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Research Article

Solution of Non-Autonomous Bloch Equation via Multistage Differential Transformation Method

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Abstract: This study explores the multistage differential transformation method (MsDTM) as an efficient approach for solving non-autonomous differential equations. The proposed method demonstrates wide-ranging applicability in fields such as computer graphics, quantum optics, biomathematics, and image processing. Specifically, the Bloch equations, which describe the interaction of a spin-1/2 system (or two-level atom) with a mono- or bichromatic laser field in the presence of an off-resonant broad-band squeezed vacuum (SV), are examined. As a system of non-autonomous ordinary differential equations, the Bloch model captures the quantum dynamics between matter and electromagnetic fields, offering insights into more complex and experimentally relevant models. The MsDTM is employed to obtain numerical solutions with high precision and computational efficiency, outperforming the classical 4th-order Runge-Kutta (RK4) method. A key advantage of the MsDTM is its adaptability; its accuracy can be further enhanced by either increasing the number of iterations or refining the time-step in the numerical scheme. Consequently, MsDTM emerges as a robust tool for computing solutions to a broad class of non-autonomous equations.

Keywords: the multistage differential transformation method, optical quantum, numerical solution, non-autonomous bloch equation

MSC: 65L05, 34K06, 34K28

1. Introduction

Mathematical modeling plays an important role across many scientific fields such as biochemistry, natural sciences, and social sciences. Ordinary differential equations (ODEs) are commonly used to model both dynamic processes and evolutionary systems in these areas [1]. Various methods exist to solve ODEs, including numerical techniques like the Tanch method [2], the shifted Gegenbauer spectral method [3], modified Lucas polynomials [4], and the Legendre-Galerkin spectral method [5]. Other methods include the Eighth-Kind Chebyshev Polynomials Collocation Algorithm [6] and the Modal Shifted Fifth-Kind Chebyshev Tau Integral Approach [7].

The Taylor series method is another widely used approach, but it involves calculating many derivatives, making it computationally difficult. To overcome these challenges, Zhou introduced the Differential Transformation Method (DTM) [8], which has since become a powerful tool in mathematical computation [9]. DTM is a semi-analytical approach

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that can solve both partial differential equations and ordinary equations by generating a convergent series with easily computed terms [10]. This method is particularly useful in analyzing both linear and nonlinear initial value problems, especially in electrical circuits. When applied to a system with boundary conditions (BCs), DTM efficiently calculates the derivatives of an analytic function at specific points, and creates a numerical solution for differential equations in the form of a polynomial.

Unlike the conventional high-order Taylor series method, which requires time-consuming symbolic computations for function derivatives, DTM is less computationally intensive. As a result, a solution to differential equations can be derived faster with DTM compared to the Taylor series approach [10]. DTM is also a versatile technique that can handle a wide variety of functional problems [11], including the Schrödinger equation [12], KdV-type equations [13], differential-algebraic equations [14], two-point boundary value problems [15], and fractional differential equations [16].

The form of the infinite series corresponding to the solution is known, DTM can closely approximate the numerical solution. However, for practical purposes, the series is usually plotted in a finite form, and its solution converges only within a limited area around the initial point, due to the nature of the local Taylor expansion [17]. The Padé approximation, developed by Her-Moths and Padé, has been widely applied in various scientific fields. More recently, this approximation has been combined with DTM to enhance its ability to provide solutions over extended domains [18]. A multistage version of DTM (MsDTM) has also been developed by authors in recent works [17]. According to these studies, the MsDTM offers faster and more precise results than the fourth-order Runge-Kutta (RK4) method, as well as improved performance compared to the Padé-DTM (PDTM).

One advantage of the MsDTM is that it allows for greater control over the accuracy of the method by adjusting the step size and number of iterations, achieving high precision without using extremely small step sizes. In extended domains, the MsDTM yields an approximate solution [19, 20]. Although MsDTM has been applied to various types of equations, it has not yet been used to solve non-autonomous equations. Thus, the novelty of this paper lies in applying MsDTM to solve autonomous equations, particularly Bloch equations.

