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Research Article

Interconnection Between Schur Stability and Structured Singular Values

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Abstract: In this paper, we present new results on the interconnections between Schur stability and structured singular values of real-valued matrices, denoted as $\mathbb{R}^{n\times n}$. Most new findings are obtained for $n=2$ and $n=3$. These novel insights into the relationship between Schur stability and structured singular values are developed by applying various tools from linear algebra, system theory, and matrix analysis. Schur stability ensures that all eigenvalues lie within the unit circle in the complex plane, which is fundamental for the boundedness and stability of system responses. Structured singular values, on the other hand, provide a measure of robustness, stability, and performance against structured perturbations in system parameters, offering valuable insights into the stability margins and performance limits under such uncertainties.

***Keywords*:** Schur stability, Schur *D*-stability, linear dynamics, singular values, structured singular values

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1. Introduction

The Schur stable matrix $A\in\mathbb{R}^{n,\,n}$ is the one whose all eigenvalues $\sigma(A)$ lie in the open unit disk. Furthermore, for $A\in\mathbb{R}^{n,\,n}$ to be a Schur stable matrix, its spectral radius $\rho(A)<1$. The concept of Schur stability is closely related to Schur *D*-stability in the sense that *A* is Schur *D*-stable if and only if for each *D*, a positive diagonal matrix, the matrix *DA* is Schur stable [1].

For a given concrete problem (see [2]) $x_{k+1}=Ax_k$, with $A\in\mathbb{R}^{n,\,n}$, x_k is the state of the linear dynamical system at a given time k. The natural choice of initial state $x=0$ acts as an arbitrary, on the other hand, if the linear dynamical system has different equilibrium points, so there is a need to make a shift in the origin by an affine change of coordinates.

Stability analysis is a fundamental concept in various areas of science and, in particular, engineering. Linear algebra plays an important role in studying the stability analysis of linear dynamical system $x_{k+1}=Ax_k$. For instance, the computation of eigenvalues and singular values discusses the stability of such a system. But, once such systems are subject to external perturbations in the form of structured or unstructured uncertainties, then one needs the computation of structured singular values to analyze the stability and instability of such systems.

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In literature [3], the most common mathematical technique to deal with Schur stability is bi-linear transformation [4, 5], followed by the use of Hurwitz stability tools. On the other hand, we can use bi-quadratic transformation [6] to study and discuss the Schur stability of the linear dynamical system. There is a vast amount of literature to study Schur stability, for instance, see [7–19] and the references therein.

A comprehensive and detailed analysis of the necessary and sufficient conditions for Schur stability, using the Schur-Cohn criterion, is presented in [20]. The investigation into the robust Schur stability of n-dimensional matrix segments, employing the bi-alternate product of matrices, is discussed in [21]. Furthermore, the authors demonstrated that the problem under consideration allows us to examine and analyze the negative spectrum (the eigenvalues) in two out of three constructed matrices, as well as the presence of the spectrum in the interval $[1, \infty)$ in a third matrix.

The structured singular value introduced by Doyle [22] describes both the stability and performance of linear dynamical systems. Unfortunately, the exact determination of structured singular values is NP-hard [23]. Due to this limitation, various mathematical techniques have been developed [24–26] to specifically address the problem in a lower-dimensional linear dynamical system.

The n-dimensional diagonal matrix is defined as:

$$diag(\delta_1, \cdots, \delta_n) : \delta_1, \cdots, \delta_n \in \mathbb{R}.$$

For a given $A \in \mathbb{R}^{n, n}$, the largest singular value is denoted by $\sigma_1(A)$. The uncertainty set, that is, the set of diagonal matrices, $\mathbb{B}_1$, is defined by

$$\mathbb{B}_1 : = \{diag(\delta_1, \cdots, \delta_n) : \delta_i \in \mathbb{R}, \quad \forall\, i = 1, 2, \cdots, n\}.$$

For each $\delta \in \mathbb{R}$, $\delta \geq 0$, we define the set X_δ as

$$X_\delta = \{diag(\Delta_1, \cdots, \Delta_1, \Delta_2, \cdots, \Delta_2, \cdots, \Delta_n, \cdots, \Delta_n,) : \sigma_1(\Delta_j) \leq \delta, \quad \forall j = 1 : n\},$$

and $\Delta_j \in \mathbb{R}^{m_j, m_j}$, $\forall j = 1, 2, \cdots, n$.

