Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

F-Hardy Rogers Type Contractions Endowed with Mann's Iterative Scheme in Convex Generalized b-Metric Spaces

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Received: 12 September 2024; **Revised:** 11 October 2024; **Accepted:** 11 November 2024

Abstract: This article presents novel fixed point results using Mann's iterative process in complete convex b-metric spaces, building upon Isa Yildirim's recent work. The author established the definition of the $\mathscr{F}$-Hardy-Rogers contraction of the Nadler type by relaxing two conditions of Wardowski's $\mathscr{F}$-mapping. Our approach employs Mann's iterative scheme in $\mathscr{G}_b$-metric spaces under convex conditions. A supporting example with detailed calculations validates our result. Furthermore, we demonstrate the applicability of our findings by solving an integral equation through fixed point equation along with the axioms of the provided result. The obtained results are generalizations of several existing results in the literature.

Keywords: metric space ($\mathfrak{MS}$), b-metric space (b-$\mathfrak{MS}$), $\mathscr{G}_b$-metric space ($\mathscr{G}_b$-$\mathfrak{MS}$), cauchy sequence ($\mathfrak{cs}$), fixed point ($\mathfrak{fp}$), banach contraction principle ($\mathfrak{BCP}$)

MSC: 47H10, 54H25

1. Introduction

Fixed point ($\mathfrak{fp}$) theory, a vital branch of functional analysis, has numerous applications in nonlinear analysis. The contraction mapping principle, also known as Banach's contraction principle ($\mathfrak{BCP}$) [1], is a fundamental tool for studying nonlinear equations. Its constructive nature enables numerical calculation of fixed points, making it an intriguing area of research. This task is achieved by converting an operator equation $\mathscr{G}\zeta = 0$ into a $\mathfrak{fp}$ equation $Q\zeta = \zeta$ with self-mapping Q and a suitable domain. In recent years, researchers have extensively generalized $\mathfrak{BCP}$ by modifying spaces, contraction conditions, or both.

Chen et al. [2] presented the notion of convex b-$\mathfrak{MS}$ and established certain $\mathfrak{fp}$ results. Ek et al. [3] applied the convex condition to Chatterjea and Hardy Roger's contractive mappings and proved some $\mathfrak{fp}$ results which are analogous to this concept.

Iterative processes are an important feature of many numerical techniques, especially for finding a $\mathfrak{fp}$. The Picard iteration scheme is the most simple and commonly used iterative scheme which is applied in $\mathfrak{BCP}$. Later, in 1953, Mann [4] presented Mann's iterative scheme to approximate $\mathfrak{fp}$ of a mapping, which is a generalization of Picard iteration. Ji et al. [5] used the convex structure endowed with $\mathscr{G}_b$-$\mathfrak{MS}$ to prove $\mathfrak{fp}$ results using Mann's iterative scheme. Moudafi [6]

DOI: https://doi.org/10.37256/cm.6120255712

used the concept of Mann's iterative scheme to generalize some $\mathfrak{fp}$ results. In his research he has presented a method for finding hierarchically a fixed-point of a nonexpansive mapping Q with respect to a nonexpansive mapping P. Unlike in the case of $\mathfrak{BCP}$, a non-expensive map Q even with a unique $\mathfrak{fp}$ may fail to converge to the $\mathfrak{fp}$ with iterative sequence $\Psi_{u+1} = Q\Psi_u$, $\Psi_0 \in \mathscr{S}$, $u \geq 0$. According to Krasnoselski [7], a convergent sequence of successive approximations can be obtained by taking the auxiliary non-expensive mapping $\frac{1}{2}(I+Q)$, where I represents the identity transformation, i.e., if the sequence of successive approximations is defined, for arbitrary $\Psi_0 \in \mathscr{D}$, by

$$\Psi_{u+1} = \frac{1}{2}(\Psi_u + Q\Psi_u), \ u \geq 0. \tag{1}$$

It is clear that the mapping Q and $\frac{1}{2}(I+Q)$ have the same set of $\mathfrak{fp}$'s so that the limit of the convergent sequence defined by (1) is necessarily a $\mathfrak{fp}$ of $\mathscr{Q}$. On the other hand, Ullah and Arshad [8] introduced a new concept known as the K^*-iterative scheme. This scheme provides some accurate results in the least iterative steps equipped with Suzuki mappings. Since then, many generalizations have been made using different iterative schemes. Some other most commonly used iterative schemes are the Picard-Mann hybrid [9], iterative methods by strictly pseudocontractive mappings [10], S^*-iteration of Karahan and Ozdemir [11], SP iteration of Pheungrattana and Suantai [12], iterative scheme of Suzuki's generalized non-expansive mappings [13].

In 2012, Wardowski [14] presented a new fixed point theorem concerning $\mathscr{F}$-contraction using a mapping $\mathscr{F} : \mathbb{R}^+ \to \mathbb{R}$. The concept of $\mathscr{F}$-contraction has inspired a bulk of research studies since its introduction. Aslam et al. [15] proved coincidence point results endowed with $\mathscr{F}$-weak contraction by using a binary relation. Cosentino et al. [16] presented the notion of $\mathscr{F}$-contractive mappings of Hardy-Rogers-type which further generalizes the $\mathscr{F}$-contraction by relaxing two condition of $\mathscr{F}$ mappings. Asif et al. [17] presented the concept of $\mathscr{F}$-Reich contraction in convex b-metric spaces.

In 2022, Yildirim [18] presented the $\mathscr{F}$-Hardy-Rogers of Nadler's type contraction by removing two condition of $\mathscr{F}$ mappings. Moreover, He established some $\mathfrak{fp}$ results using Mann's iterative scheme in convex b-$\mathfrak{M}\mathscr{S}$. Motivated by the idea of Yildirim [18], this article encompasses some $\mathfrak{fp}$ results on the platform of convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ using the Mann's iterative scheme by further weakening the conditions on $\mathscr{F}$.

The article is structured as follows:

1. Necessary definitions and preliminaries.
2. Existence and uniqueness of $\mathfrak{fp}$ theorems using $\mathscr{F}$-Hardy Roger's type contraction.
3. Example and application.
4. Conclusion.

2. Preliminaries

In the current section, we will recollect some basics for the best understanding of this article.

Definition 1 [19] Let $\mathscr{D} \neq \phi$ and $d : \mathscr{D} \times \mathscr{D} \to [0, +\infty)$ be a mapping, which fulfills the subsequent properties for every $\Psi, \zeta, \eta \in \mathscr{D}$:

(1): $d(\Psi, \zeta) = 0 \iff \Psi = \zeta$;

(2): $d(\Psi, \zeta) = d(\zeta, \Psi)$;

(3): $d(\Psi, \eta) \leq \mathfrak{s}[d(\Psi, \zeta) + d(\zeta, \eta)]$ for $\mathfrak{s} \geq 1$,

then d and $(\mathscr{D}, d)$ represents b-metric and b-$\mathfrak{M}\mathscr{S}$ respectively.

In [20], Aghajani et al. presented the idea of $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ as follows.

Definition 2 [20] Let $\mathscr{D} \neq \phi$ and $\mathscr{G} : \mathscr{D} \times \mathscr{D} \times \mathscr{D} \to [0, +\infty)$ be a mapping, which fulfills the subsequent properties for each $\Psi, \zeta, \eta \in \mathscr{D}$:

(1): $\mathscr{G}(\Psi, \zeta, \eta) = 0$ if $\Psi = \zeta = \eta$;

(2): $\mathscr{G}(\Psi, \Psi, \zeta) > 0$ for every $\Psi, \zeta \in \mathscr{D}$ with $\Psi \neq \zeta$;

(3): $\mathscr{G}(\Psi,\ \Psi,\ \zeta) \leq \mathscr{G}(\Psi,\ \zeta,\ \eta)$ for every $\Psi,\ \zeta,\ \eta \in \mathscr{D}$ with $\zeta \neq \eta$;

(4): $\mathscr{G}(\Psi,\ \zeta,\ \eta) = \mathscr{G}(p\{\Psi,\ \zeta,\ \eta\})$, where p is a permutation of $(\Psi,\ \zeta,\ \eta)$ (symmetry);

(5): there exists $\mathfrak{s} \geq 1$ such that $\mathscr{G}(\Psi,\ \zeta,\ \eta) \leq \mathfrak{s}[\mathscr{G}(\Psi,\ \zeta,\ \zeta) + \mathscr{G}(\zeta,\ \zeta,\ \eta)]$ for every $\Psi,\ \zeta,\ \eta,\ \zeta \in \mathscr{D}$, then $\mathscr{G}$ and $(\mathscr{D},\ \mathscr{G})$ are called $\mathscr{G}_b$-metric and $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ respectively.

Remark 1 [20] It is important to note that b-$\mathfrak{M}\mathscr{S}$ and $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ are equivalent topologically. To get benefit from this fact, we can use many results of b-$\mathfrak{M}\mathscr{S}$ into $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$.

Definition 3 [21] Let $(\mathscr{D},\ \mathscr{G})$ denotes a $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$. We say that $\{\Psi_{\mathrm{u}}\} \subseteq \mathscr{D}$ is a $\mathscr{G}$-Cauchy sequence ($\mathfrak{cs}$) if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $\mathfrak{l},\ \mathfrak{m},\ \mathfrak{n} \geq N$, $\mathscr{G}(\Psi_{\mathfrak{l}},\ \Psi_{\mathfrak{m}},\ \Psi_{\mathfrak{n}}) < \varepsilon$.

Definition 4 [20] Let $(\mathscr{D},\ \mathscr{G})$ denotes a $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$. If there exists $\Psi_0 \in \mathscr{D}$ such that $\lim\limits_{\mathrm{u},\ k \to +\infty} \mathscr{G}(\Psi_{\mathrm{u}},\ \Psi_k, \Psi_0) = 0$, then $\{\Psi_{\mathrm{u}}\} \subseteq \mathscr{D}$ is called a convergent sequence in $\mathscr{D}$.

Remark 2 If every $\mathfrak{cs}$ is convergent in $\mathscr{D}$ then $(\mathscr{D},\ \mathscr{G})$ is called complete $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$.

Definition 5 [20] A $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ is called symmetric if $\mathscr{G}(\Psi_{\mathrm{u}},\ \Psi_k,\ \Psi_k) = \mathscr{G}(\Psi_k,\ \Psi_{\mathrm{u}},\ \Psi_{\mathrm{u}})$ for every $\Psi_{\mathrm{u}},\ \Psi_k \in \mathscr{D}$.

Definition 6 [22] Consider two $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ defined as $(\mathscr{D}_1,\ \mathscr{G}_1)$ and $(\mathscr{D}_2,\ \mathscr{G}_2)$. Then $f : (\mathscr{D}_1,\ \mathscr{G}_1) \to (\mathscr{D}_2,\ \mathscr{G}_2)$ is $\mathscr{G}$-continuous at a point $\Psi_0 \in \mathscr{D}$ if for every $\Psi_1,\ \Psi_2 \in \mathscr{D}$ and $\varepsilon > 0$, there exists $\delta > 0$, such that $\mathscr{G}_1(\Psi_0,\ \Psi_1,\ \Psi_2) < \delta \implies \mathscr{G}_2(f\Psi_0,\ f\Psi_1,\ f\Psi_2) < \varepsilon$.

Proposition 1 [20] Consider two $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ defined as $(\mathscr{D}_1,\ \mathscr{G}_1)$ and $(\mathscr{D}_2,\ \mathscr{G}_2)$. Then $f : (\mathscr{D}_1,\ \mathscr{G}_1) \to (\mathscr{D}_2,\ \mathscr{G}_2)$ is $\mathscr{G}$-continuous at a point $\Psi_0 \in \mathscr{D} \iff f(\Psi_{\mathrm{u}})$ is $\mathscr{G}$-convergent to $f(\Psi_0)$ whenever $\{\Psi_{\mathrm{u}}\}$ is $\mathscr{G}$-convergent to Ψ_0.