This paper presents an analytical approach for solving a non-autonomous linear system of ODEs (Bloch equations), which model the interaction between a spin-1/2 system (representing a two-level atom) and a monochromatic laser field in an off-resonant broad-band normal vacuum (NV) field. The Bloch equations are derived based on certain physical assumptions and approximations. We closely follow the derivation presented by Hassan et al. [21] in both the Heisenberg and Schrödinger frameworks. The model involves coupling nonlinear complex conjugate absorption and dispersion of atoms, along with a third equation representing atomic inversion. These equations are widely used in fields such as quantum optics, quantum information, and signal processing. The Bloch equations have previously been explored using the iterative perturbation method for both monochromatic and bichromatic cases [22], but no exact solutions have been found due to the periodic nature of the system. Additionally, these equations have not been solved numerically. This study aims to explore the use of MsDTM to solve the Bloch equations.

The structure of the paper is as follows: Section 2 describes the multistage differential transformation method (MsDTM). Section 3 introduces the Bloch model as applied in the optical quantum field. Section 4 discusses the quantum mechanics of the model. Section 5 presents the results and discussion, followed by the conclusion in Section 6.

2. The methodology

Assume the analytical function $v(x)$, where $x \in X$ and X is the given domain. The function $v(x)$ transforms to $\phi(x,\ h)$ using differential transformation (DT) as follows

$$\frac{d^h v(x)}{dx^h} = \Phi(x,\ h),\ \forall x \in X, \tag{1}$$

for $x = x_j$, $\Phi(x,\ h) = \Phi(x_j,\ h)$, where $h \in H$ and H is the domain belongs $\mathbb{N}$. Thus, the differential transform is defined by

$$V_j(h) = \Phi(x_j,\ h) = \left[\frac{d^h v(x)}{dx^h}\right]_{x=x_j}, \tag{2}$$

for all $h \in H$ where $V_j(h)$ is the spectrum of $v(x)$ at $x = x_j$.

On the other hand, the inverse transformation of $V(h)$ is given as

$$V(h) = B(h)\left[\frac{d^h u(x)v(x)}{dx^h}\right]_{x=x_j}, \tag{3}$$

where $h = 0,\ 1,\ 2,\ ...,\ \infty$, the function $v(x)$ is described as:

$$v(x) = \frac{1}{u(x)}\sum_{h=0}^{\infty}\frac{(x-x_j)^h}{h!}\frac{V(h)}{B(h)}, \tag{4}$$

where $B(h) \neq 0,\ u(x) \neq 0$. The functions $B(h)$ and $u(t)$ are weighting factor and and we will assume $B(h) = \dfrac{1}{h!}$ and $u(t) = 1$. Then, Eq. (3) becomes

$$V(h) = \frac{1}{h!}\left[\frac{d^h v(x)}{dx^h}\right],\ \text{where } h = 0,\ 1,\ ...,\ \infty. \tag{5}$$

A differential equation can be converted into an algebraic equation in the H domain using the DT as in the Table 1 [23–25]. One of the ways to improve DTM is by splitting the given domain into subdomains and then, applying the DTM for each sub-domain. Thus, the accuracy of the method is based on the iteration number and a time step, for more details see the references [19, 20].

Table 1. Differential transformations' primary operations

Original domain	DT domain
$v(x) = w(x) + u(x)$	$V(h) = W(h) + U(h)$
$v(x) = cw(x)$	$V(h) = cW(h)$
$v(x) = w(x)\cdot u(x)$	$V(h) = \sum_{q=0}^{h} W(q)U(h-q)$
$v(x) = \dfrac{w(x)}{u(x)}$	$V(h) = \dfrac{W(h) - \sum_{q=0}^{h} W(q)V(h-q)}{U(0)}$
$v(x) = \dfrac{dw(x)}{dx}$	$V(h) = (h+1)W(h+1)$
$v(x) = \dfrac{d^2 w(x)}{dx^2}$	$V(h) = (h+1)(h+2)W(h+2)$
$v(x) = \dfrac{d^n w(x)}{dx^n}$	$V(h) = \dfrac{(h+n)!}{h!}W(h+n)$
$v(x) = x^m$	$V(h) = \delta(h-m) = \begin{cases} 1 & \text{h=m} \\ 0 & \text{h}\neq \text{m} \end{cases}$

3. The approximate solution

3.1 *Non-autonomous bloch equation*

We discuss the foundations and framework of quantum mechanics (QM) to derive the non-autonomous Bloch equation [21]. In the presence of an off-resonant $SV(q \neq 0)$ field and a monochromatic laser field, the model Bloch equations for a single two-level atom have the form.

