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Research Article

Integration of Theoretical Foundations and Practical Implications for Neutrosophic Set Theory in Real-Time Maintenance Decision Systems

Settu K, Jayalakshmi M*

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, India
E-mail: m.jayalakshmi@vit.ac.in

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Abstract: In the present era, the concepts of neutrosophic set theory are essential for handling uncertain data with three distinct components. Researchers across various fields widely use these concepts due to their significant applications. Our world is filled with unpredictability, ambiguity, and vagueness, making it crucial to replace items at the appropriate time. This research paper focuses on the importance of addressing the replacement issue to improve reliability in maintenance scheduling. Ambiguity and uncertainty were present challenges in resolving maintenance issues. For example, the group replacement model has been solved using single-valued unique hexagonal neutrosophic numbers and the enhanced score function for hexagonal neutrosophic numbers (HNN) discussed in this paper. Additionally, the removal area method is used to determine the de-eutrophication of the linear neutrosophic hexagonal number, showing significant improvement in the clarification of HNN. MATLAB code is employed for de-eutrophication and to assess the effectiveness of this method. Numerical examples are provided to validate the proposed method. Using this enhanced score function, the replacement problem has been solved in a hexagonal neutrosophic environment. A comparative study was conducted between the established and proposed methods, which will benefit researchers in the field of neutrosophic domain in the future.

***Keywords*:** area removal method, MATLAB, replacement model, score function, single-valued unique hexagonal neutrosophic number

MSC: 65L05, 34K06, 34K28

Abbreviation

FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
NS	Neutrosophic Set
RAM	Removal Area Method
HNN	Hexagonal Neutrosophic Number
SVNN	Single Valued Neutrosophic Number
SVLNN	Single Valued Linear Neutrosophic Number

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SVLPNN	Single Valued Pentagonal Neutrosophic Number
USVHNN	Unique Single Valued Hexagonal Neutrosophic Number
SVNWA	single-valued neutrosophic weighted average
SVNHN	Single Valued Neutrosophic Hexagonal Number

1. Introduction

The replacement problem requires prompt decision-making to replace deteriorating equipment. This involves determining the best time to replace machines, equipment, and other components at risk of sudden failures. When the replacement time is uncertain, a replacement model can be established using neutrosophic numbers. Hexagonal neutrosophic numbers provide precise and informative descriptions of data, allowing for accurate and reasonable models. They also enable more accurate classification of facts and data, which helps decision-makers keep track of details and make well-informed choices. This approach can offer advantages in decision-making, scheduling maintenance, and addressing economic issues [1]. If each of the three components is represented by six numbers, then triangular or trapezoid neutrosophic numbers may not be the best choice. Using hexagonal neutrosophic numbers can effectively tackle problems and offer solutions. Furthermore, the application of hexagonal neutrosophic numbers extends beyond mere replacement decisions; it can significantly enhance the overall management of operational systems [2]. By incorporating this advanced mathematical framework, organizations can better quantify the uncertainty and imprecision inherent in their data. This leads to more robust models that can adapt to changing conditions and unforeseen challenges.

In practice, the implementation of hexagonal neutrosophic numbers allows for a nuanced analysis of various scenarios, enabling decision-makers to weigh the risks associated with equipment failure against the costs of premature replacement [3]. This balance is crucial in optimizing resource allocation and minimizing downtime, thereby improving productivity and operational efficiency. Moreover, the application of these advanced mathematical constructs enables organizations to simulate various scenarios, allowing for a deeper understanding of how different variables interact. For instance, decision-makers can evaluate the effects of varying maintenance schedules, operational conditions, and resource availability on equipment longevity. This capability empowers them to devise tailored maintenance strategies that align with both budgetary constraints and operational demands [4].

In addition, using hexagonal neutrosophic numbers fosters collaborative decision-making among stakeholders by providing a common language for discussing uncertainties. This inclusivity can lead to more comprehensive evaluations, as diverse perspectives contribute to a richer understanding of the challenges at hand. As a result, organizations can cultivate a culture of proactive risk management, where potential issues are identified and addressed before they escalate into costly disruptions [5].