For a given $A \in \mathbb{R}^{n, n}$, and $\mathbb{B}_1$, as defined above, the structured singular value is denoted by $\mu_{\mathbb{B}_1}(A)$. The quantity $\mu_{\mathbb{B}_1}(A) = 0$ if there there exists no $\Delta \in \mathbb{B}_1$ such that $det(I - A\Delta) = 0, \forall \Delta \in \mathbb{B}_1 :$ otherwise,

$$\mu_{\mathbb{B}_1}(A) : = (min\{\sigma_1(\Delta) : det(I - A\Delta) = 0\})^{-1}, \quad \forall\, \Delta \in \mathbb{B}_1,$$

where minimum is over all perturbations $\Delta \in \mathbb{B}_1$.

The spectral radius of $A \in \mathbb{R}^{n, n}$ is denoted by $\rho(\cdot)$ and is defined as

$$\rho(A) : = max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\},$$

with $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of matrix A. The main contribution of this article is to present some new results on the interconnection between Schur stability and structured singular values for $A \in \mathbb{R}^{n, n}$. Our results on Schur stability are more general in the sense that they apply to both complex-valued matrices as well as the real-valued matrices.

2. Preliminaries results on $\mu_{\mathbb{B}_1}(A)$

In this section, we recall some important properties and results on structured singular values.

2.1 *Properties of structured singular values*

Some of the basic properties of structured singular values are taken from [22] (properties P_1 to P_4) and from [27] (properties P_5 to P_8), and they are as follows:

P_1: $\mu(\alpha A) = |\alpha|\mu(A)$ for a square matrix $A \in \mathbb{R}^{n,n}$.

P_2: $\mu(I) = 1$, for an identity matrix I.

P_3: $\mu(AB) \le \sigma_1(A)\mu(B)$, where $\sigma_1(\cdot)$ is the largest singular value of matrix, and B is also a square matrix.

P_4: $\mu(\Delta) = \sigma_1(\Delta)$, $\forall\, \Delta \in X_\delta$.

P_5: Let $\Delta_0 = \{\lambda I : \lambda \in \mathbb{C}\}$, then $\mu(A) = \rho(A)$, where $\rho(A)$ is the spectral radius of A.

P_6: Let $\Delta = \{diag(\Delta_1, \Delta_2, \cdots, \Delta_n) : \Delta_i \in \mathbb{C}^{n,n}\}$, then $\mu_\Delta(A) = \mu_\Delta(D^{-1}AD)$, where $D = \{diag(d_1, \cdots, d_n),\ |d_i| > 0\}$.

P_7: Let $\Delta_0 = diag(\Delta_1, \Delta_2, \cdots, \Delta_n)$, $\Delta_i \in \mathbb{C}^{n,n}$, then $\rho(A) < \mu(A) < \sigma_1(A)$.

P_8: From P_6 and P_7, we have that

$$\mu(A) = \mu(D^{-1}AD) \le inf\, \sigma_1(D^{-1}AD),$$

where **inf** is taken over D.

2.2 *Results on the computation of structured singular values*

Next, we recall some of the well-known results on the computation of structured singular values. There is an alternative expression concerning the computation of structured singular values for a given $A \in \mathbb{R}^{n,n}$, and $\mathbb{B}_1$. This fact is provided in Lemma 3.7 taken from [28], which shows that the computation of structured singular values is equivalent to the computation of the spectral radius of an admissible perturbation from $\mathbb{B}_1$ times the given $A \in \mathbb{R}^{n,n}$.

Lemma 1 Let $A \in \mathbb{R}^{n,n}$, then

$$\mu_{\mathbb{B}_1}(A) = max\, \rho(\Delta A),$$

where $\Delta \in \mathbb{B}_1$, and the **max** is taken over all such Δ's.

For $Q \in \mathbb{B}_1$, we define a set $\mathbb{Q}_1$ as

$$\mathbb{Q}_1 = \{Q \in \mathbb{B}_1 : Q^*Q = I_n\},$$

and consider a positive diagonal matrix D. Then, the computation of structured singular value is given by following Theorem 3.8, taken from [28].

Theorem 2 Let $Q \in \mathbb{Q}_1$, and $D > 0$, a positive diagonal matrix. Then

$$\mu_{\mathbb{B}_1}(AQ) = \mu_{\mathbb{B}_1}(QA) = \mu_{\mathbb{B}_1}(A) = \mu_{\mathbb{B}_1}(D^{\frac{1}{2}}AD^{\frac{-1}{2}}).$$

3. New results on Schur stability and $\mu_{\mathbb{B}_1}(A)$

In this section, we present some new results on the interconnection between Schur stability, and $\mu_{\mathbb{B}_1}(A)$, where $A \in \mathbb{R}^{n,n}$. The set $\mathbb{B}_1$, the set of uncertainties is defined in the introductory section.