Definition 7 [5] Let $(\mathscr{D},\ \mathscr{G})$ be a $\mathscr{G}_b$- $\mathfrak{M}\mathscr{S}$ and a mapping $\mathrm{Q} : \mathscr{D} \to \mathscr{D}$. We say that $\{\Psi_{\mathrm{u}}\}$ is a Mann sequence if

$$\Psi_{\mathrm{u}+1} = v(\Psi_{\mathrm{u}},\ \mathscr{Q}\Psi_{\mathrm{u}};\ \mu_{\mathrm{u}}),\ \mathrm{u} \in \mathbb{N}_0,$$

where $\Psi_0 \in \mathscr{D}$ and $\mu_{\mathrm{u}} \in [0,\ 1]$.

However, Iterative methods have an important role in finding $\mathfrak{fp}$'s of non-expansive mappings. In particular, Mann iterative is one of the well-known methods to find the approximations of the problems by using iteration schemes. Mann's iterative scheme is defined as

$$\Psi_{\mathrm{u}+1} = \mu_{\mathrm{u}}\Psi_{\mathrm{u}} + (1 - \mu_{\mathrm{u}})\mathrm{Q}\Psi_{\mathrm{u}},\ \mu_{\mathrm{u}} \in [0,\ 1].$$

Definition 8 [5] Let $(\mathscr{D},\ \mathscr{G})$ be a $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ with constant $\mathfrak{s} \geq 1$ and $I = [0,\ 1]$. A mapping $v : \mathscr{D} \times \mathscr{D} \times I \to \mathscr{D}$ is called a convex structure on $\mathscr{D}$ if for each $\Psi_1,\ \Psi_2,\ \Psi_3,\ \eta,\ \zeta \in \mathscr{D}$ and $\mu \in I$

$$\mathscr{G}(\eta,\ \zeta,\ v(\Psi_1,\ \Psi_2;\ \mu)) \leq \mu\mathscr{G}(\eta,\ \zeta,\ \Psi_1) + (1 - \mu)\mathscr{G}(\eta,\ \zeta,\ \Psi_2) \tag{2}$$

holds, then $(\mathscr{D},\ \mathscr{G},\ v)$ is called a convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$.

Next, we present an example of convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$.

Example 1 Let $\mathscr{D} = \mathbb{R}^{\mathrm{n}}$ and define a b-metric $d : \mathscr{D} \times \mathscr{D} \to [0,\ +\infty)\ \forall\ \zeta,\ \Psi \in \mathscr{D}$ by

$$d(\zeta,\ \Psi) = \sum_{i=1}^{\mathrm{n}} (\zeta_i - \Psi_i)^2,$$

for each $\zeta = (\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_{\mathrm{n}}) \in \mathscr{D}$, $\Psi = (\Psi_1,\ \Psi_2,\ \cdots,\ \Psi_{\mathrm{n}}) \in \mathscr{D}$ and define the mapping $v : \mathscr{D} \times \mathscr{D} \times [0,\ 1] \to \mathscr{D}$ by

$$v(\Psi,\ \zeta;\ \mu) = \frac{\Psi + \zeta}{2}.$$

Then $(\mathscr{D},\ d)$ is a convex b-$\mathfrak{M}\mathscr{S}$ with $\mathfrak{s} = 2$. Define a metric $\mathscr{G} : \mathscr{D} \times \mathscr{D} \times \mathscr{D} \to [0,\ +\infty)$ by

$$\mathscr{G}(\Psi,\ \zeta,\ \eta) = \max\{d(\Psi,\ \zeta),\ d(\Psi,\ \eta),\ d(\eta,\ \zeta)\}\ \forall\ \Psi,\ \zeta,\ \eta \in \mathscr{D}.$$

For each $\Psi,\ \zeta,\ \alpha,\ \beta \in \mathscr{D}$, we have

$$\begin{aligned}\mathscr{G}(\Psi,\ \zeta,\ v(\alpha,\ \beta;\ \mu)) &= \max\{d(\Psi,\ \zeta),\ d(\Psi,\ v(\alpha,\ \beta;\ \mu)),\ d(\zeta,\ v(\alpha,\ \beta;\ \mu))\} \\ &\le \max\{d(\Psi,\ \zeta),\ \mu d(\Psi,\ \alpha) + (1-\mu)d(\Psi,\ \beta),\ \mu d(\zeta,\ \alpha) + (1-\mu)d(\zeta,\ \beta)\} \\ &\le \mu \max\{d(\Psi,\ \zeta),\ d(\Psi,\ \alpha),\ d(\zeta,\ \alpha)\} + (1-\mu)\max\{d(\Psi,\ \zeta),\ d(\Psi,\ \beta),\ d(\zeta,\ \beta)\} \\ &= \mu\mathscr{G}(\Psi,\ \zeta,\ \alpha) + (1-\mu)\mathscr{G}(\Psi,\ \zeta,\ \beta).\end{aligned}$$

Hence $(\mathscr{G},\ \mathscr{D},\ v)$ is a convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ with $\mathfrak{s} = 2^{p-1}$.

Remark 3 A convex $\mathscr{G}_b$- $\mathfrak{M}\mathscr{S}$ becomes a convex $\mathscr{G}$-$\mathfrak{M}\mathscr{S}$ for $\mathfrak{s} = 1$.

Wardowski [14] introduced the $\mathscr{F}$-contraction in 2012, which plays a crucial role in recent trends of research in the area of $\mathfrak{fp}$ theory. Cosentino et al. [23] presented the following.

Definition 9 [23] Let $\mathfrak{s} \ge 1$ be a real number. $\mathscr{F} : (0,\ +\infty) \to \mathbb{R}$ be a mapping which fulfills the subsequent conditions:

(F_1): $\mathscr{F}$ is strictly increasing,

(F_2): for every sequence $\{\Psi_\mathrm{u}\}_{\mathrm{u}\in\mathbb{N}}$ of positive numbers $\lim_{\mathrm{u}\to+\infty} \Psi_\mathrm{u} = 0 \iff \lim_{\mathrm{u}\to+\infty} \mathscr{F}(\Psi_\mathrm{u}) = -\infty$,

(F_3): there exists $k \in (0,\ 1)$ such that $\lim_{\Psi\to 0^+} \Psi^k \mathscr{F}(\Psi) = 0$,

(F_4): for every sequence $\{\Psi_\mathrm{u}\} \subset \mathbb{R}^+$

$$\text{if } \tau + \mathscr{F}(\mathfrak{s}\Psi_\mathrm{u}) \le \mathscr{F}(\mathfrak{s}\Psi_{\mathrm{u}-1})\ \forall\ \mathrm{u} \in \mathbb{N},\ \tau \in \mathbb{R}^+,\ \text{ then}$$

$$\tau + \mathscr{F}(\mathfrak{s}^\mathrm{u}\Psi_\mathrm{u}) \le \mathscr{F}(\mathfrak{s}^{\mathrm{u}-1}\Psi_{\mathrm{u}-1})\ \forall\ \mathrm{u} \in \mathbb{N}.$$

Definition 10 [14] Let $(\mathscr{D},\ d)$ be a $\mathfrak{M}\mathscr{S}$. A mapping $\mathrm{Q} : \mathscr{D} \to \mathscr{D}$ is said to be $\mathscr{F}$-contraction if there exists $\tau > 0$ such that $d(\mathrm{Q}\Psi_1,\ \mathrm{Q}\Psi_2) > 0$

$$\implies \tau + \mathscr{F}(d(\mathrm{Q}\Psi_1,\ \mathrm{Q}\Psi_2)) \le \mathscr{F}(d(\Psi_1,\ \Psi_2)) \quad \text{for each } \Psi_1,\ \Psi_2 \in \mathscr{D}.$$

Popescu and Stan [24] proved fixed point results by applying weaker symmetrical conditions on the self-map of a complete metric space, Wadowski's control function $\mathscr{F}$, and the contractions defined by Wardowski. Vujakovic et al. [25] proved Wardowski type results within $\mathscr{G}$-$\mathfrak{M}\mathscr{S}$ using only the condition F_1. Fabiano et al. [26] presented a beautiful survey on $\mathscr{F}$ mappings and suggested some improvements on the conditions of $\mathscr{F}$ mapping involved in the contractive condition.

We now state a property [25, 26] of the function $\mathscr{F}$ which is the consequence of the condition F_1. This paper is a third chapter of the book (see, [27]).

• At each point $\mathrm{u} \in (0, +\infty)$ there exist its left and right limits $\lim_{\zeta \to \mathrm{u}^-} \mathscr{F}(\zeta) = \mathscr{F}(\mathrm{u}^-)$ and $\lim_{\zeta \to \mathrm{u}^+} \mathscr{F}(\zeta) = \mathscr{F}(\mathrm{u}^+)$. Moreover, for the function $\mathscr{F}$ one of the subsequent two properties hold: $\mathscr{F}(0^+) = m \in \mathbb{R}$ or $\mathscr{F}(0^+) = -\infty$.

The collection of functions that satisfy condition (F_1) are denoted by $\mathbb{F}$.

3. Main results

Definition 11 Assume that $\mathscr{F} \in \mathbb{F}$ and $(\mathscr{D}, \mathscr{G}, v)$ is a complete convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ with $\mathfrak{s} > 1$. Then $\mathrm{Q} : \mathscr{D} \to \mathscr{D}$ is said to be a $\mathscr{F}$-Hardy Rogers type contraction if for $\mathrm{f}, \mathrm{g}, \mathrm{h} : \mathscr{D} \times \mathscr{D} \to \left[0, \frac{1}{2}\right)$ the subsequent equation hold:

$$\tau + \mathscr{F}(\mathfrak{s}\mathscr{G}(\mathrm{Q}\Psi, \mathrm{Q}\zeta, \mathrm{Q}\gamma)) \leq \mathscr{F}\Big(\mathrm{f}(\Psi, \zeta, \gamma)\mathscr{G}(\Psi, \zeta, \gamma) + \mathrm{g}(\Psi, \zeta, \gamma)\Big[\mathscr{G}(\Psi, \mathrm{Q}\Psi, \mathrm{Q}\Psi) + \mathscr{G}(\zeta, \mathrm{Q}\zeta, \mathrm{Q}\zeta)$$

$$+ \mathscr{G}(\gamma, \mathrm{Q}\gamma, \mathrm{Q}\gamma)\Big] + \mathrm{h}(\Psi, \zeta, \gamma)\Big[\mathscr{G}(\mathrm{Q}\Psi, \mathrm{Q}\Psi, \zeta) + \mathscr{G}(\mathrm{Q}\zeta, \mathrm{Q}\zeta, \Psi) \tag{3}$$

$$+ \mathscr{G}(\mathrm{Q}\gamma, \mathrm{Q}\gamma, \Psi)\Big]\Big),$$

for every $\Psi, \zeta, \gamma \in \mathscr{D}$, with $p \neq q$ and $q \neq r$.