Ultimately, the adoption of hexagonal neutrosophic numbers in decision-making processes represents a significant advancement in the quest for operational excellence. By embracing this innovative approach, organizations can not only enhance their ability to navigate uncertainties but also position themselves to capitalize on opportunities for continuous improvement. In an increasingly competitive landscape, such strategic agility is paramount for sustaining long-term success and resilience.

Currently, a significant experimental study of this era focuses on unpredictability and uncertainty [6]. In this case, Zadeh introduced the concept of the fuzzy set as a more effective method for dealing with ambiguity and clarifying vagueness [7]. Belatedly, Atanassov proposed the study of intuitionistic fuzzy numbers, which help us make decisions even when we are unsure. They are used in various ways to help us understand things better. This exploration of fuzzy sets and intuitionistic fuzzy numbers has opened new avenues for research and application across multiple disciplines [8]. In particular, these concepts have been found to be relevant in areas such as artificial intelligence, decision-making processes, and control systems. By allowing for degrees of truth rather than the binary true/false dichotomy, fuzzy logic enables systems to operate more effectively in real-world scenarios characterized by uncertainty. His flexibility not only enhances the modeling of complex systems but also aligns more closely with human reasoning, which often involves nuance and uncertainty. Researchers are increasingly leveraging fuzzy sets and intuitionistic fuzzy numbers to refine algorithms, optimize processes, and create more resilient frameworks that adapt to varied inputs and conditions. For instance, in the realm of artificial intelligence, these mathematical tools facilitate machine learning models that better interpret ambiguous data, aligning their outputs with human-like decision-making.

As the research community continues to expand the horizons of fuzzy mathematics, the potential applications seem boundless. Future investigations may delve deeper into the synthesis of fuzzy logic with other emerging technologies, such as quantum computing and neural networks, possibly leading to even more sophisticated forms of reasoning and problem-solving. As we unlock these new frontiers, it becomes evident that fuzzy sets and intuitionistic fuzzy numbers are not merely theoretical constructs but vital instruments for grappling with the complexities of our increasingly intricate world.

1.1 *Survey of literature*

In the field of mathematical modeling, researchers have been exploring innovative approaches to solve replacement problems using various forms of fuzzy intuitionistic and fuzzy sets. Prof. Florentin Samandrance introduced the concept of neutrosophic sets in 1998, which effectively combine elements of traditional fuzzy and intuitionistic fuzzy sets. Ongoing research in the field of neutrosophic environments has led to numerous models and methods being applied in diverse fields such as economy, medicine, engineering, optimization, and decision-making. Neutrosophic sets provide a broader framework that encompasses classical sets, FS, and IFS, enabling the analysis of different types like uncertainty, vagueness, and ambiguous data. Nivatha and Varadharajan introduced a new approach to address fuzzy replacement method issues, including factors like maintenance, salvage, and initial costs [9]. Additionally, Shanmuga Sundari and Saranya proposed methods to identify optimal solutions for equipment replacement without affecting its fundamental characteristics [10]. Chakraborty et al. introduced the concept of bipolar neutrosophic numbers and the linear pentagonal neutrosophic number and its divisions [11]. They have successfully devised a method to convert linear pentagonal neutrosophic numbers using the removal area technique and have effectively applied it to solve a minimal spanning tree problem [12]. They also conducted a comparative analysis with existing methods. Rajkumar and Richard developed MATLAB built-in functions for the de-eutrophication technique, which are used to solve optimization models involving single-valued linear neutrosophic numbers. These functions can obtain a feasible ideal solution using optimization programming for the neutrosophic number under constrained conditions, taking into account practical production scenarios [13–15]. Khalifa and Kumar developed a model for assignment problems using inter-valued neutrosophic numbers. They employed order relations to optimize the objective function expressed in interval form, transforming the model into a multi-objective one based on the decision-makers' preferences regarding interval profits or costs [16].

Chakraborty and Broumi et.al, examined the pentagonal fuzzy number, covering its diverse representations, properties, ranking, defuzzification, and application in-game problems. Additionally, they introduced the foundational concept of bipolar triangular neutrosophic numbers, thereby expanding the understanding of bipolar neutrosophic numbers. Both linear and nonlinear forms were examined, incorporating truth, false, and hesitant functions to connect membership functions [17]. Avishek Chakraborty and Broumi provided properties for pentagonal neutrosophic numbers. Their study included the introduction of logical scores and accuracy functions for pentagonal neutrosophic numbers and they applied it to solve transportation problems, shortest path, networking, and machine inventory problems which were addressed using pentagonal neutrosophic numbers in this study [18–22]. The newly designed neutrosophic membership function [23] and a multi-attribute decision-making (MADM) technique contingent on neutrosophic trapezoid linguistic weighting for information analysis and decision-making [24–27]. Murshid Kamal examined the issue of multi-faceted disparities in a neutrosophic setting. By incorporating fuzzy parameters, they successfully identified the required system maintenance and replacement of components [28, 29].