Assumption 3 For $A \in \mathbb{R}^{2,2}$, the spectrum $\sigma(A) = \{\lambda_i\}_{i=1}^{2}$ does not contain the zero eigenvalue.

The following Theorem 4 discusses an interconnection between 2-dimensional Schur stable matrices, and structured singular value $\mu_{\mathbb{B}_1}(A)$, where $\mathbb{B}_1$, is the set of block diagonal matrices.

Theorem 4 Let $A = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathbb{R}^{2,2}$. Then A is Schur stable if

$$0 \leq \mu_{\mathbb{B}_1}\left[(I_2 + A)^{-1}(I_2 - A)\right] < 1,$$

where I_2 is a 2-dimensional identity matrix.

Proof. Let $\Delta = (I_2 - D)(I_2 + D)^{-1}$ is with 2-dimensional diagonal structure. The matrix $D = diag(d_{ii})$, for all $i = 1, 2$, is a real positive diagonal matrix. The matrix D in term of $\Delta \in \mathbb{B}_1$ is of the form

$$D = (I_2 + \Delta)^{-1}(I_2 - \Delta), \ \forall \Delta \in \mathbb{B}_1.$$

To show that $0 \leq \mu_{\mathbb{B}_1}\left[(I_2 + A)^{-1}(I_2 - A)\right] < 1$, it is necessary to show that

$$\lambda_1\left[I_2 - (I_2 + A)^{-1}(I_2 - A)\Delta\right] \neq 0, \quad \forall \Delta \in \mathbb{B}_1$$

and

$$\lambda_2\left[I_2 - (I_2 + A)^{-1}(I_2 - A)\Delta\right] \neq 0, \quad \forall \Delta \in \mathbb{B}_1.$$

The $rank\left[A + (I_2 + \Delta)^{-1}(I_2 - \Delta)\right]$ will lead us to proof, that is,

$$rank\left[A + (I_2 + \Delta)^{-1}(I_2 - \Delta)\right] = rank\left[(I_2 + A) - (I_2 - A)\Delta\right], \quad \forall \Delta \in \mathbb{B}_1.$$

This leads us to matrix $\left(I_2 - (I_2 + A)^{-1}(I_2 - A)\Delta\right)$, $\forall \Delta \in \mathbb{B}_1$. From this, it is further obvious that

$$\lambda_1\left[I_2 - (I_2 + A)^{-1}(I_2 - A)\Delta\right] \neq 0, \quad \forall \Delta \in \mathbb{B}_1,$$

and

$$\lambda_2\left[I_2 - (I_2 + A)^{-1}(I_2 - A)\Delta\right] \neq 0, \quad \forall \Delta \in \mathbb{B}_1,$$

which is the necessary condition that

$$0 \leq \mu_{\mathbb{B}_1} \left[(I_2 + A)^{-1}(I_2 - A) \right] < 1.$$

□

Theorem 5 provides an interconnection between Schur stable matrix $A \in \mathbb{R}^{2,\,2}$ and the structured singular value of $(I_2 + DA + A^T D)^{-1}(I_2 - DA - A^T D)$, where A^T is the transpose of A.

Theorem 5 Let $A = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathbb{R}^{2,\,2}$. Then A is Schur Stable if

$$0 \leq \mu_{\mathbb{B}_1} \left[(I_2 + DA + A^T D)^{-1}(I_2 - DA - A^T D) \right] < 1,$$

where $D = diag(d_{ii})$, $d_{ii} > 0, \forall\, i = 1,\, 2$.

Proof. Let $\Delta \in \mathbb{B}_1$, a block diagonal structure, that is, $\Delta = (I_2 - D)(I_2 + D)^{-1}$, $D = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$ such that $d_{11} > 0$, $d_{22} > 0$. As, $\lambda_1(DA + A^T D) \neq 0, \forall D$, and $\lambda_2(DA + A^T D) \neq 0, \forall D$. This allows us to have that

$$\lambda_{1,\,2} \left[DA + A^T D + (\mathbf{i}I_2 + \Delta)^{-1}(I_2 - \Delta) \right] \neq 0, \quad \forall D, \ \forall \Delta \in \mathbb{B}_1.$$

