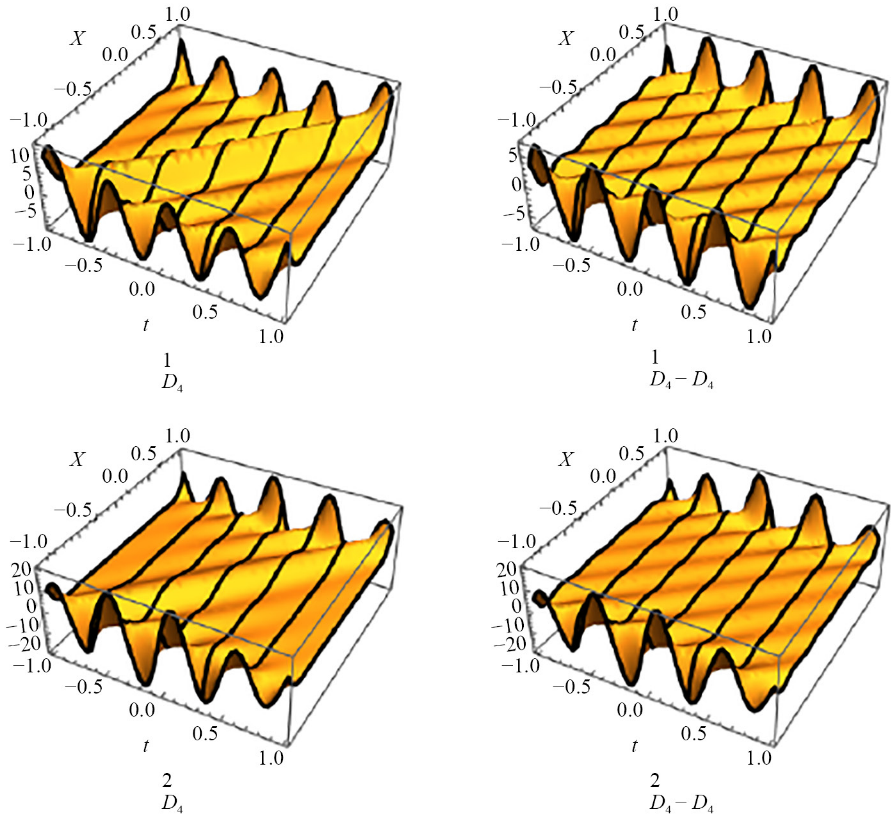

Figure 1. On the left are the graphs of the kernel of Algorithm A_1 for $n = 4$, $m = 1$, and $m = 2$. The right shows how different these kernels are from the corresponding Dirichlet kernel D_4. The mesh on the surface corresponds to the values $t = const$

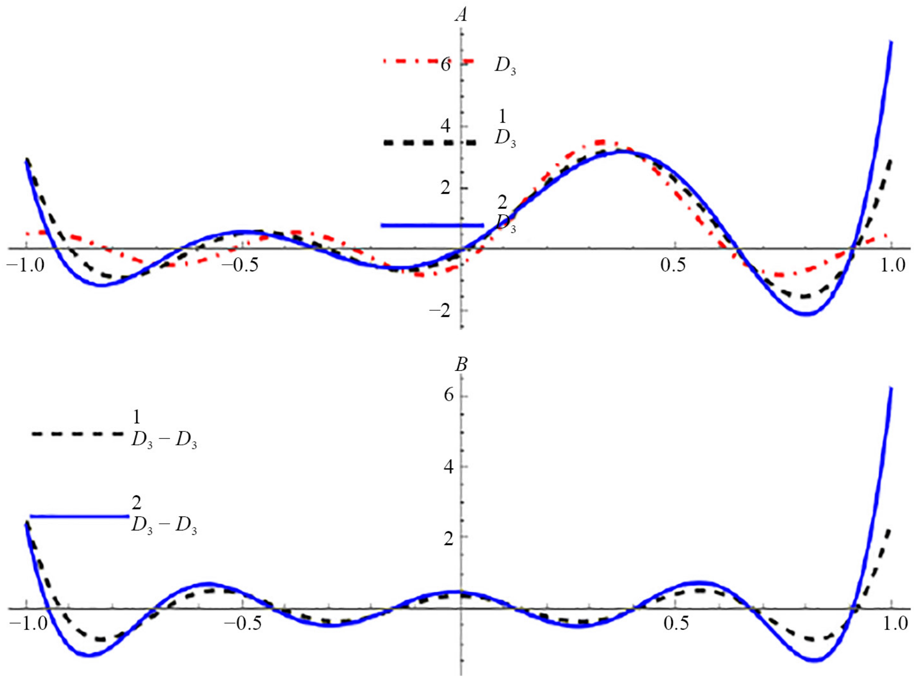

Figure 2. On the top are the graph of the kernel of algorithm A_1 for $t = 1/3,\ n = 3,\ m = 0,\ m = 1$, and $m = 2$. The bottom shows how different these functions are from the corresponding Dirichlet kernel $D_4(x - 1/3)$

For clarity, we considered the cases of small n and m. In the case of $n \geq 4$ and $m \simeq n$, it turns out that on the graphs of delta-shaped maxima, the values on the boundary sometimes significantly exceed the values inside the region at $x \simeq t$ (i.e., at the maximum of the Dirichlet kernel D).