$$\langle \hat{S_+} \rangle^{\cdot} = -(\Gamma + i\Delta)\langle \hat{S_+} \rangle - \gamma M e^{iqt} \langle \hat{S_-} \rangle + \Omega \langle \hat{S_z} \rangle, \tag{6}$$

$$= (\langle \hat{S_-} \rangle^{\cdot})^*, \tag{7}$$

$$\langle \hat{S_z} \rangle^{\cdot} = -\frac{\gamma}{2} - 2\Gamma \langle \hat{S_z} \rangle - \frac{\Omega}{2}(\langle \hat{S_+} \rangle + \langle \hat{S_-} \rangle). \tag{8}$$

Where $\langle \hat{S}_{\pm,\, z} \rangle = \dfrac{d}{dt}\langle \hat{S}_{\pm,\, z} \rangle$, is the derivative relative to time t. The variables $\langle \hat{S}_{\pm,\, z}(t) \rangle$ are the averaged atomic polarization elements and the inversion, with $\hat{S}_{\pm,\, z}$ being the spin $\dfrac{1}{2}$-Pauli operators according to the algebra

$$[\hat{S}_+,\, \hat{S}_-] = 2\hat{S}_z,\, [\hat{S}_\pm,\, \hat{S}_z] = \mp \hat{S}_\pm, \tag{9}$$

$$\hat{S_\pm}^2 = 0\ \hat{S}_\pm \hat{S}_\mp = \frac{1}{2} \pm \hat{S}_z. \tag{10}$$

The notations in Eqs. (6-8) are Ω is the laser field associated with the (real) Rabi frequency, $\Delta = \omega_1 - \omega_0$ is the atomic detuning, where ω_1 and ω_0 represent the laser field and the atom's corresponding frequencies, $q = 2(\omega_p - \omega_1)$ is the SV detuning parameter, and ω_p is the SV field's central frequency, $\Gamma = \dfrac{\gamma(1+2N)}{2}$, γ is the A-coefficient for Einstein damping.$M = \mid M \mid e^{i\varnothing}$ and N are the characteristics of the SV field: N is the average photon number, M (complex) is a squeezing strength indicator, and is the SV field's phase about the laser field. According to Eqs. (6-8) the physical parameters are specified as follows:

In Eq. (6) the term $-(\Gamma + i\Delta)\langle \hat{S_+} \rangle$ explains the damping process $(\Gamma \langle \hat{S_+} \rangle)$ (section includes the processes for both spontaneous and stimulated radiation damping $-\dfrac{\gamma}{2}$, $-\gamma N$ respectively) process, and dispersion $(-i\Delta \langle \hat{S_+} \rangle)$ for atomic detuning with non-zero $(\Delta \neq 0)$. The term $(-\gamma M e^{iqt} \langle \hat{S_-} \rangle)$ reflects the interaction between the atomic polarization components in quadrature $\langle \hat{S_\pm} \rangle$ a result of the SV field $(M \neq 0)$. The driving term $(\Omega \langle \hat{S_\pm} \rangle)$ resulting from the laser field

In Eq. (8), the term $(-2\Gamma \langle \hat{S_z} \rangle)$ is a representation of the naturally occurring and induced radiation damping processes, and the inhomogeneous term $(-\dfrac{\gamma}{2})$ is an eigenvalue that represents the equilibrium value (the lowest energy level's eigenvalue). The last term, $(-\dfrac{\Omega}{2}(\langle \hat{S_+} \rangle + \langle \hat{S_-} \rangle))$, depicts the interaction between the laser field and the atomic polarized field.

3.2 *Numerical solutions in the resonant case (q = 0)*

The resonance case ($q=0$), Eq. (6-8) are of constant coefficients, making it possible to find precise solutions [26]. In a normalized form we let $\gamma=1$ and hence it is appropriate to consider the system parameters Δ, Ω, and q to be normalized parameters $\frac{\Delta}{\gamma}$, $\frac{\Omega}{\gamma}$ and $\frac{q}{\gamma}$ where t is a normalized time γt. Thus, Eq. (6-8) for $q=0$ in normalized form is given by:

$$\langle \hat{S_+}(t)\rangle^{\cdot} = -(\Gamma+i\Delta)\langle \hat{S_+}(t)\rangle - \gamma M\langle \hat{S_-}(t)\rangle + \Omega\langle \hat{S}_z(t)\rangle, \tag{11}$$

$$\langle \hat{S_-}(t)\rangle^{\cdot} = -(\Gamma-i\Delta)\langle \hat{S_-}(t)\rangle - \gamma M^*\langle \hat{S_+}(t)\rangle + \Omega\langle \hat{S}_z(t)\rangle, \tag{12}$$

$$\langle \hat{S}_z(t)\rangle^{\cdot} = -\frac{\gamma}{2} - 2\Gamma\langle \hat{S}_z(t)\rangle - \frac{\Omega}{2}(\langle \hat{S_+}(t)\rangle + \langle \hat{S_-}(t)\rangle). \tag{13}$$