Using MATLAB built-in functions, researchers developed neutrosophic numbers to address optimization models. They successfully found a feasible best solution for the optimization program using neutrosophic numbers within practical constraints [30]. Rajesh and Sunay proposed a novel approach for evaluating safety and risk in intricate systems utilizing neutrosophic logic [31]. A mathematical programming strategy has been introduced to address the selective maintenance issue in systems operating in uncertain environments. The goal is to enhance system reliability by creating a model that accounts for different uncertainties within the system [32]. Various techniques for calculating the system's reliability function were discussed, such as estimators based on maximum likelihood and methods that minimize variance while remaining unbiased. This study introduces a combined method for optimizing reliability allocation and examines the ideal quantity of components for a specific system [33]. Caglar Karamasa and Ezgi Demir proposed a novel approach utilizing the neutrosophic Analytic Hierarchy Process (AHP) to identify the variables influencing third-party logistics (3PL) outsourcing [34].

In observation, the envisioned technique was suggested to recognize the athletes who made the greatest contributions to the winning team through their exceptional efforts. The new players are rated using the weighted averaging operator [35]. Recently, the introduction of Plithogenic sets by Smarandache had a profound impact on contemporary research in diverse areas including IoT-based dilemmas, hybrid MCDM problems, and forecasting difficulties [36–39]. Several researches in the field of neutrosophic theory have been published across various domains, including dynamic intervals, Q-rung linguistic variables, and the MCDM method to assess safety and risk in complex systems [40–44]. In the present era, neutrosophic set theory is utilized in various aspects of the replacement method to assess the optimal path for selecting deteriorating inventory. Replacement stands as a crucial concept in the realm of operations research [45–49]. A few more articles published in the neutrosophic domain are discussed here, highlighting its significant role in addressing uncertainty within the optimization and inventory model research arena [50–54]. A comprehensive review study was conducted to highlight the advantages of the replacement policy method utilizing neutrosophic environment optimization of machine selection over the period. The study drew upon diverse sources, including research articles, academic journals, and book chapters, to demonstrate the merits of utilizing replacement theory in such an optimization framework problems [55]. The findings indicated that the integration of neutrosophic logic within the replacement policy framework not only enhances decision-making processes but also accommodates the inherent uncertainty and imprecision often encountered in real-world scenarios. By applying this method, decision-makers can effectively assess the performance and viability of different machines based on fluctuating operational conditions and financial considerations [56]. Furthermore, the study advocates for the continuous adaptation of replacement policies as a proactive measure to mitigate risks associated with equipment failure and operational downtimes. Scalability and computational complexity are critical considerations when designing and managing large systems, particularly in fields like distributed computing, big data, and machine learning. In conclusion, the embracing of replacement theory coupled with neutrosophic optimization presents a promising avenue for research and practical application, paving the way for more resilient and efficient industrial practices in the face of uncertainty.

1.2 *Motivation*

In our daily lives, the ability to make the right decision is often uncertain, so a neutrosophic environment is an essential part of analyzing our decisions in uncertainty, and it helps in constructing mathematical models, statistical problems, and big data analysis. The construction of neutrosophic numbers deals with the concept of uncertainty, which is a significant part of the neutrosophic environment. Uncertainty is a common factor in various aspects of life; often making decisions is a challenging task. The neutrosophic theory aims to reduce different forms of uncertainty in the decision-making process. Researchers are actively exploring and expanding the neutrosophic concept to be applicable across various disciplines like Decision-Making, image processing, Engineering and Optimization, and Medical diagnosis [6]. The distinct neutrosophic single-valued number is a particular example of a neutrosophic single-valued number. Can we define this unique case? How would a visual depiction of this figure appear? In what areas can it be utilized? These are some of the inquiries that arise when delving into this subject, and this article provides answers to these questions.