Theorem 1 Let $(\mathscr{D}, \mathscr{G}, v)$ be a complete convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ with a convex structure v and $\mathrm{Q} : \mathscr{D} \to \mathscr{D}$ is a $\mathscr{F}$-Hardy Rogers type contraction. Assume that the sequence $\{\Psi_\mathrm{u}\}$ is defined as

$$\Psi_\mathrm{u} = v(\Psi_{\mathrm{u}-1}, \mathrm{Q}\Psi_{\mathrm{u}-1}, \mu_{\mathrm{u}-1}), \text{ where } 0 < \mu_{\mathrm{u}-1} < \frac{1}{4\mathfrak{s}^2} \ \forall\, \mathrm{u} \in \mathbb{N}, \tag{4}$$

then a unique $\mathfrak{fp}$ of $\mathscr{Q}$ exists, provided that

$$\mathrm{f}(\Psi, \zeta, \gamma) + 3\mathrm{g}(\Psi, \zeta, \gamma) + 4\mathrm{h}(\Psi, \zeta, \gamma) \leq \frac{1}{4\mathfrak{s}^4}. \tag{5}$$

Proof. By Equation (4) and convex structure of the $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$

$$\begin{aligned} \mathscr{G}(\Psi_\mathrm{u}, \Psi_\mathrm{u}, \Psi_{\mathrm{u}+1}) &= \mathscr{G}(\Psi_\mathrm{u}, \Psi_\mathrm{u}, v(\Psi_\mathrm{u}, \mathrm{Q}\Psi_\mathrm{u}; \mu_\mathrm{u})) \\ &\leq (1 - \mu_\mathrm{u})\mathscr{G}(\Psi_\mathrm{u}, \Psi_\mathrm{u}, \mathrm{Q}\Psi_\mathrm{u}) \end{aligned} \tag{6}$$

and

$$
\begin{aligned}
\mathscr{G}(\Psi_{u},\, Q\Psi_{u},\, Q\Psi_{u}) &= \mathscr{G}(v(\Psi_{u-1},\, Q\Psi_{u-1};\, \mu_{u-1}),\, Q\Psi_{u},\, Q\Psi_{u})\\
&\leq (\mu_{u-1})\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u}) + (1-\mu_{u-1})\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u})\\
&\leq (\mu_{u-1})\mathfrak{s}\Big\{\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + \mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u})\Big\}\\
&\quad + (1-\mu_{u-1})\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u})\\
&\leq (\mu_{u-1})\mathfrak{s}\Big\{\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + \mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u})\Big\}\\
&\quad + \mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u})\\
&= \mu_{u-1}\mathfrak{s}\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + (1+\mu_{u-1}\mathfrak{s})\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u})\\
&\leq \mu_{u-1}\mathfrak{s}\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + \mathfrak{s}(1+\mu_{u-1})\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u}).
\end{aligned}
$$

Therefore,

$$
\mathscr{G}(\Psi_{u},\, Q\Psi_{u},\, Q\Psi_{u}) \leq \mu_{u-1}\mathfrak{s}\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + \mathfrak{s}(1+\mu_{u-1})\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u}). \tag{7}
$$

By using contraction

$$
\begin{aligned}
\tau + \mathscr{F}\Big(\mathfrak{s}\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u},\, Q\Psi_{u}\Big) \leq \mathscr{F}\Big(&\mathrm{f}(\Psi_{u-1},\, \Psi_{u},\, \Psi_{u})\mathscr{G}(\Psi_{u-1},\, \Psi_{u},\, \Psi_{u}) + \mathrm{g}(\Psi_{u-1},\, \Psi_{u},\, \Psi_{u})\\
&\Big[\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + \mathscr{G}(\Psi_{u},\, Q\Psi_{u},\, Q\Psi_{u})\\
&+ \mathscr{G}(\Psi_{u},\, Q\Psi_{u},\, Q\Psi_{u})\Big] + \mathrm{h}(\Psi_{u-1},\, \Psi_{u},\, \Psi_{u})\Big[\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u-1},\, \Psi_{u})\\
&+ \mathscr{G}(Q\Psi_{u},\, Q\Psi_{u},\, \Psi_{u-1}) + \mathscr{G}(Q\Psi_{u},\, Q\Psi_{u},\, \Psi_{u-1})\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
\implies \mathscr{F}\Big(\mathfrak{s}\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_u,\, Q\Psi_u\Big) \leq \mathscr{F}\Big(& f(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\mathscr{G}(\Psi_{u-1},\, \Psi_u,\, \Psi_u) + g(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\\
&\Big[\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + \mathscr{G}(\Psi_u,\, Q\Psi_u,\, Q\Psi_u)\\
&+\mathscr{G}(\Psi_u,\, Q\Psi_u,\, Q\Psi_u)\Big] + h(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\Big[\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u-1},\, \Psi_u)\\
&+\mathscr{G}(Q\Psi_u,\, Q\Psi_u,\, \Psi_{u-1}) + \mathscr{G}(Q\Psi_u,\, Q\Psi_u,\, \Psi_{u-1})\Big]\Big) - \tau\\
\leq \mathscr{F}\Big(& f(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\mathscr{G}(\Psi_{u-1},\, \Psi_u,\, \Psi_u) + g(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\\
&\Big[\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + \mathscr{G}(\Psi_u,\, Q\Psi_u,\, Q\Psi_u)\\
&+\mathscr{G}(\Psi_u,\, Q\Psi_u,\, Q\Psi_u)\Big] + h(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\Big[\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u-1},\, \Psi_u)\\
&+\mathscr{G}(Q\Psi_u,\, Q\Psi_u,\, \Psi_{u-1}) + \mathscr{G}(Q\Psi_u,\, Q\Psi_u,\, \Psi_{u-1})\Big]\Big).
\end{aligned}
$$

With the help of property $\mathscr{F}_1$, we have

$$
\begin{aligned}
\mathfrak{s}\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_u,\, Q\Psi_u) \leq\ & f(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\mathscr{G}(\Psi_{u-1},\, \Psi_u,\, \Psi_u) + g(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\\
&\Big[\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + 2\mathscr{G}(\Psi_u,\, Q\Psi_u,\, Q\Psi_u)\Big] \qquad (8)\\
&+h(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\Big[\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u-1},\, \Psi_u) + 2\mathscr{G}(Q\Psi_u,\, Q\Psi_u,\, \Psi_{u-1})\Big].
\end{aligned}
$$

By using Equation (8) in Equation (7),

$$
\begin{aligned}
\mathscr{G}(\Psi_u,\, Q\Psi_u,\, Q\Psi_u) \leq\ & \mu_{u-1}\mathfrak{s}\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + (1+\mu_{u-1})\Big\{f(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\mathscr{G}(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\\
&+g(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\Big[\mathscr{G}(\Psi_{u-1},\, Q\Psi_{u-1},\, Q\Psi_{u-1}) + 2\mathscr{G}(\Psi_u,\, Q\Psi_u,\, Q\Psi_u)\Big]\\
&+h(\Psi_{u-1},\, \Psi_u,\, \Psi_u)\Big[\mathscr{G}(Q\Psi_{u-1},\, Q\Psi_{u-1},\, \Psi_u) + 2\mathscr{G}(Q\Psi_u,\, Q\Psi_u,\, \Psi_{u-1})\Big]\Big\}.
\end{aligned}
$$

From Equation(6),

$$\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) \leq \mu_{u-1}\mathfrak{s}\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + (1+\mu_{u-1})(1-\mu_{u-1})f(\Psi_{u-1}, \Psi_u, \Psi_u)$$

$$\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + (1+\mu_{u-1})g(\Psi_{u-1}, \Psi_u, \Psi_u)\Big[\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1})$$

$$+ 2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)\Big] + (1+\mu_{u-1})h(\Psi_{u-1}, \Psi_u, \Psi_u)\Big[\mathscr{G}(Q\Psi_{u-1}, Q\Psi_{u-1}, \Psi_u)$$

$$+ 2\mathscr{G}(Q\Psi_u, Q\Psi_u, \Psi_{u-1})\Big].$$

Then

$$\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) \leq \mu_{u-1}\mathfrak{s}\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + (1-\mu_{u-1}^2)f(\Psi_{u-1}, \Psi_u, \Psi_u)$$

$$\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + (1+\mu_{u-1})g(\Psi_{u-1}, \Psi_u, \Psi_u)\Big[\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1})$$

$$+ 2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)\Big] + (1+\mu_{u-1})h(\Psi_{u-1}, \Psi_u, \Psi_u)\Big[\mathscr{G}(\Psi_u, Q\Psi_{u-1}, Q\Psi_{u-1})$$

$$+ 2\mathscr{G}(\Psi_{u-1}, Q\Psi_u, Q\Psi_u)\Big]$$

$$\leq \mu_{u-1}\mathfrak{s}\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + (1-\mu_{u-1}^2)f(\Psi_{u-1}, \Psi_u, \Psi_u)$$

$$\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + (1+\mu_{u-1})g(\Psi_{u-1}, \Psi_u, \Psi_u)\Big[\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1})$$

$$+ 2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)\Big] + \mathfrak{s}(1+\mu_{u-1})h(\Psi_{u-1}, \Psi_u, \Psi_u)\Big[\mathscr{G}(\Psi_u, \Psi_{u-1}, \Psi_{u-1})$$

$$+ \mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + 2\mathscr{G}(\Psi_{u-1}, \Psi_u, \Psi_u) + 2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)\Big]$$

$$\leq \mu_{u-1}\mathfrak{s}\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + (1-\mu_{u-1}^2)f(\Psi_{u-1}, \Psi_u, \Psi_u)$$

$$\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + (1+\mu_{u-1})g(\Psi_{u-1}, \Psi_u, \Psi_u)\Big[\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1})$$

$$+ 2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)\Big] + \mathfrak{s}(1+\mu_{u-1})h(\Psi_{u-1}, \Psi_u, \Psi_u)\Big[(1-\mu_{u-1})$$

$$\mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1}) + \mathscr{G}(\Psi_{u-1}, Q\Psi_{u-1}, Q\Psi_{u-1})$$

$$+2\mathscr{G}(\Psi_{u},\ Q\Psi_{u},\ Q\Psi_{u})+2(1-\mu_{u-1})\mathscr{G}(\Psi_{u-1},\ Q\Psi_{u-1},\ Q\Psi_{u-1}))\Big]$$

$$=\Big[\mu_{u-1}\mathfrak{s}+(1-\mu_{u-1}^{2})f(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+\mathfrak{s}(1+\mu_{u-1})$$

$$h(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})(4-3\mu_{u-1})\Big]\mathscr{G}(\Psi_{u-1},\ Q\Psi_{u-1},\ Q\Psi_{u-1})+\Big[2(1+\mu_{u-1})$$

$$g(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})\Big]\mathscr{G}(\Psi_{u},\ Q\Psi_{u},\ Q\Psi_{u}),$$

which implies that

$$\Big[1-\Big\{2(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})\Big\}\Big]\mathscr{G}(\Psi_{u},\ Q\Psi_{u},\ Q\Psi_{u})$$

$$\leq\Big[\mu_{u-1}\mathfrak{s}+(1-\mu_{u-1}^{2})f(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+\mathfrak{s}(1+\mu_{u-1})$$

$$h(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})(4-3\mu_{u-1})\Big]\mathscr{G}(\Psi_{u-1},\ Q\Psi_{u-1},\ Q\Psi_{u-1}).$$

By the hypothesis, we know that

$$f(\Psi,\ \zeta,\ \gamma)+3g(\Psi,\ \zeta,\ \gamma)+4h(\Psi,\ \zeta,\ \gamma)\leq\frac{1}{4\mathfrak{s}^{4}}\text{ and }\mu_{u-1}\in\left(0,\ \frac{1}{4\mathfrak{s}^{2}}\right].$$

Consider

$$2(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})$$

$$\leq 2(1+\mu_{u-1})\mathfrak{s}g(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})$$

$$=2\,(1+\mu_{u-1})\,\mathfrak{s}[g(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})+h(\Psi_{u-1},\ \Psi_{u},\ \Psi_{u})]$$

$$\leq 2\left(1+\frac{1}{4\mathfrak{s}^{2}}\right)\mathfrak{s}\times\frac{1}{4\mathfrak{s}^{4}}$$

$$<1,$$

and

$$
\begin{aligned}
&(1-\mu_{u-1}^2)f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+\mathfrak{s}(1+\mu_{u-1})\\
&h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)(4-3\mu_{u-1})\\
\leq&(1-\mu_{u-1}^2)f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+4\mathfrak{s}(1+\mu_{u-1})\\
&h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\\
=&(1+\mu_{u-1})(1-\mu_{u-1})f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+4\mathfrak{s}(1+\mu_{u-1})\\
&h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\\
\leq&(1+\mu_{u-1})\Big[f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+4h\mathfrak{s}(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\Big]\\
\leq&\mathfrak{s}(1+\mu_{u-1})\Big[f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+4h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\Big]\\
\leq&\left(1+\frac{1}{4\mathfrak{s}^2}\right)\mathfrak{s}\times\frac{1}{4\mathfrak{s}^4}\\
<&\ 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathscr{G}(\Psi_u,\ Q\Psi_u,\ Q\Psi_u)\\
&\leq\frac{\begin{array}{c}\mu_{u-1}\mathfrak{s}+(1-\mu_{u-1}^2)f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\\ +\mathfrak{s}(1+\mu_{u-1})h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)(4-3\mu_{u-1})\end{array}}{1-\{2(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\}}\\
&\times\mathscr{G}(\Psi_{u-1},\ Q\Psi_{u-1},\ Q\Psi_{u-1}).
\end{aligned}
$$