Using the MsDTM:

$$\langle \hat{S_+}(h+1)\rangle^{\cdot} = -\frac{1}{h+1}[(\Gamma+i\Delta)\langle \hat{S_+}(h)\rangle - \gamma M\langle \hat{S_-}(h)\rangle + \Omega\langle \hat{S}_z(h)\rangle], \tag{14}$$

$$\langle \hat{S_-}(h+1)\rangle^{\cdot} = -\frac{1}{h+1}[(\Gamma-i\Delta)\langle \hat{S_-}(h)\rangle - \gamma M^*\langle \hat{S_+}(h)\rangle + \Omega\langle \hat{S}_z(h)\rangle], \tag{15}$$

$$\langle \hat{S}_z(h+1)\rangle^{\cdot} = -\frac{1}{h+1}[\frac{\gamma}{2} - 2\Gamma\langle \hat{S}_z(h)\rangle - \frac{\Omega}{2}(\langle \hat{S_+}(h)\rangle + \langle \hat{S_-}(h)\rangle)]. \tag{16}$$

for $h=1$

$$\langle \hat{S_+}(2)\rangle^{\cdot} = -\frac{1}{2}[(\Gamma+i\Delta)\langle \hat{S_+}(1)\rangle - \gamma M\langle \hat{S_-}(1)\rangle + \Omega\langle \hat{S}_z(1)\rangle], \tag{17}$$

$$\langle \hat{S_-}(2)\rangle^{\cdot} = -\frac{1}{2}[(\Gamma-i\Delta)\langle \hat{S_-}(1)\rangle - \gamma M^*\langle \hat{S_+}(1)\rangle + \Omega\langle \hat{S}_z(1)\rangle], \tag{18}$$

$$\langle \hat{S}_z(2)\rangle^{\cdot} = -\frac{1}{2}[\frac{\gamma}{2} - 2\Gamma\langle \hat{S}_z(1)\rangle - \frac{\Omega}{2}(\langle \hat{S_+}(1)\rangle + \langle \hat{S_-}(1)\rangle)]. \tag{19}$$

for $h=2$

$$\langle \hat{S_+}(3)\rangle^{\cdot} = -\frac{1}{3}[(\Gamma + i\Delta)\langle \hat{S_+}(2)\rangle - \gamma M\langle \hat{S_-}(2)\rangle + \Omega\langle \hat{S_z}(2)\rangle], \tag{20}$$

$$\langle \hat{S_-}(3)\rangle^{\cdot} = -\frac{1}{3}[(\Gamma - i\Delta)\langle \hat{S_-}(2)\rangle - \gamma M^*\langle \hat{S_+}(2)\rangle + \Omega\langle \hat{S_z}(2)\rangle], \tag{21}$$

$$\langle \hat{S_z}(3)\rangle^{\cdot} = -\frac{1}{3}[\frac{\gamma}{2} - 2\Gamma\langle \hat{S_z}(2)\rangle - \frac{\Omega}{2}(\langle \hat{S_+}(2)\rangle + \langle \hat{S_-}(2)\rangle)]. \tag{22}$$

for $h = 3$

$$\langle \hat{S_+}(4)\rangle^{\cdot} = -\frac{1}{4}[(\Gamma + i\Delta)\langle \hat{S_+}(3)\rangle - \gamma M\langle \hat{S_-}(3)\rangle + \Omega\langle \hat{S_z}(3)\rangle], \tag{23}$$

$$\langle \hat{S_-}(4)\rangle^{\cdot} = -\frac{1}{4}[(\Gamma - i\Delta)\langle \hat{S_-}(3)\rangle - \gamma M^*\langle \hat{S_+}(3)\rangle + \Omega\langle \hat{S_z}(3)\rangle], \tag{24}$$

$$\langle \hat{S_z}(4)\rangle^{\cdot} = -\frac{1}{4}[\frac{\gamma}{2} - 2\Gamma\langle \hat{S_z}(3)\rangle - \frac{\Omega}{2}(\langle \hat{S_+}(3)\rangle + \langle \hat{S_-}(3)\rangle)]. \tag{25}$$

4. Computational results

Here, we present the figures of the results for the approximation solutions and compare them with the solution by RK4 method. Initial atomic conditions are considered (ground state). Results are shown for the SV parameters $N = 1$, $|M| = \sqrt{N(N+1)}$.