This implies

$$\lambda_{1,\,2} \left[(I_2 + DA + A^T D) - (I_2 - DA - A^T D)\Delta \right] \neq 0, \quad \forall D, \ \forall \Delta \in \mathbb{B}_1.$$

Thus,

$$\lambda_{1,\,2} \left[I_2 - (I_2 + DA + A^T D)^{-1}(I_2 - DA - A^T D)\Delta \right] \neq 0, \quad \forall D, \ \forall \Delta \in \mathbb{B}_1.$$

The last expression for $\lambda_{1,\,2}$ is precisely the necessary condition that

$$0 \leq \mu_{\mathbf{B}_1} \left[(I_2 + DA + A^T D)^{-1}(I_2 - DA - A^T D) \right] < 1.$$

□

In Theorem 6, the inequalities present the conditions for Schur stability and structured singular values for a three-dimensional real-valued matrix.

Theorem 6 Let $A = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \in \mathbb{R}^{3,\,3}$. Then A is Schur stable if

1. $\left| \sum_{i=1}^{3} \lambda_i(A) \prod_{i=1}^{3} \lambda_i(A) - (m_{11} + m_{22} + m_{33}) \right| < 1 - \prod_{i=1}^{3} \lambda_i(A)$, and

2. $0 \leq \mu_S\left(A_S(v)\right) < \sigma_1\left(A_S(v)\right) \leq 1$, where $\sigma_1\left(A_S(v)\right)$ is the largest singular value of $A_S(v) := S^{-1}A$, $S = \begin{pmatrix} I & 0 \\ 0 & v^T v \end{pmatrix}$, $v \in \mathbb{R}^{3,\,1}$.

Proof. Consider that $\sum_{i=1}^{3}\lambda_i(A)\prod_{i=1}^{3}\lambda_i(A)-(m_{11}+m_{22}+m_{33})\geq 0$, and $\prod_{i=1}^{3}\lambda_i(A)>0$. For $A\in\mathbb{R}^{3,\,3}$, we know that

$$\sum_{i=1}^{3}\lambda_i(A)+\prod_{i=1}^{3}\lambda_i(A)-1<(m_{11}+m_{22}+m_{33})$$

or

$$\sum_{i=1}^{3}\lambda_i(A)+\prod_{i=1}^{3}\lambda_i(A)<1+(m_{11}+m_{22}+m_{33})$$

subtract $(m_{11}+m_{22}+m_{33})$ from the last inequality, we have

$$0\leq\sum_{i=1}^{3}\lambda_i(A)\prod_{i=1}^{3}\lambda_i(A)-(m_{11}+m_{22}+m_{33})$$

$$\leq\prod_{i=1}^{3}\lambda_i(A)+\prod_{i=1}^{3}\lambda_i(A)(m_{11}+m_{22}+m_{33})-(m_{11}+m_{22}+m_{33})-\prod_{i=1}^{3}\lambda_i^2(A)$$

$$=\prod_{i=1}^{3}\lambda_i(A)-(m_{11}+m_{22}+m_{33}).$$

This imply

$$0\leq\sum_{i=1}^{3}\lambda_i(A)\prod_{i=1}^{3}\lambda_i(A)-(m_{11}+m_{22}+m_{33})<1-\prod_{i=1}^{3}\lambda_i^2(A).$$

Since, $1-\prod_{i=1}^{3}\lambda_i^2(A)>0$ because $|\prod_{i=1}^{3}\lambda_i(A)|<1$.
(ii) The matrix decomposition of $A\in\mathbb{R}^{n,\,n}$, ($n=3$ can be taken) can be written as

$$A=U\begin{pmatrix}\sigma_1 & 0\\ 0 & T\end{pmatrix}V^T,$$

where U, V are unitary matrices. Let $\sigma_1(A_S)=||AQ_1||_2=||A||_2$, $||Q_1||_2=1$, with $||\cdot||_2$ is matrix 2-norm defined over the real-valued matrix.

Let $u_1=\dfrac{AQ_1}{\sigma_1}$, this yields $||u_1||_2=\dfrac{||AQ_1||_2}{||A||_2}=1$. Also, take $U=(u_1|U_2)$, and $V=(v_1|V_2)$. Then the matrix product U^TAV becomes

$$(u_1|U_2)A(v_1|V_2)=\begin{pmatrix}\sigma_1 & a^T\\ 0 & B\end{pmatrix},$$

where $u_1^T u_1 = 1$, $U_2^T u_1 = 0$, $a = V_2^T A u_1$, and $B = U_2^T A V_2$.