Denote,

$$
\theta_{u-1}=\frac{\begin{array}{c}\mu_{u-1}\mathfrak{s}+(1-\mu_{u-1}^2)f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\\ +\mathfrak{s}(1+\mu_{u-1})h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)(4-3\mu_{u-1})\end{array}}{1-\{2(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\}}.
$$

Then,

$$
\begin{aligned}
\theta_{u-1} &= \frac{\begin{array}{c}\mu_{u-1}\mathfrak{s}+(1-\mu_{u-1}^2)f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\\ +\mathfrak{s}(1+\mu_{u-1})h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)(4-3\mu_{u-1})\end{array}}{1-\{2(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\}}\\
&\leq \frac{\begin{array}{c}\mu_{u-1}\mathfrak{s}+(1-\mu_{u-1}^2)f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\\ +\mathfrak{s}(1+\mu_{u-1})4h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\end{array}}{1-\{2(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\}}\\
&< \frac{\begin{array}{c}\mu_{u-1}\mathfrak{s}+(1+\mu_{u-1})f(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\\ +\mathfrak{s}(1+\mu_{u-1})4h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\end{array}}{1-\{2(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\}}\\
&< \frac{1+\frac{1}{4\mathfrak{s}}}{1-\{2(1+\mu_{u-1})g(\Psi_{u-1},\ \Psi_u,\ \Psi_u)+2(1+\mu_{u-1})\mathfrak{s}h(\Psi_{u-1},\ \Psi_u,\ \Psi_u)\}}-1\\
&< \frac{1+\frac{1}{4\mathfrak{s}}}{1-\mathfrak{s}\left(1+\frac{1}{4\mathfrak{s}^2}\right)\frac{1}{4\mathfrak{s}^4}}-1\\
&= \frac{\frac{4\mathfrak{s}+1}{4\mathfrak{s}}}{1-\frac{1}{4\mathfrak{s}^3}-\frac{1}{16\mathfrak{s}^5}}-1\\
&= \frac{4\mathfrak{s}^4+4\mathfrak{s}^2+1}{16\mathfrak{s}^5-4\mathfrak{s}^2-1}\\
&< 1.
\end{aligned}
$$

Implies

$$
\begin{aligned}
\mathscr{G}(\Psi_u,\ Q\Psi_u,\ Q\Psi_u) &\leq \theta_{u-1}\mathscr{G}(\Psi_{u-1},\ Q\Psi_{u-1},\ Q\Psi_{u-1})\\
&< \frac{4\mathfrak{s}^4+4\mathfrak{s}^2+1}{16\mathfrak{s}^5-4\mathfrak{s}^2-1}\mathscr{G}(\Psi_{u-1},\ Q\Psi_{u-1},\ Q\Psi_{u-1}).
\end{aligned} \tag{9}
$$

By using (3), (7) and (9), we have

$$\tau+\mathscr{F}(d_{\mathrm{u}}) = \tau+\mathscr{F}(\mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\Psi_{\mathrm{u}},\, \mathrm{Q}\Psi_{\mathrm{u}}))$$

$$\leq \mathscr{F}\Big(\frac{4\mathfrak{s}^4+4\mathfrak{s}^2+1}{16\mathfrak{s}^5-4\mathfrak{s}^2-1}\mathscr{G}(\Psi_{\mathrm{u}-1},\, \mathrm{Q}\Psi_{\mathrm{u}-1},\, \mathrm{Q}\Psi_{\mathrm{u}-1})\Big) = \mathscr{F}(d_{\mathrm{u}-1})$$

$$\Longrightarrow \mathscr{F}(\mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\Psi_{\mathrm{u}},\, \mathrm{Q}\Psi_{\mathrm{u}})) \leq \mathscr{F}\Big(\frac{4\mathfrak{s}^4+4\mathfrak{s}^2+1}{16\mathfrak{s}^5-4\mathfrak{s}^2-1}\mathscr{G}(\Psi_{\mathrm{u}-1},\, \mathrm{Q}\Psi_{\mathrm{u}-1},\, \mathrm{Q}\Psi_{\mathrm{u}-1})\Big) - \tau$$

that is,

$$\mathscr{F}(d_{\mathrm{u}}) < \mathscr{F}(d_{\mathrm{u}-1}) - \tau \text{ for all } p \in \mathbb{N}. \tag{10}$$

Since $\mathscr{F}$ is strictly increasing, then $d_{\mathrm{u}} < d_{\mathrm{u}-1}$. Thus, we conclude that the sequence $\{d_{\mathrm{u}}\}$ is strictly decreasing, so there exists $\lim\limits_{u\to+\infty} d_{\mathrm{u}} = d$. Suppose that $d > 0$. Since $\mathscr{F}$ is increasing mapping there exists $\lim\limits_{\zeta\to d^+} \mathscr{F}(\zeta) = \mathscr{F}(d^+)$, so taking limit as $p \to +\infty$ in inequality (10), we get

$$\tau+\mathscr{F}(d^+) \leq \mathscr{F}(d^+),$$

a contradiction. Therefore $\lim\limits_{u\to+\infty} d_{\mathrm{u}} = 0$,

$$\lim_{\mathrm{u}\to+\infty} \mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\Psi_{\mathrm{u}},\, \mathrm{Q}\Psi_{\mathrm{u}}) = 0.$$

Then

$$\begin{aligned} \lim_{\mathrm{u}\to+\infty} \mathscr{G}(\Psi_{\mathrm{u}},\, \Psi_{\mathrm{u}},\, \Psi_{\mathrm{u}+1}) &= \lim_{\mathrm{u}\to+\infty} \mathscr{G}(\Psi_{\mathrm{u}},\, \Psi_{\mathrm{u}},\, \nu(\Psi_{\mathrm{u}},\, \mathrm{Q}\Psi_{\mathrm{u}};\, \mu_{\mathrm{u}})) \\ &\leq \lim_{\mathrm{u}\to+\infty} (1-\mu_{\mathrm{u}})\mathscr{G}(\Psi_{\mathrm{u}},\, \Psi_{\mathrm{u}},\, \mathrm{Q}\Psi_{\mathrm{u}}), \end{aligned} \tag{11}$$

which implies that

$$\lim_{\mathrm{u}\to+\infty} \mathscr{G}(\Psi_{\mathrm{u}},\, \Psi_{\mathrm{u}},\, \Psi_{\mathrm{u}+1}) = 0. \tag{12}$$

Now, we will check the Cauchyness of the sequence $\{\Psi_{\mathrm{u}}\}$. For this, we proceed by a contradiction. Assume that $\{\Psi_{\mathrm{u}}\}$ is not a Cauchy sequence. Then there exists an ε and two subsequences $\{\Psi_{\nu(\hat{\lambda})}\}$ and $\{\Psi_{\mathrm{w}(\hat{\lambda})}\}$ of $\{\Psi_{\mathrm{u}}\}$ such that

$$\nu(\hat{\lambda}) > \mathrm{w}(\hat{\lambda}) > \hat{\lambda},$$

and

$$\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})}) \geq \varepsilon,$$

and

$$\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})}) \leq \varepsilon.$$

Then,

$$\begin{aligned}\varepsilon &\leq \mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})}) \\ &\leq \mathfrak{s}\left\{\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) + \mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})})\right\}.\end{aligned}$$

By using (12)

$$\varepsilon \leq \mathfrak{s} \lim_{\hat{\lambda}\to+\infty} \mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}).$$

Therefore,

$$\begin{aligned}\frac{\varepsilon}{\mathfrak{s}} &\leq \lim_{\hat{\lambda}\to+\infty} \mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) \\ &\leq \lim_{\hat{\lambda}\to+\infty} \sup \mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}).\end{aligned}$$

Also,

$$
\begin{aligned}
\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) &= \mathscr{G}(\nu(\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}; \mu_{\nu(\hat{\lambda})-1}), \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) \\
&\leq \mu_{\nu(\hat{\lambda})-1}\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) \\
&\quad + (1-\mu_{\nu(\hat{\lambda})-1})\mathscr{G}(Q\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) \\
&\leq \mu_{\nu(\hat{\lambda})-1}\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) \\
&\quad + \mathfrak{s}(1-\mu_{\nu(\hat{\lambda})-1})\Big[\mathscr{G}(Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1}) \\
&\quad + \mathscr{G}(Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big].
\end{aligned} \tag{13}
$$

By using the contraction condition,

$$
\begin{aligned}
&\mathscr{F}(\mathfrak{s}\mathscr{G}(Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1})) \\
\leq &\mathscr{F}\Big(\mathrm{f}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) \\
&+ \mathrm{g}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}) \\
&+ \mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})-1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1}) + \mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})+1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1})\Big] \\
&+ \mathrm{h}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[\mathscr{G}(Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) \\
&+ \mathscr{G}(Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})-1}) + \mathscr{G}(Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, Q\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})-1)}\Big]\Big) - \tau \\
\leq &\mathscr{F}\Big(\mathrm{f}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}) \\
&+ \mathrm{g}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1})
\end{aligned}
$$

$$+\mathscr{G}(\Psi_{w(\hat{\lambda})-1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}) + \mathscr{G}(\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1})\Big]$$

$$+ h(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\Big[\mathscr{G}(Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1})$$

$$+\mathscr{G}(Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})-1}) + \mathscr{G}(Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})-1)}\Big]\Big).$$

By using $\mathscr{F}_1$,

$$\mathfrak{s}\mathscr{G}(Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1})$$

$$\leq f(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})$$

$$+ g(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1})$$

$$+\mathscr{G}(\Psi_{w(\hat{\lambda})-1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}) + \mathscr{G}(\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1})\Big]$$

$$+ h(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\Big[\mathscr{G}(Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1})$$

$$+\mathscr{G}(Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})-1}) + \mathscr{G}(Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})-1)}\Big].$$

Use the above equation in (13),

$$\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})$$

$$\leq (\mu_{\nu(\hat{\lambda})-1})\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1}) + (1-\mu_{\nu(\hat{\lambda})-1})$$

$$\Big(\Big[f(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})$$

$$+ g(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1})$$

$$+\mathscr{G}(\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}) + \mathscr{G}(\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1})\Big]$$

$$
\begin{aligned}
&+\mathrm{h}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\nu(\hat{\lambda})-1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1})\\
&+\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\nu(\hat{\lambda})-1})+\mathscr{G}(\mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})-1})\Big]\\
&+\mathfrak{s}\mathscr{G}(\mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big]\Big)\\
\leq\ &\mathfrak{s}(\mu_{\nu(\hat{\lambda})-1})[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})})+\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})]+(1-\mu_{\nu(\hat{\lambda})-1})\\
&\Big[\mathrm{f}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\mathfrak{s}[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})})+\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})]\\
&+\mathrm{g}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \mathrm{Q}\Psi_{\nu(\hat{\lambda})-1}, \mathrm{Q}\Psi_{\nu(\hat{\lambda})-1})\\
&+2\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1})\Big]\\
&+\mathrm{h}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[2\mathfrak{s}\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda}))}\\
&+2\mathfrak{s}\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1})+\mathfrak{s}\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})}, \Psi_{\nu(\hat{\lambda})})\Big]\\
&+\mathfrak{s}\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \mathrm{Q}\Psi_{\nu(\hat{\lambda})-1}, \mathrm{Q}\Psi_{\nu(\hat{\lambda})-1})\Big]+\mathfrak{s}\mathscr{G}(\mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\\
\leq\ &\mathfrak{s}(\mu_{\nu(\hat{\lambda})-1})[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})})+\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})]+(1-\mu_{\nu(\hat{\lambda})-1})\\
&\Big[\mathrm{f}(\Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\mathfrak{s}[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})})+\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})]\\
&+\mathrm{g}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \mathrm{Q}\Psi_{\nu(\hat{\lambda})-1}, \mathrm{Q}\Psi_{\nu(\hat{\lambda})-1})\\
&+2\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1})\Big]\\
&+\mathrm{h}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[2\mathfrak{s}\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})})\\
&+2\mathfrak{s}^2\Big[\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})}, \Psi_{\mathrm{w}(\hat{\lambda})+1}, \Psi_{\mathrm{w}(\hat{\lambda})+1})+\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1}, \mathrm{Q}\Psi_{\mathrm{w}(\hat{\lambda})+1})\Big]
\end{aligned}
$$

$$+\mathfrak{s}\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\nu(\hat{\lambda})}, \Psi_{w(\hat{\lambda})+1}) + \mathfrak{s}^2\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\nu(\hat{\lambda})-1}, \Psi_{\nu(\hat{\lambda})-1})$$

$$+\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1})\Big] + \mathfrak{s}\mathscr{G}(Q\Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1}).$$