Where the ground state is, $\langle \hat{S}_z(0)\rangle = -\frac{1}{2}$, $\langle \hat{S}_\pm(0)\rangle = 0$, and classify the discussions in strong laser, fields $\Omega = 10$ and non-resonant detuning $\Delta = 5$. The atomic inversion behavior $\langle \hat{S}_z(t)\rangle$ identical with the numerical (RMK4) solution. For the dispersive, $Im\langle \hat{S}_+(t)\rangle$, and absorptive, $Re\langle \hat{S}_+(t)\rangle$, polarization components, the behavior oscillates towards the steady state values, and the difference of the oscillations is up to (10^{-4}), both absorptive and dispersive polarization components exhibit irregular oscillation at early time (see the blue curves in (Figure 1)). For stronger field ($\Omega = 10$) at non-resonance ($\Delta = -5$), the proposed method and the numerical solutions coincide exactly for large t, Figure 2 and Figure 3, both the appropriate and the numerical solutions for the atomic inversion $\langle \hat{S}_z(t)\rangle$ coincide for all times, (Figure 1, 2, 3a). For early time, the polarization components $Re\langle \hat{S}_+(t)\rangle$ show oscillations due to the Strong Rabi frequency (Ω), (Figure 1, 2, 3(b, c)). Changing the frequency from Γ from 0.1 in Figure 1 to 0.5 in Figure 3 give the same results.