Take $a = 0$, yields

$$\sigma_1^2(A_S) = \max_{x \neq 0} \frac{\left\| \begin{pmatrix} \sigma_1 & u^T \\ 0 & B \end{pmatrix} x \right\|_2^2}{||x||_2^2}.$$

Replacing $x \to a$, gives

$$\sigma_1^2 = \sigma_1^2 + a^T a, \quad \Longrightarrow \; a = 0$$

and thus,

$$U^T A V = \begin{pmatrix} \sigma_1 & 0 \\ 0 & B \end{pmatrix}$$

or

$$A = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & B \end{pmatrix} V^T.$$

To show that $0 \leq \mu_S(A_S(v)) \leq \sigma_1(A_S(v))$, we have that

$$A_S(v) \colon = \begin{pmatrix} M_{11} & M_{12} \\ \frac{1}{v} M_{11} & \frac{1}{v} M_{22} \end{pmatrix}.$$

Also,

$$\begin{pmatrix} I & A_S(v) \\ A_S^T(v) & I \end{pmatrix} > 0 \iff I - A_S(v) I A_S^T(v) \geq 0.$$

From this, it follows that

$$\lambda_i(I - A_S(v) A_S^T(v)) \geq 0, \; \forall i.$$

implies $0 \leq \sigma_1(A_S(v)) \leq 1$. □

The following Theorem 7 show that the given real-value d Hermitian matrix $A \in \mathbb{R}^{3,\,3}$ is Schur stable if it can decomposed as $A = P - Q$.

Theorem 7 Let $A \in \mathbb{R}^{3,\,3}$ be a singular matrix, and let $A = P - Q$. Then, the matrix $P^{-1} Q$ is Schur stable, that is,

$$\max_i |\lambda_i(P^{-1}Q)| < 1.$$

Proof. As $A = P - Q$, with P, Q being as the Hermitian matrices such that P^{-1} exists, then $A^{-1}Q$ can be written as

$$A^{-1} = (P-Q)^{-1} = (I_3 - P^{-1}Q)P^{-1}Q.$$

The expression for $P^{-1}Q$ is

$$P^{-1}Q = (A+Q)^{-1}Q = (I_3 + A^{-1}Q)^{-1}A^{-1}Q.$$

For $x \neq 0$, $x \in \mathbb{R}^{3,\,1}$, we have

$$A^{-1}Px = \frac{1}{1 - \max_i |\lambda_i(P^{-1}Q)|}x.$$

Finally,

$$\begin{aligned}
& A^{-1}Qx \geq P^{-1}Qx \\
\iff & \max_i |\lambda_i(P^{-1}Q)|x\left(1 - \max_i |\lambda_i(P^{-1}Q)|\right)^{-1} \geq \max_i |\lambda_i(P^{-1}Q)|x \\
\iff & \max_i |\lambda_i(P^{-1}Q)| < 1.
\end{aligned}$$

□

Theorem 8 gives the condition for Schur stability of given matrix while taking into account the computations of its eigenvectors, and an admissible perturbation $\varepsilon \in [0,\,1)$.

Theorem 8 Let $A \in \mathbb{R}^{n\times n}$. Then, A is Schur stable for $\varepsilon > 0$, $\varepsilon x - Ax > 0$, where $x > 0$, is an eigenvector, such that

$$\max_i |\lambda_i(A)| < \varepsilon, \quad \varepsilon \in [0,\,1).$$

Proof. The proof is straightforward by letting $y > 0$, the left eigenvector such that

$$y^T(\varepsilon x - Ax) > 0 \iff \left(\varepsilon - \max_i |\lambda_i(A)|\right) y^T x > 0.$$

Since, $x > 0$, $y^T > 0$, the last inequality becomes

$$\max_i |\lambda_i(A)| < \varepsilon, \quad \varepsilon \in [0, 1).$$

□

4. Conclusion

The present work introduces novel findings on the relationship between Schur stability and structured singular values of real-valued *n*-dimensional matrices. We derive these findings by using a range of mathematical techniques from linear algebra, matrix analysis, and system theory. Both Schur's stability and structured singular values play a pivotal role in modern system theory and the design of robust systems. Schur stability ensures the boundedness and stability of system responses by requiring that all eigenvalues lie within the unit circle. In contrast, structured singular values provide a means to quantify a system's resilience, stability, and performance under organized disturbances in system parameters. This quantification offers critical insights into the stability margins and performance boundaries in the presence of uncertainties.

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Conflict of interest

The authors declare no competing financial interest.

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