By applying $\lim_{\hat{\lambda}\to+\infty}$ in above inequality,

$$\lim_{\hat{\lambda}\to+\infty} \sup \mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})$$

$$\leq \lim_{\hat{\lambda}\to+\infty} \Big\{ \sup \mathfrak{s}(\mu_{\nu(\hat{\lambda})-1})[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda})}) + \mathscr{G}(\Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})] + (1-\mu_{\nu(\hat{\lambda})-1})$$

$$\Big[f(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\mathfrak{s}[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda}))} + \mathscr{G}(\Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})]$$

$$+ g(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1})$$

$$+ 2\mathscr{G}(\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1})\Big]$$

$$+ h(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\Big[2\mathfrak{s}\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda}))}$$

$$+ 2\mathfrak{s}^2\Big[\mathscr{G}(\Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1}) + \mathscr{G}(\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1}, Q\Psi_{w(\hat{\lambda})+1})\Big]$$

$$+ \mathfrak{s}\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\nu(\hat{\lambda})}, \Psi_{w(\hat{\lambda})+1}) + \mathfrak{s}^2\Big[\mathscr{G}(\Psi_{\nu(\hat{\lambda})}, \Psi_{\nu(\hat{\lambda})-1}, \Psi_{\nu(\hat{\lambda})-1})$$

$$+ \mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1}, Q\Psi_{\nu(\hat{\lambda})-1})\Big] + \mathfrak{s}\mathscr{G}(Q\Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\Big\}$$

$$\leq \lim_{\hat{\lambda}\to+\infty} \Big\{ \sup \mathfrak{s}(\mu_{\nu(\hat{\lambda})-1})\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda})})$$

$$+ \Big[f(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\mathfrak{s}[\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda})})]$$

$$+ h(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})+1}, \Psi_{w(\hat{\lambda})+1})\Big[2\mathfrak{s}\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1}, \Psi_{w(\hat{\lambda})}, \Psi_{w(\hat{\lambda})})$$

$$+ \mathfrak{s}\mathscr{G}(\Psi_{w(\hat{\lambda})+1}, \Psi_{\nu(\hat{\lambda})}, \Psi_{\nu(\hat{\lambda})})\Big]\Big\}$$

$$\leq \lim_{\hat{\lambda}\to+\infty}\Big\{\Big[\sup \mathfrak{s}(\mu_{\nu(\hat{\lambda})-1}) + \mathrm{f}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1})\mathfrak{s}\Big]\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})$$

$$+\mathrm{h}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[2\mathfrak{s}\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})$$

$$+\mathfrak{s}^3\mathscr{G}(\Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\nu(\hat{\lambda})-1})\Big]\Big\}$$

$$\leq \lim_{\hat{\lambda}\to+\infty}\Big\{\Big[\sup \mathfrak{s}(\mu_{\nu(\hat{\lambda})-1}) + \mathrm{f}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1})\mathfrak{s}\Big]\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})$$

$$+\mathrm{h}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[(2\mathfrak{s}+\mathfrak{s}^3)\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})\Big\}$$

$$\leq \lim_{\hat{\lambda}\to+\infty}\Big\{\Big[\sup \mathfrak{s}(\mu_{\nu(\hat{\lambda})-1}) + \mathrm{f}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1})\mathfrak{s}^3\Big]\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})$$

$$+\mathrm{h}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1})\Big[(2\mathfrak{s}^3+\mathfrak{s}^3)\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})\Big\}$$

$$\leq \lim_{\hat{\lambda}\to+\infty}\Big\{\sup(\mathfrak{s}(\frac{1}{4\mathfrak{s}^2})+\mathfrak{s}^3(\mathrm{f}+3\mathrm{h}))\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})\Big\}$$

$$\leq \lim_{\hat{\lambda}\to+\infty}\Big\{\sup(\mathfrak{s}(\frac{1}{4\mathfrak{s}^2})+\mathfrak{s}^3(\frac{1}{4\mathfrak{s}^4})\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})\Big\}$$

$$= \lim_{\hat{\lambda}\to+\infty}\sup\Big\{\frac{1}{2\mathfrak{s}}\mathscr{G}(\Psi_{\nu(\hat{\lambda})-1},\ \Psi_{\mathrm{w}(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})})\Big\}.$$

Implies

$$\frac{\varepsilon}{\mathfrak{s}} \leq \lim_{\hat{\lambda}\to+\infty}\sup \mathscr{G}(\Psi_{\nu(\hat{\lambda})},\ \Psi_{\mathrm{w}(\hat{\lambda})+1},\ \Psi_{\mathrm{w}(\hat{\lambda})+1}) \leq \frac{\varepsilon}{2\mathfrak{s}},$$

which leads to a contradiction. Thus $\{\Psi_{\mathrm{u}}\}$ is a Cauchy sequence in $\mathscr{D}$. Since $(\mathscr{D},\ \mathscr{G},\ \nu)$ is a complete convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$, there exists $\hat{\Psi}\in\mathscr{D}$ such that $\Psi_{\mathrm{u}}\to\hat{\Psi}\in\mathscr{D}$ as $\mathrm{u}\to+\infty$. Now, we will show that $\hat{\Psi}$ is a $\mathfrak{fp}$ of Q. Note that

$$\begin{aligned}\mathscr{G}(\hat{\Psi},\ \mathrm{Q}\hat{\Psi},\ \mathrm{Q}\hat{\Psi}) &\leq \mathfrak{s}\Big[\mathscr{G}(\hat{\Psi},\ \Psi_{\mathrm{u}},\ \Psi_{\mathrm{u}})+\mathscr{G}(\Psi_{\mathrm{u}},\ \mathrm{Q}\hat{\Psi},\ \mathrm{Q}\hat{\Psi})\Big]\\ &\leq \mathfrak{s}\mathscr{G}(\hat{\Psi},\ \Psi_{\mathrm{u}},\ \Psi_{\mathrm{u}})+\mathfrak{s}^2\Big[\mathscr{G}(\Psi_{\mathrm{u}},\ \mathrm{Q}\Psi_{\mathrm{u}},\ \mathrm{Q}\Psi_{\mathrm{u}})+\mathscr{G}(\mathrm{Q}\Psi_{\mathrm{u}},\ \mathrm{Q}\hat{\Psi},\ \mathrm{Q}\hat{\Psi})\Big]\end{aligned} \tag{14}$$

and

$$\begin{aligned}\mathscr{F}(\mathfrak{s}\mathscr{G}(Q\Psi_u, Q\hat{\Psi}, Q\hat{\Psi})) \leq& \mathscr{F}\Big(f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi}) + g(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)\\ &+ \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi}) + \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})\Big] + h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(Q\Psi_u, Q\Psi_u, \hat{\Psi})\\ &+ \mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u) + \mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u)\Big]\Big) - \tau\\ \leq& \mathscr{F}\Big(f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi}) + g(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)\\ &+ \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi}) + \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})\Big] + h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(Q\Psi_u, Q\Psi_u, \hat{\Psi})\\ &+ \mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u) + \mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u)\Big]\Big),\end{aligned}$$

with the help of $\mathscr{F}_1$,

$$\begin{aligned}\mathfrak{s}\mathscr{G}(Q\Psi_u, Q\hat{\Psi}, Q\hat{\Psi}) \leq& f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi}) + g(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})\\ &+ \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})\Big] + h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(Q\Psi_u, Q\Psi_u, \hat{\Psi}) + \mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u)\\ &+ \mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u)\Big].\end{aligned} \tag{15}$$

From (14) and (15),

$$\begin{aligned}\mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi}) \leq& \mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathfrak{s}^2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathfrak{s}^2\mathscr{G}(Q\Psi_u, Q\hat{\Psi}, Q\hat{\Psi})\\ \leq& \mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathfrak{s}^2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathfrak{s}\Big\{f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi})\\ &+ g(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi}) + \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})\Big]\\ &+ h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(Q\Psi_u, Q\Psi_u, \hat{\Psi}) + \mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u) + \mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u)\Big]\Big\}\\ =& \mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathfrak{s}^2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathfrak{s}\Big\{f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi})\end{aligned}$$

$$+ g(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + 2\mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})\Big]$$

$$+ h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(Q\Psi_u, Q\Psi_u, \hat{\Psi}) + 2\mathscr{G}(Q\hat{\Psi}, Q\hat{\Psi}, \Psi_u)\Big]\Big\}$$

$$\leq \mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathfrak{s}^2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathfrak{s}\Big\{f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi})$$

$$+ g(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + 2\mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})\Big]$$

$$+ h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathfrak{s}[\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)]$$

$$+ 2\mathfrak{s}[\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi}) + \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})]\Big]\Big\}.$$

Hence

$$\mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi}) \leq \mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathfrak{s}^2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathfrak{s}\Big\{f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi}) + g(\Psi_u, \hat{\Psi}, \hat{\Psi})$$

$$\Big[\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + 2\mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})\Big] + h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[\mathfrak{s}[\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u)$$

$$+ \mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u)] + 2\mathfrak{s}[\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi}) + \mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})]\Big]\Big\},$$

$$\Longrightarrow [1 - 2\mathfrak{s}g(\Psi_u, \hat{\Psi}, \hat{\Psi}) - 2\mathfrak{s}^2h(\Psi_u, \hat{\Psi}, \hat{\Psi})]\mathscr{G}(\hat{\Psi}, Q\hat{\Psi}, Q\hat{\Psi})$$

$$\leq \mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathfrak{s}^2\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathfrak{s}f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi}) + \mathfrak{s}g(\Psi_u, \hat{\Psi}, \hat{\Psi})$$

$$\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathfrak{s}h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[2\mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathfrak{s}\mathscr{G}(\Psi_u, Q\Psi_u, Q\Psi_u) + \mathfrak{s}\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big]$$

$$< \mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u) + \mathfrak{s}^2\left(\frac{4\mathfrak{s}^4 + 4\mathfrak{s}^2 + 1}{16\mathfrak{s}^5 - 4\mathfrak{s}^2 - 1}\right)^u\mathscr{G}(\Psi_0, Q\Psi_0, Q\Psi_0) + \mathfrak{s}f(\Psi_u, \hat{\Psi}, \hat{\Psi})\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi})$$

$$+ \mathfrak{s}g(\Psi_u, \hat{\Psi}, \hat{\Psi})\left(\frac{4\mathfrak{s}^4 + 4\mathfrak{s}^2 + 1}{16\mathfrak{s}^5 - 4\mathfrak{s}^2 - 1}\right)^u\mathscr{G}(\Psi_0, Q\Psi_0, Q\Psi_0) + \mathfrak{s}h(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big[2\mathfrak{s}\mathscr{G}(\hat{\Psi}, \Psi_u, \Psi_u)$$

$$+ \mathfrak{s}\left(\frac{4\mathfrak{s}^4 + 4\mathfrak{s}^2 + 1}{16\mathfrak{s}^5 - 4\mathfrak{s}^2 - 1}\right)^u\mathscr{G}(\Psi_0, Q\Psi_0, Q\Psi_0) + \mathfrak{s}\mathscr{G}(\Psi_u, \hat{\Psi}, \hat{\Psi})\Big].$$