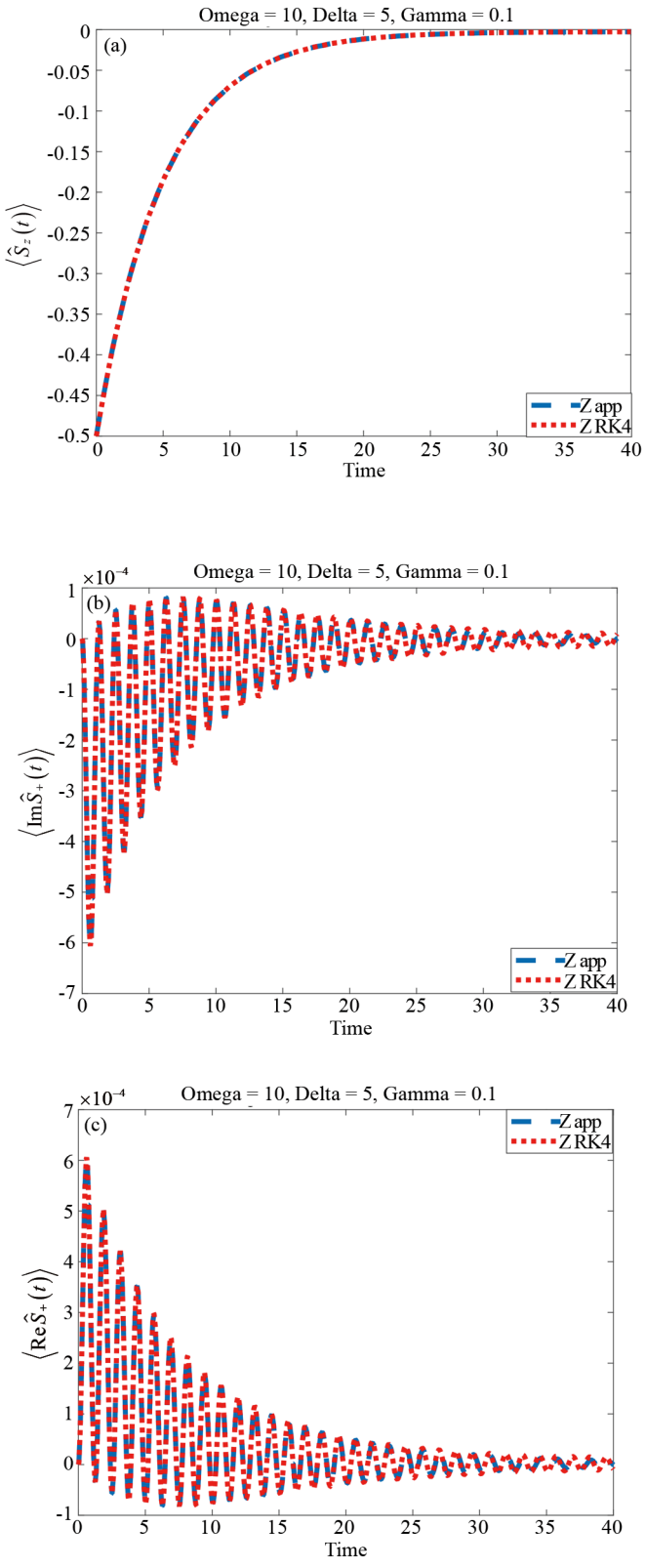
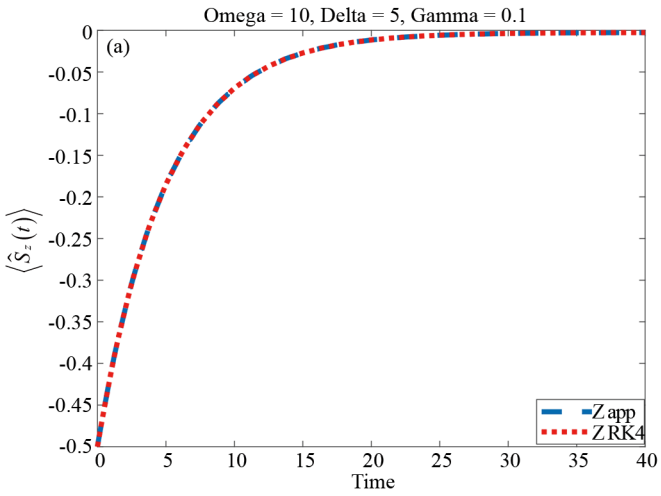
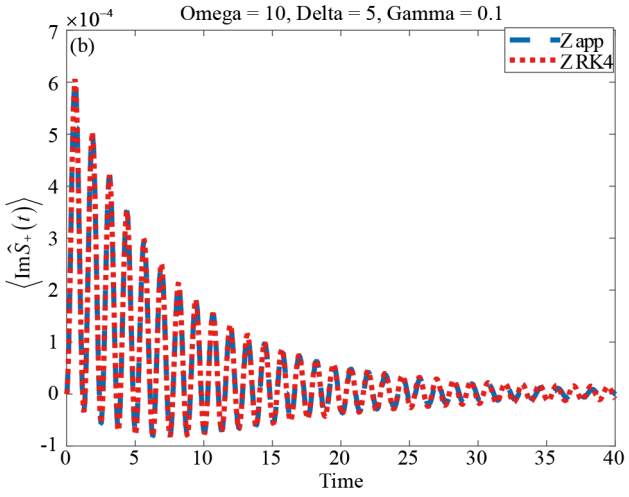


Figure 1. (a) The atomic inversion, $\langle \hat{S}_z(t) \rangle$ (red line) and the approximate solutions (blue line) against the normalized time t and for $\Omega = 10,\ \Delta = 5$, SV parameters $N = 1,\ |M| = \sqrt{N(N+1)}$. (b) Same as (a) but for the dispersive polarization components, $Im\langle \hat{S}_+(t) \rangle$. (c) Same as (a) but for the absorptive polarization components, $Re\langle \hat{S}_+(t) \rangle$