4.3 *Properties of the algorithms*

First of all, we note that in Section 4.3 of [9] algorithm A_1 was tested for $1 \leq m \leq 8$ when $n = 24$. Our tests show that for $12 \leq n \leq 24$, A_1 can be successfully used up to $m \simeq n$. At the same time, it is interesting that, as a rule, when m grows near n, the error stabilizes. This phenomenon has little to do with the accumulation of errors, since it does not respond to an increase in the bit depth of computations. In [9] this was not noticed but when the same thing was repeated here when algorithm A_2 was applied, it can be concluded that may be such stabilization is inherent in all "traditional" algorithms. In particular, this means that these algorithms are stable.

It should also be noted that a significant part of the computational complexity of the discussed algorithms at $m \simeq n$ consists of operations with products in formula (6).

4.4 *Properties of the considered kernels*

Figure 1 clarifies the situation to a great extent. On the graphs on the left, one can see that the kernels of our algorithms in their "middle" part are similar to functions that depend on the difference in their arguments, but along the edges are very different. After subtracting the Dirichlet kernel D_4, the "remainders" of these kernels can be seen on the right. The plots of the "cut curves" at $t = 1/3$ are shown in more detail in Figure 2 (this time at $n = 3$). The results presented in Figure 1 and Figure 2 show that the algorithm A_1 and A_2 actively uses the values of a function $f(x)$ near the ends of the segment $x \in [-1,\ 1]$, although only first $(2n + 1)$ Fourier coefficients $\{f_k\}$ are known.

A similar situation is observed with an increase in n. Therefore, it can be argued that the principle of localization of Fourier series in its classical sense is violated here. Namely, it turns out that in the case of algorithms A_1 and A_2, the behavior of approximation $F_{m,\,n}$ (see (8)) at the point $x_0 \in (-1,\ 1)$ if $m = const. > 0,\ n \to \infty$ depends only on the properties of a smooth $f(x)$ both in neighborhood of the point $x = x_0$ and $x = \pm 1$. [It is convenient to identify the ends of segment $[-1,\ 1]$].

5. Conclusion

We found that the algorithms under study successfully cope with the Gibbs phenomenon by such "violating" the localization principle. They seem to somehow "use" the approximated function $f(x)$ values at $x = \pm 1$. An interesting situation arises when the kernels of classical summation methods (see [1, 10–13]), called "good" even in textbooks, turn out to be "bad" in terms of practical efficiency for sufficient smooth functions.

Of course, the specific mechanism of the A-type algoritms operation is yet to be determined, but the possibilities of numerical analysis of "traditional" algorithms are far from exhausted. In general, the efficiency of algorithms A_1 and A_2 is undeniable, especially considering their simplicity (see Remark 4).

Remark 8 The term "fast summation" generally refers to the estimate of the algorithm's accuracy relative to the number of Fourier coefficients used, which in Tables 2-4 at $m = n$ ranges from 10^{-7} to 10^{-19}. At the same time, our computer operating time was increased by only a few dozen times (from 0.3 sec. to 14 sec).

Let's sum up some results.

5.1 *About the method and possible applications*

According to not only the experiments presented here but also our other experiments, the proposed algorithms are stable and provide relative L_2-accuracy (see (13)) in the range of $10^{-5} - 10^{-15}$, depending on the use of 17 to 49 Fourier coefficients for $f(x) \in C^7[-1,\ 1]$. This includes functions that do not oscillate vigorously on the interval $[-1,\ 1]$ and are analytic in the neighborhood. We reached these conclusions after testing functions with different properties (see, for example, (30) and (31) in [9]). As shown in Figure 2 in [9], algorithm A_1 can perform approximate analytical continuation.

Overall, we did not notice any significant difference between the results of algorithms A_1 and A_2. Algorithms, based on formulas (2-8) are simple enough to be implemented in standard computers and used, for example, by many engineers, scientists, and students. It is enough to use computer programs such as MatLab, Wolfram Mathematica, Maple and even simpler means to do this.

Note that technically, the method for constructing efficient "traditional" summation algorithms has already been described for multidimensional Fourier series and other generalizations (see Section 5.2 in [9] and the report [18]). In this case, the numerical implementation has yet to be realized. Still, analogs of Dirichlet-type kernels arise naturally, and the issues raised here remain relevant in theoretical and applied aspects.

5.2 *About the next steps*

Formulas (10) and (11) define only two particular algorithms of the type A. It is natural to try to improve them to increase their efficiency further. When applying future computational methods to this problem and its generalization, the following directions can be noted:

• The immediate goal is to speed up the algorithms in their stabilization zone (see Section 4.3). Here, we can believe that such a problem will be solved if we assume that a fixed k λ_k depends not only on n but also on m.

• It would be interesting to find algorithms not inferior in efficiency to those discussed but based on a completely different behavior of the parameters λ_k at $n \to \infty$.

• In [7], in particular, a fast algorithm for accelerating the convergence of the Fourier series in cosines was obtained based on a sequence somewhat different from (2). The possible "traditional" summation was not considered here, although it promises to be "faster" than A_1 or A_2. But how much? An interesting problem.

• Implementing the above algorithms in the multidimensional case is possible, as a rule, only with powerful workstations or supercomputers. For example, in the simplest case of a two-dimensional Fourier series in a square

$n \times n$ region, each row (column) of the series is a partial sum $S_n(x)$ of a one-dimensional Fourier series. Therefore, the complexity of such an algorithm will be $2n$ times greater. Here, all problems may be solved by using full parallelization.

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Conflict of interest

The author declares no conflicts of interest regarding the publication of this paper.

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