As we know that $\frac{4\mathfrak{s}^4+4\mathfrak{s}^2+1}{16\mathfrak{s}^5-4\mathfrak{s}^2-1}<1$. By applying limit

$$\lim_{\mathrm{u}\to+\infty}\Big\{\mathfrak{s}\mathscr{G}(\hat{\Psi},\ \Psi_{\mathrm{u}},\ \Psi_{\mathrm{u}})+\mathfrak{s}^2\Big(\frac{4\mathfrak{s}^4+4\mathfrak{s}^2+1}{16\mathfrak{s}^5-4\mathfrak{s}^2-1}\Big)^{\mathrm{u}}\mathscr{G}(\Psi_0,\ \mathrm{Q}\Psi_0,\ \mathrm{Q}\Psi_0)$$

$$+\mathfrak{s}\mathrm{f}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})\mathscr{G}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})+\mathfrak{s}\mathrm{g}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})\Big(\frac{4\mathfrak{s}^4+4\mathfrak{s}^2+1}{16\mathfrak{s}^5-4\mathfrak{s}^2-1}\Big)^{\mathrm{u}}\mathscr{G}(\Psi_0,\ \mathrm{Q}\Psi_0,\ \mathrm{Q}\Psi_0)$$

$$+\mathfrak{s}\mathrm{h}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})\Big[2\mathfrak{s}\mathscr{G}(\hat{\Psi},\ \Psi_{\mathrm{u}},\ \Psi_{\mathrm{u}})$$

$$+\mathfrak{s}\Big(\frac{4\mathfrak{s}^4+4\mathfrak{s}^2+1}{16\mathfrak{s}^5-4\mathfrak{s}^2-1}\Big)^{\mathrm{u}}\mathscr{G}(\Psi_0,\ \mathrm{Q}\Psi_0,\ \mathrm{Q}\Psi_0)+\mathfrak{s}\mathscr{G}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})\Big]\Big\}$$

$$=0.$$

It implies that

$$\lim_{\mathrm{u}\to+\infty}[1-2\mathfrak{s}\mathrm{g}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})-2\mathfrak{s}^2\mathrm{h}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})]\mathscr{G}(\hat{\Psi},\ \mathrm{Q}\hat{\Psi},\ \mathrm{Q}\hat{\Psi})=0. \tag{16}$$

By assumption,

$$\mathrm{f}(\Psi,\ \zeta,\ \gamma)+3\mathrm{g}(\Psi,\ \zeta,\ \gamma)+4\mathrm{h}(\Psi,\ \zeta,\ \gamma)\le\frac{1}{4\mathfrak{s}^4}\text{ and }\mu_{\mathrm{u}-1}\in\Big(0,\ \frac{1}{4\mathfrak{s}^2}\Big]\ \forall\, n\in\mathbb{N}\text{ and }\Psi,\ \zeta,\ \gamma\in\mathscr{D}. \tag{17}$$

This implies that

$$2\mathfrak{s}\mathrm{g}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})+2\mathfrak{s}^2\mathrm{h}(\Psi_{\mathrm{u}},\ \hat{\Psi},\ \hat{\Psi})<1\quad\forall\, n\in\mathbb{N}.$$

From (16), we obtain $\mathscr{G}(\hat{\Psi},\ \mathrm{Q}\hat{\Psi},\ \mathrm{Q}\hat{\Psi})=0$. Hence it is proved that $\hat{\Psi}$ is a $\mathfrak{fp}$ of the mapping Q. Next to prove the uniqueness we proceed by a contradiction. Assume that, ζ is also $\mathfrak{fp}$ of the mapping Q. Then

$$\mathscr{F}(\mathfrak{s}\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta}))$$

$$=\mathscr{F}(\mathfrak{s}\mathscr{G}(\mathrm{Q}\hat{\Psi},\ \mathrm{Q}\hat{\zeta},\ \mathrm{Q}\hat{\zeta}))$$

$$\le\mathscr{F}\Big(\mathrm{f}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})+\mathrm{g}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\Big[\mathscr{G}(\hat{\Psi},\ \mathrm{Q}\hat{\Psi},\ \mathrm{Q}\hat{\Psi})+\mathscr{G}(\hat{\zeta},\ \mathrm{Q}\hat{\zeta},\ \mathrm{Q}\hat{\zeta})$$

$$+\mathscr{G}(\hat{\zeta},\ Q\hat{\zeta},\ Q\hat{\zeta})\Big]+h(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\Big[\mathscr{G}(Q\hat{\Psi},\ Q\hat{\Psi},\ \hat{\zeta})+\mathscr{G}(Q\hat{\zeta},\ Q\hat{\zeta},\ \hat{\Psi})$$

$$+\mathscr{G}(Q\hat{\zeta},\ Q\hat{\zeta},\ \hat{\Psi})\Big]\Big)-\tau$$

$$=\mathscr{F}\Big(f(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})+h(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\Big[\mathscr{G}(Q\hat{\Psi},\ Q\hat{\Psi},\ \hat{\zeta})+\mathscr{G}(Q\hat{\zeta},\ Q\hat{\zeta},\ \hat{\Psi})$$

$$+\mathscr{G}(Q\hat{\zeta},\ Q\hat{\zeta},\ \hat{\Psi})\Big]\Big)-\tau$$

$$=\mathscr{F}\Big(f(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})+h(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\Big[\mathscr{G}(\hat{\Psi},\ \hat{\Psi},\ \hat{\zeta})+\mathscr{G}(\hat{\zeta},\ \hat{\zeta},\ \hat{\Psi}) \tag{18}$$

$$+\mathscr{G}(\hat{\zeta},\ \hat{\zeta},\ \hat{\Psi})\Big]\Big)-\tau$$

$$=\mathscr{F}\Big(f(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})+h(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\Big[\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})+\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})$$

$$+\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\Big]\Big)-\tau=\mathscr{F}\Big([f(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})+3h(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})]\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\Big)-\tau$$

$$\leq\mathscr{F}\Big([f(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})+3h(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})]\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\Big).$$

$\mathscr{F}_1$ implies

$$\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})\leq[f(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})+3h(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})]\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})$$

$$\leq\frac{1}{4\mathfrak{s}^5}\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta}),$$

which is a contradiction. Therefore, $\mathscr{G}(\hat{\Psi},\ \hat{\zeta},\ \hat{\zeta})=0$. This proves the uniqueness of the $\mathfrak{fp}$. That is, $\hat{\Psi}=\hat{\zeta}$. □

Remark 4 Choosing $\mu_u=0$ and $b=1$ in Theorem 1 with suitable values for $f(x,\ y,\ z)$, $g(x,\ y,\ z)$ and $h(x,\ y,\ z)$ we get the results of [28] and [29].

Theorem 2 Let $(\mathscr{D},\mathscr{G},v)$ be a complete convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ with a convex structure v and $Q:\mathscr{D}\to\mathscr{D}$ be a $\mathscr{F}$-Chatterjea type contraction, that is there exists $h:\mathscr{D}\times\mathscr{D}\to\left[0,\ \frac{1}{4\mathfrak{s}^4}\right)$ the following hold:

$$\tau+\mathscr{F}(\mathfrak{s}\mathscr{G}(Q\Psi,\ Q\zeta,\ Q\gamma))\leq\mathscr{F}\Big(h(\Psi,\ \zeta,\ \gamma)\Big[\mathscr{G}(Q\Psi,\ Q\Psi,\ \zeta)+\mathscr{G}(Q\zeta,\ Q\zeta,\ \Psi)+\mathscr{G}(Q\gamma,\ Q\gamma,\ \Psi)\Big]\Big),$$

for every $\Psi,\ \zeta,\ \gamma\in\mathscr{D}$. Assume that the sequence $\{\Psi_u\}$ is defined as

$$\Psi_u = v(\Psi_{u-1},\ Q\Psi_{u-1},\ \mu_{u-1}),\ \text{ where } 0 < \mu_{u-1} < \frac{1}{4\mathfrak{s}^2}\ \forall\, u \in \mathbb{N},$$

then Q has a unique $\mathfrak{fp}$ in $\mathscr{D}$, provided that

$$4h(\Psi,\ \zeta,\ \gamma) \leq \frac{1}{4\mathfrak{s}^4}.$$

By choosing $h = 0$ in the Theorem (3.we obtain the subsequent result.

Theorem 3 Let $(\mathscr{D}, \mathscr{G}, v)$ be a complete convex $\mathscr{G}_b$-$\mathfrak{MS}$ with a convex structure v and $Q : \mathscr{D} \to \mathscr{D}$ be a $\mathscr{F}$-Riech type contraction is as follows. Suppose there exists f, $g : \mathscr{D} \times \mathscr{D} \to \left[0,\ \frac{1}{4\mathfrak{s}^4}\right)$ the following hold:

$$\tau + \mathscr{F}(\mathfrak{s}\mathscr{G}(Q\Psi,\ Q\zeta,\ Q\gamma)) \leq \mathscr{F}\Big(f(\Psi,\ \zeta,\ \gamma)\mathscr{G}(\Psi,\ \zeta,\ \gamma)$$
$$+ g(\Psi,\ \zeta,\ \gamma)\Big[\mathscr{G}(\Psi,\ Q\Psi,\ Q\Psi) + \mathscr{G}(\zeta,\ Q\zeta,\ Q\zeta) + \mathscr{G}(\gamma,\ Q\gamma,\ Q\gamma)\Big]\Big),$$

for every $\Psi,\ \zeta,\ \gamma \in \mathscr{D}$. Assume that the sequence $\{\Psi_u\}$ is defined as

$$\Psi_u = v(\Psi_{u-1},\ Q\Psi_{u-1},\ \mu_{u-1}),\ \text{ where } 0 < \mu_{u-1} < \frac{1}{4\mathfrak{s}^2} \quad \forall\, u \in \mathbb{N},$$

then Q has a unique $\mathfrak{fp}$ in $\mathscr{D}$, provided that

$$f(\Psi,\ \zeta,\ \gamma) + 3g(\Psi,\ \zeta,\ \gamma) \leq \frac{1}{4\mathfrak{s}^4}.$$

Example 2 Assume $\mathscr{D} = \{1,\ 2,\ 3\}$ and $\mathscr{G} : \mathscr{D} \times \mathscr{D} \times \mathscr{D} \to [0,\ +\infty)$ defined by

$$\mathscr{G}(\Psi,\ \zeta,\ \gamma) = \max\{d(\Psi,\ \zeta),\ d(\Psi,\ \gamma),\ d(\gamma,\ \zeta)\} \qquad \forall\, \zeta,\ \Psi,\ \eta \in \mathscr{D},$$

be a mapping for each $\Psi,\ \zeta,\ \gamma \in \mathscr{D}$ such that $\mathscr{G}(\Psi,\ \zeta,\ \gamma) = \mathscr{G}(\zeta,\ \Psi,\ \gamma) = \mathscr{G}(\gamma,\ \zeta,\ \Psi) = \cdots$ and $\mathscr{G}(1,\ 1,\ 1) = \mathscr{G}(2,\ 2,\ 2) = \mathscr{G}(3,\ 3,\ 3) = 0, \mathscr{G}(1,\ 1,\ 2) = \mathscr{G}(2,\ 2,\ 1) = 1, \mathscr{G}(1,\ 1,\ 3) = \mathscr{G}(3,\ 3,\ 1) = 4, \mathscr{G}(2,\ 2,\ 3) = \mathscr{G}(3,\ 3,\ 2) = 1, \mathscr{G}(1,\ 2,\ 3) = 4$. Then $(\mathscr{D},\ \mathscr{G})$ is a complete $\mathscr{G}_b$-$\mathfrak{MS}$ with $\mathfrak{s} = 2$. Define a mapping Q such that $Q\theta = \frac{\theta}{15}$ for any $\theta \in \mathscr{D}$. Also the mapping $v : \mathscr{D} \times \mathscr{D} \times [0,\ 1] \to \mathscr{D}$ by

$$v(\zeta,\ \Psi;\ \mu) \leq \mu\zeta + (1-\mu)\Psi,$$

for each $\zeta,\ \Psi \in \mathscr{D}$ and $\mu \in [0,\ 1]$.