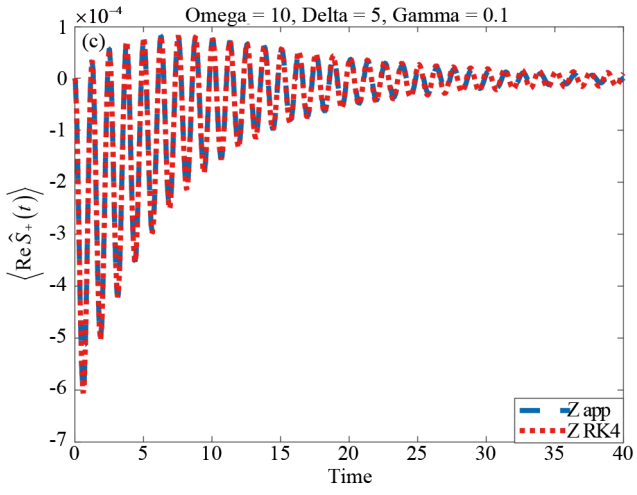
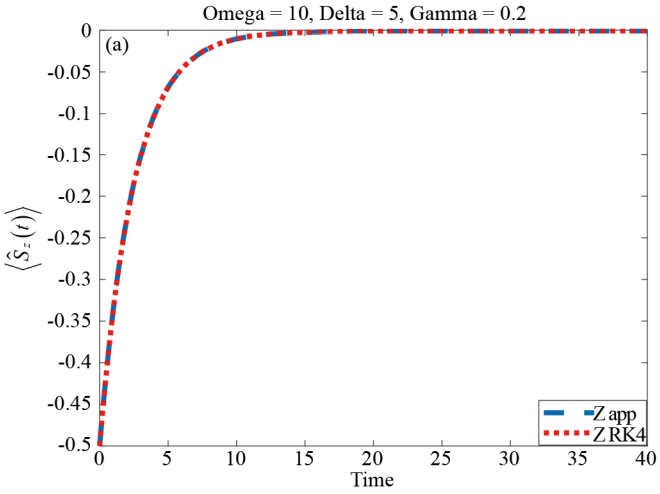
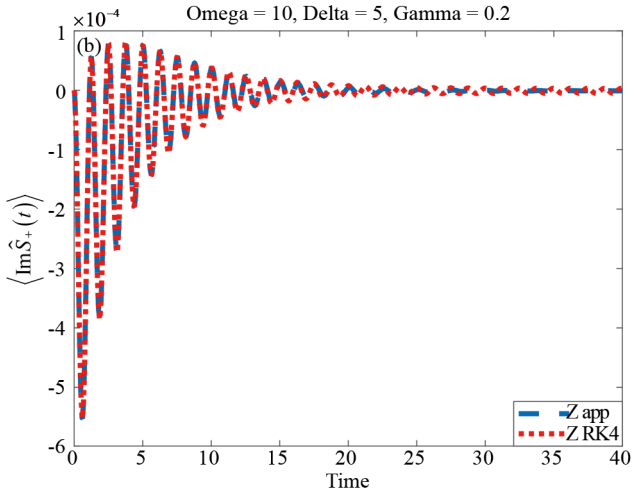


Figure 2. Same as Figure 1 but for $\Delta = -5$





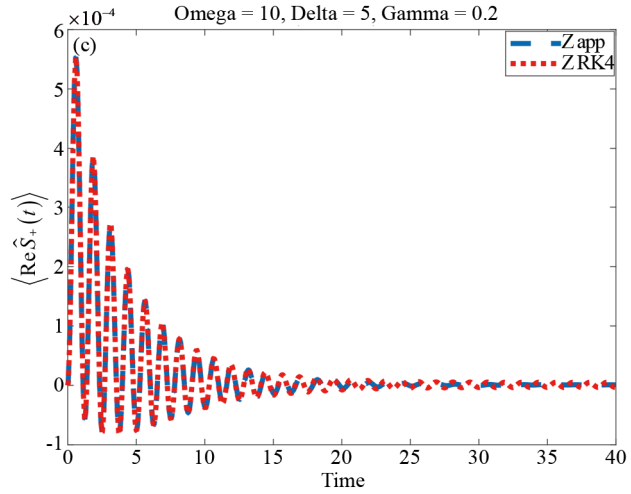


Figure 3. Same as 1 but for $\Gamma = 0.2$

5. Conclusion

The core focus of this paper is the application of the MsDTM (Modified series Decomposition Transformation Method) to numerically solve non-autonomous linear systems of ordinary differential equations, specifically the Bloch equations, which model the dynamics of a two-level atom interacting with a monochromatic laser field. This scenario includes an off-resonant broad-band normal vacuum (NV) field with $N = 0$. The MsDTM provides accurate numerical solutions even for large detuning parameters (SV), as demonstrated by the results. Additionally, the method is effective for arbitrary values of the laser field strength ω. Numerical solutions are presented for various parameter values and compared to results obtained using the classical RK4 (Runge-Kutta) method. The approximation of the Bloch equations using MsDTM for non-zero SV detuning parameters shows that the solutions are nearly identical to those from RK4, with discrepancies as small as $O(10^4)$. The MsDTM could also be extended to address fractional differential equations in future work such as equations in the refrences [27] and [6].

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Conflict of interest

The authors declare no competing financial interest.

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