Set

$$\Psi_u = v(\Psi_{u-1},\ Q\Psi_{u-1},\ \mu_{u-1}) \text{ and } \mu_{u-1} = \frac{1}{24}.$$

Then $(\mathscr{D},\ \mathscr{G},\ v)$ be a complete convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ with $\mathfrak{s} = 2$, where $d(a,\ b) = (a-b)^2$ for each $a,\ b \in \mathscr{D}$. Next, define f, g, h : $\mathscr{D} \times \mathscr{D} \to \left[0,\ \frac{1}{2}\right)$ as

$$\mathfrak{f}(\Psi,\ \zeta,\ \gamma) = \begin{cases} \frac{1}{128}, & \text{if } \Psi < \zeta,\ \zeta < \gamma \\ \frac{1}{138}, & \text{otherwise }, \end{cases}$$

$\mathfrak{g}(\Psi,\ \zeta,\ \gamma) = \frac{1}{384}$ and $\mathrm{h}(\Psi,\ \zeta,\ \gamma) = 0$ for every $\Psi,\ \zeta,\ \gamma \in \mathscr{D}$. It is clear that

$$\mathfrak{f}(\Psi,\ \zeta,\ \gamma) + 3\mathfrak{g}(\Psi,\ \zeta,\ \gamma) + 4\mathrm{h}(\Psi,\ \zeta,\ \gamma) \leq \frac{1}{64}.$$

Then all conditions of Theorem (1) are fulfilled. That is, a unique $\mathfrak{fp}$ of Q exists. Indeed,

$$\begin{aligned} \tau + \mathscr{F}(\mathfrak{s}\mathscr{G}(Q\Psi,\ Q\zeta,\ Q\gamma)) \leq \mathscr{F}\Big(&\mathrm{f}(\Psi,\ \zeta,\ \gamma)\mathscr{G}(\Psi,\ \zeta,\ \gamma) + \mathrm{g}(\Psi,\ \zeta,\ \gamma)\Big[\mathscr{G}(\Psi,\ Q\Psi,\ Q\Psi) + \mathscr{G}(\zeta,\ Q\zeta,\ Q\zeta) \\ &+ \mathscr{G}(\gamma,\ Q\gamma,\ Q\gamma)\Big] + \mathrm{h}(\Psi,\ \zeta,\ \gamma)\Big[\mathscr{G}(Q\Psi,\ Q\Psi,\ \zeta) + \mathscr{G}(Q\zeta,\ Q\zeta,\ \Psi) \\ &+ \mathscr{G}(Q\gamma,\ Q\gamma,\ \Psi)\Big]\Big). \end{aligned} \tag{19}$$

Next, consider the following cases in (19)

- Case 1: If $\Psi = 1,\ \zeta = 2,\ \gamma = 3$, then

$$\begin{aligned} \tau + \ln[2\mathscr{G}(Q1,\ Q2,\ Q3)] &= \tau + \ln\left[2\max\left\{\mathrm{d}\left(\frac{1}{15},\ \frac{2}{15}\right),\ \mathrm{d}\left(\frac{1}{15},\ \frac{3}{15}\right),\ \mathrm{d}\left(\frac{2}{15},\ \frac{3}{15}\right)\right\}\right] \\ &= \tau + \ln\left[2\mathrm{d}\left(\frac{1}{15},\ \frac{3}{15}\right)\right] \\ &= \tau + \ln\left[\frac{2}{225}(1-3)^2\right] \\ &= \tau - 3.337 \end{aligned}$$

$$\leq \ln \Big[f(1,\ 2,\ 3)\mathscr{G}(1,\ 2,\ 3)$$

$$+ g(1,\ 2,\ 3)\Big[\mathscr{G}(1,\ Q1,\ Q1)+\mathscr{G}(2,\ Q2,\ Q2)+\mathscr{G}(3,\ Q3,\ Q3)\Big]$$

$$+ h(1,\ 2,\ 3)\Big[\mathscr{G}(Q1,\ Q1,\ 2)+\mathscr{G}(Q2,\ Q2,\ 1)+\mathscr{G}(Q3,\ Q3,\ 1)\Big]\Big]$$

$$= \ln \left[\frac{1}{128}(4) + \frac{1}{384}\left[\left(1-\frac{1}{15}\right)^2 + \left(2-\frac{2}{15}\right)^2 + \left(3-\frac{3}{15}\right)^2 \right] \right]$$

$$= \ln[0.0631]$$

$$= -2.763.$$

For $\tau \leq 0.574$, our contraction condition is satisfied.

• Case 2: If $\Psi = 1,\ \zeta = 1,\ \gamma = 2$, then

$$\tau + \ln[2\mathscr{G}(Q1,\ Q1,\ Q2)] = \tau + \ln\left[2\max\left\{ d\left(\frac{1}{15},\ \frac{1}{15}\right),\ d\left(\frac{1}{15},\ \frac{2}{15}\right),\ d\left(\frac{1}{15},\ \frac{2}{15}\right)\right\}\right]$$

$$= \tau + \ln\left[2d\left(\frac{1}{15},\ \frac{2}{15}\right)\right]$$

$$= \tau - 4.7230$$

$$\leq \ln \Big[f(1,\ 1,\ 2)\mathscr{G}(1,\ 1,\ 2)$$

$$+ g(1,\ 1,\ 2)\Big[\mathscr{G}(1,\ Q1,\ Q1)+\mathscr{G}(1,\ Q1,\ Q1)+\mathscr{G}(2,\ Q2,\ Q2)\Big]$$

$$+ h(1,\ 1,\ 2)\Big[\mathscr{G}(Q1,\ Q1,\ 1)+\mathscr{G}(Q1,\ Q1,\ 1)+\mathscr{G}(Q2,\ Q2,\ 1)\Big]\Big]$$

$$= \ln \left[\frac{1}{138} + \frac{1}{384}\left[\left(1-\frac{1}{15}\right)^2 + \left(1-\frac{1}{15}\right)^2 + \left(2-\frac{2}{15}\right)^2 \right] \right]$$

$$= \ln[0.02084]$$

$$= -3.8709.$$

For $\tau \leq 0.8521$, our contraction condition is satisfied.

• Case 3: If $\Psi = 1,\ \zeta = 1,\ \gamma = 3$, then

$$\tau + \ln[2\mathscr{G}(Q1,\ Q1,\ Q3)] = \tau + \ln\left[2\max\left\{d\left(\frac{1}{15},\ \frac{1}{15}\right),\ d\left(\frac{1}{15},\ \frac{3}{15}\right),\ d\left(\frac{1}{15},\ \frac{3}{15}\right)\right\}\right]$$

$$= \tau + \ln\left[2d\left(\frac{1}{15},\ \frac{3}{15}\right)\right]$$

$$= \tau - 3.337$$

$$\leq \ln\Big[f(1,\ 1,\ 3)\mathscr{G}(1,\ 1,\ 3)$$

$$+ g(1,\ 1,\ 3)\Big[\mathscr{G}(1,\ Q1,\ Q1) + \mathscr{G}(1,\ Q1,\ Q1) + \mathscr{G}(3,\ Q3,\ Q3)\Big]$$

$$+ h(1,\ 1,\ 3)\Big[\mathscr{G}(Q1,\ Q1,\ 1) + \mathscr{G}(Q1,\ Q1,\ 1) + \mathscr{G}(Q3,\ Q3,\ 1)\Big]\Big]$$

$$= \ln\left[\frac{1}{138} + \frac{1}{384}\left[\left(1 - \frac{1}{15}\right)^2 + \left(1 - \frac{1}{15}\right)^2 + \left(3 - \frac{3}{15}\right)^2\right]\right]$$

$$= \ln[0.0539]$$

$$= -2.921.$$

For $\tau \leq 2.4315$, our contraction condition is satisfied.

• Case 4: If $\Psi = 2,\ \zeta = 2,\ \gamma = 3$, then

$$\tau + \ln[2\mathscr{G}(Q2,\ Q2,\ Q3)] = \tau + \ln\left[2\max\left\{d\left(\frac{2}{15},\ \frac{2}{15}\right),\ d\left(\frac{2}{15},\ \frac{3}{15}\right),\ d\left(\frac{2}{15},\ \frac{3}{15}\right)\right\}\right]$$

$$= \tau + \ln\left[2d\left(\frac{2}{15},\ \frac{3}{15}\right)\right]$$

$$= \tau - 4.7230$$

$$\leq \ln\Big[f(2,\ 2,\ 3)\mathscr{G}(2,\ 2,\ 3)$$

$$+ g(2,\ 2,\ 3)\Big[\mathscr{G}(2,\ Q2,\ Q2) + \mathscr{G}(2,\ Q2,\ Q2) + \mathscr{G}(3,\ Q3,\ Q3)\Big]$$

$$+\mathrm{h}(2,\ 2,\ 3)\Big[\mathscr{G}(\mathrm{Q}2,\ \mathrm{Q}2,\ 2)+\mathscr{G}(\mathrm{Q}2,\ \mathrm{Q}2,\ 2)+\mathscr{G}(\mathrm{Q}3,\ \mathrm{Q}3,\ 2)\Big]\Big]$$

$$=\ln\left[\frac{1}{138}+\frac{1}{384}\left[\left(2-\frac{2}{15}\right)^2+\left(2-\frac{2}{15}\right)^2+\left(3-\frac{3}{15}\right)^2\right]\right]$$

$$=\ln[0.02085]$$

$$=-3.870.$$

For $\tau \leq 0.8530$, our contraction condition is satisfied.

We choose $\Psi_0 \in \mathscr{D}/\{0\}$. Combining with

$$\Psi_{\mathrm{u}} = \nu(\Psi_{\mathrm{u}-1},\ \mathrm{Q}\Psi_{\mathrm{u}-1},\ \mu_{\mathrm{u}-1}),\ \mu_{\mathrm{u}-1} = \frac{1}{24} \text{ and } \mathrm{Q}\Psi = \frac{\Psi}{15},$$

we obtain

$$\Psi_{\mathrm{u}} = \nu(\Psi_{\mathrm{u}-1},\ \mathrm{Q}\Psi_{\mathrm{u}-1},\ \mu_{\mathrm{u}-1}),\ \mu_{\mathrm{u}-1}$$

$$= \mu_{\mathrm{u}-1}\Psi_{\mathrm{u}-1} + (1-\mu_{\mathrm{u}-1})\mathrm{Q}\Psi_{\mathrm{u}-1}$$

$$= \frac{1}{24}\Psi_{\mathrm{u}-1} + \left(1-\frac{1}{24}\right)\frac{\Psi_{\mathrm{u}-1}}{15}$$

$$= \frac{19}{180}\Psi_{\mathrm{u}-1}.$$

Proceeding in the same way, we obtain

$$\Psi_{\mathrm{u}-1} = \frac{19}{180}\Psi_{\mathrm{u}-2},\ \Psi_{\mathrm{u}-2} = \frac{19}{180}\Psi_{\mathrm{u}-3},\ \cdots.\ \Psi_1 = \frac{19}{180}\Psi_0.$$

Therefore,

$$\Psi_{\mathrm{u}} = \left(\frac{19}{180}\right)^{\mathrm{u}}\Psi_0,\ \mathrm{Q}\Psi_{\mathrm{u}} = \frac{1}{15}\left(\frac{19}{180}\right)^{\mathrm{u}}\Psi_0.$$

By applying $\lim\limits_{\mathrm{u}\to+\infty}$, we get $\Psi_{\mathrm{u}} \to 0$ and $\mathrm{Q}\Psi_{\mathrm{u}} \to 0$. That is, 0 is a $\mathfrak{fp}$ of Q.

Due to the vast applications of integral equations in many real-life problems, the solution of integral equations and their existence has become an important topic for researchers. A huge literature is present on the existence of the solution to such integral equations using the fixed point technique. Gnanaprakasam et al. [30] applied their results to prove the existence of the solution to the integral equation by incorporating F-Khan contraction. Similarly, Panda et al. [31] presented fixed point results and their application to find the solution of Volterra integral equations to verify their results on the platform of dislocated extended b-metric spaces. Gupta et al. [32] applied their results to find the solution of Fredholm integral equation in the framework of $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$.

4. Application

To ensure the existence of a solution to the subsequent integral equation, we apply Theorem 2.

$$\Psi_{\mathrm{u}}(\mathfrak{q}) = f(\mathfrak{q}) + \gamma \int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{q},\ \zeta)\mathfrak{K}_1(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta \int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{q},\ \zeta)\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta \quad \text{for all } \mathrm{u} \in \mathbb{N} \tag{20}$$

for any $\mathfrak{q} \in [\mathfrak{l}_1,\ \mathfrak{l}_2]$, where $f : [\mathfrak{l}_1,\ \mathfrak{l}_2] \to \mathbb{R}$, $w : [\mathfrak{l}_1,\ \mathfrak{l}_2] \times [\mathfrak{l}_1,\ \mathfrak{l}_2] \to \mathbb{R}$ and $\mathfrak{K}_1,\ \mathfrak{K}_2 : [\mathfrak{l}_1,\ \mathfrak{l}_2] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $\mathscr{D} = C([\mathfrak{l}_1,\ \mathfrak{l}_2],\ \mathbb{R})$ represent the space of continuous functions on $[\mathfrak{l}_1,\ \mathfrak{l}_2]$. Define

$$\mathscr{G}(\Psi_{\mathrm{u},\ \beta_{\mathrm{u}},\ \eta_{\mathrm{u}}}) = \Big(\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]} |\Psi_{\mathrm{u}}(\mathfrak{q}) - \beta_{\mathrm{u}}(\mathfrak{q})| + \sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]} |\beta_{\mathrm{u}}(\mathfrak{q}) - \eta_{\mathrm{u}}(\mathfrak{q})| + \sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]} |\Psi_{\mathrm{u}}(\mathfrak{q}) - \eta_{\mathrm{u}}(\mathfrak{q})| \Big)^2, \text{ for all } \mathrm{u} \in \mathbb{N}$$

while the function $v : \mathscr{D} \times \mathscr{D} \times (0,\ 1) \to \mathscr{D}$ is presented as $v(\Psi_{\mathrm{u}},\ \beta_{\mathrm{u}};\ \theta) = \theta\Psi_{\mathrm{u}} + (1-\theta)\beta_{\mathrm{u}}$. Then, $(\mathscr{D},\ \mathscr{G},\ v)$ represents a complete convex $\mathscr{G}_b$-$\mathfrak{M}\mathscr{S}$ with $\mathfrak{s} = 2$. Consider a mapping $\mathscr{Q} : \mathscr{S} \to \mathscr{S}$ by

$$\mathrm{Q}\Psi_{\mathrm{u}}(\mathfrak{q}) = f(\mathfrak{q}) + \gamma \int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{q},\ \zeta)\mathfrak{K}_1(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta \int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{q},\ \zeta)\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta. \tag{21}$$

Q is well-defined. To obtain the solution for (20), it is equivalent to finding a $\mathfrak{fp}$ of Q. Next, we state the subsequent theorem.

Theorem 4 Suppose that the subsequent conditions are fulfilled:

(1): $\gamma \le \frac{1}{\mathfrak{s}}$;

(2): $\int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{q},\ \zeta)d\zeta \le 1$;

(3): $|\mathfrak{K}_i(\zeta,\ \Psi_{\mathrm{u}}(\zeta)) - \mathfrak{K}_i(\zeta,\ \beta_{\mathrm{u}}(\zeta))| \le \frac{\sqrt{3}}{3}\sqrt{h(\beta,\ \zeta,\ \eta)}\Big|\Psi_{\mathrm{u}}(\zeta) - \mathrm{Q}\beta_{\mathrm{u}}(\zeta)\Big|$, $i = 1,\ 2$, $\mathrm{u} \in \mathbb{N}$ and

$$\int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{q},\ \zeta)|\mathfrak{K}_1(\zeta,\ \beta_{\mathrm{u}}(\zeta)) + \mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))|d\zeta \le 1.$$

Then, the unique solution of Equation (20) exists.

Proof. Clearly, any $\mathfrak{fp}$ of (21) is solution of (20). Using conditions (1)-(3), we have

$$\mathfrak{s}\mathscr{G}(\mathrm{Q}\Psi_{\mathrm{u}},\ \mathrm{Q}\beta_{\mathrm{u}},\ \mathrm{Q}\beta_{\mathrm{u}})$$

$$=\mathfrak{s}\Big(2\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}|\mathrm{Q}\Psi_{\mathrm{u}}(\mathfrak{q})-\mathrm{Q}\beta_{\mathrm{u}}(\mathfrak{q})|\Big)^2$$

$$=4\gamma\left(\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\left|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_1(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta\right.\right.$$

$$\left.\left.-\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_1(\zeta,\ \beta_{\mathrm{u}}(\zeta))d\zeta\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_2(\zeta,\ \beta_{\mathrm{u}}(\zeta))d\zeta\right|\right)^2$$

$$\leq4\gamma\left(\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\left|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\right|\mathfrak{K}_1(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta-\mathfrak{K}_1(\zeta,\ \beta_{\mathrm{u}}(\zeta))\right|d\zeta\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta$$

$$\left.+\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_1(\zeta,\ \beta_{\mathrm{u}}(\zeta))d\zeta\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\Big|\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta-\mathfrak{K}_2(\zeta,\ \beta_{\mathrm{u}}(\zeta))\Big|d\zeta\right)^2$$

$$\leq\gamma\left(\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\sup_{\zeta\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\Big|\mathfrak{K}_1(\zeta,\ \Psi_{\mathrm{u}}(\zeta))-\mathfrak{K}_1(\zeta,\ \beta_{\mathrm{u}}(\zeta))\Big|\left|\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)d\zeta\right.\right.$$

$$\left.\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta\right|+\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\sup_{\zeta\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\Big|\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))-\mathfrak{K}_2(\zeta,\ \beta_{\mathrm{u}}(\zeta))\Big|$$

$$\left.\left|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_1(\zeta,\ \beta_{\mathrm{u}}(\zeta))d\zeta\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)d\zeta\right|\right)^2$$

$$\leq\gamma\left(\frac{\sqrt{3}}{3}\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\sqrt{h(\beta,\ \zeta,\ \eta)}|\Psi_{\mathrm{u}}-\mathscr{Q}\beta_{\mathrm{u}}|\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\left|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)d\zeta\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta\right.\right.$$

$$\left.\left.+\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_1(\zeta,\ \beta_{\mathrm{u}}(\zeta))d\zeta\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)d\zeta\right|\right)^2$$

$$\leq\frac{1}{3}\gamma\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\Big(\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)d\zeta\Big)^2\left(\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\sqrt{h(\beta,\ \zeta,\ \eta)}|\Psi_{\mathrm{u}}-\mathscr{Q}\beta_{\mathrm{u}}|\sup_{\mathfrak{q}\in[\mathfrak{l}_1,\ \mathfrak{l}_2]}\left|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_2(\zeta,\ \Psi_{\mathrm{u}}(\zeta))d\zeta\right.\right.$$

$$\left.\left.+\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{q},\ \zeta)\mathfrak{K}_1(\zeta,\ \beta_{\mathrm{u}}(\zeta))d\zeta\right|\right)^2$$

$$\leq \frac{1}{3}\gamma\left(\sup_{\mathfrak{q}\in[\mathfrak{t}_1,\, \mathfrak{t}_2]} \sqrt{h(\beta,\, \zeta,\, \eta)}|\Psi_{\mathrm{u}} - \mathrm{Q}\beta_{\mathrm{u}}|\right)^2$$

$$\leq \frac{1}{3\mathfrak{s}}\left(\sup_{\mathfrak{q}\in[\mathfrak{t}_1,\, \mathfrak{t}_2]} \sqrt{h(\beta,\, \zeta,\, \eta)}|\Psi_{\mathrm{u}} - \mathrm{Q}\beta_{\mathrm{u}}|\right)^2$$

$$\leq \frac{1}{3}h(\beta,\, \zeta,\, \eta)\Big(2\sup_{\mathfrak{q}\in[\mathfrak{t}_1,\, \mathfrak{t}_2]}|\Psi_{\mathrm{u}} - \mathrm{Q}\beta_{\mathrm{u}}|\Big)^2 + \frac{1}{3}h(\beta,\, \zeta,\, \eta)\Big(2\sup_{\mathfrak{q}\in[\mathfrak{t}_1,\, \mathfrak{t}_2]}|\beta_{\mathrm{u}} - \mathrm{Q}\eta_{\mathrm{u}}|\Big)^2$$

$$+ \frac{1}{3}h(\beta,\, \zeta,\, \eta)\Big(2\sup_{\mathfrak{q}\in[\mathfrak{t}_1,\, \mathfrak{t}_2]}|\Psi_{\mathrm{u}} - \mathrm{Q}\eta_{\mathrm{u}}|\Big)^2$$

$$= \frac{1}{3}h(\beta,\, \zeta,\, \eta)\left(\mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\beta_{\mathrm{u}},\, \mathrm{Q}\beta_{\mathrm{u}}) + \mathscr{G}(\beta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}}) + \mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}})\right).$$

Hence

$$\tau + \mathscr{F}(\mathscr{G}(\mathrm{Q}\Psi_{\mathrm{u}},\, \mathrm{Q}\beta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}})) \leq \frac{1}{3}h(\beta,\, \zeta,\, \eta)\Big(\mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\beta_{\mathrm{u}},\, \mathrm{Q}\beta_{\mathrm{u}}) + \mathscr{G}(\beta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}}) + \mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}})\Big),$$

where $F(t) = \ln t$ and $\tau \in \left(0,\, \ln\left(\frac{1}{3}\frac{h(\beta,\, \zeta,\, \eta)(\mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\beta_{\mathrm{u}},\, \mathrm{Q}\beta_{\mathrm{u}}) + \mathscr{G}(\beta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}}) + \mathscr{G}(\Psi_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}}))}{\mathscr{G}(\mathrm{Q}\Psi_{\mathrm{u}},\, \mathrm{Q}\beta_{\mathrm{u}},\, \mathrm{Q}\eta_{\mathrm{u}})}\right)\right)$.

All conditions of Theorem 2 with $\beta_{\mathrm{u}} = \eta_{\mathrm{u}}$ are satisfied. Which enables us to know that a fixed point for Q exists. Thus, the solution of the integral equation exists. Hence, we obtain that Equation (20) gives a unique solution, where the sequence satisfies the convex condition with $\mu_{\mathrm{u}} \in \left(0,\, \frac{1}{4}\right)$. □

5. Conclusion

• In 2022, Yildirim [18] presented certain fixed point results using Mann's iterative scheme tailored with b-metric spaces.

• In the present research, the existence and uniqueness of the fixed points are established with Mann's iterative scheme in convex $\mathscr{G}_b$-metric spaces using $\mathscr{F}$-contraction of Hardy Rogers type.

• This task is achieved by further weakening the conditions of Wardowski's $\mathscr{F}$ mappings.

• An example is provided to support our results. Eventually, an application is given for the validity of our results.

• The obtained results are generalizations of several existing results in the literature [28, 29].

• Future research endeavors may focus on establishing the above result:

(i) in the setting of controlled $\mathscr{G}_b$-metric spaces and double controlled $\mathscr{G}_b$-metric spaces.

(ii) by using the Picard-Mann hybrid iterative scheme.

Availability of data and materials

No underlying data was collected or produced in this study.

Funding

No funds were received.

Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Confilict of interest

The authors declare that they have no competing interests.

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