1.3 *Novelty*

Numerous studies have been conducted on the topic of neutrosophic theory [22–24]. Researchers from diverse fields have applied this concept to a wide range of areas and delving into uncharted territories with ongoing explorations. The concept of Hexagonal neutrosophic numbers is new to the domain, with limited research in neutrosophic environments. Our goal is to explore unpublished points on this topic and apply them to various fields, which are described below.

- Enhanced score function for unique hexagonal single-valued neutrosophic numbers.
- Removal area method for unique hexagonal single-valued neutrosophic numbers.

• Practical execution in replacement problems with MATLAB software.

1.4 *Structure of the paper*

The research layout is structured as follows: it begins with an introduction to the topic, followed by a literature review and a discussion of the motivation and novelty in Section 1. Section 2 presents the necessary preliminary. The third section focuses on aggregator operations for single-valued unique hexagonal neutrosophic numbers. In Section 4, a proposed score function is introduced. Section 5 outlines a replacement model formulation and procedure for machine selection using the Removal Area Method (RAM). Section 6 includes a case study on machine selection, demonstrating the use of RAM and the score function with MATLAB code. Section 7 provides a comparative analysis of single-valued neutrosophic numbers (SVNN) and unique single-valued hexagonal neutrosophic numbers (USVHNN). Finally, the results are discussed, and the research concludes with suggestions for future directions.

2. Preliminaries

This section outlines the essential concepts of FS, IFS, and NS along with the relevant terminology. It also presents a combined score function for the SVNNs, which is vital for the systematic review.

Definition 2.1 [9] A set $\tilde{A}$ in a universe of discourse X is defined by

$$\tilde{A} = \left\{\alpha,\ \mu_{\tilde{A}}(\alpha) \mid \alpha \in X\right\}$$

where $\mu_{\tilde{A}}(\alpha):\ X \to [0,\ 1]$ is called the membership functioof $\tilde{A}$, and $\mu_{\tilde{A}}(\alpha)$ is the degree of membership to which $\alpha \in \tilde{A}$. Thus, the set $\tilde{A}$ is defined as a fuzzy set.

Definition 2.2 [11] Intuitionistic fuzzy sets are defined on a non-empty set X as objects having the form

$$\tilde{A} = \{(\alpha,\ \mu_{\tilde{A}}(\alpha),\ \gamma_{\tilde{A}}(\alpha)) \mid \alpha \in X\}$$

where the functions $\mu_{\tilde{A}}(\alpha):\ X \to [0,\ 1]$ and $\gamma_{\tilde{A}}(\alpha):\ X \to [0,\ 1]$ denote the degree of membership and the degree of non-membership of each element $\alpha \in X$ to the set $\tilde{A}$, respectively. Furthermore, the following condition holds for all $\alpha \in X$:

$$0 \leq \mu_{\tilde{A}}(\alpha) + \gamma_{\tilde{A}}(\alpha) \leq 1$$

Definition 2.3 [14] A neutrosophic set $\tilde{A}$ in a set X is characterized by three functions: a truth membership function $T_{\tilde{A}}(\alpha)$, an indeterminacy membership function $I_{\tilde{A}}(\alpha)$, and a falsity membership function $F_{\tilde{A}}(\alpha)$. Each of these functions, $T_{\tilde{A}}(\alpha)$, $I_{\tilde{A}}(\alpha)$, and $F_{\tilde{A}}(\alpha)$, can be real standard or real nonstandard subsets of the interval $[0,\ 1]$. There are no constraints on the sum of these functions, so it holds that

$$0 \leq T_{\tilde{A}}(\alpha) \leq \sup I_{\tilde{A}}(\alpha) \leq F_{\tilde{A}}(\alpha) \leq 3.$$

Definition 2.4 Single Valued Linear Neutrosophic Set (SVLNS): A neutrosophic set $\tilde{A}$ is defined as a single-valued linear neutrosophic set (denoted as $sneu\tilde{A}$) when a is a single-valued independent variable. It is represented as:

$$sneu\tilde{A} = \{\alpha : < [\beta_{\tilde{A}(a)},\ \delta_{\tilde{A}(a)},\ \gamma_{\tilde{A}(a)}] > \mid a \in X\}$$

where $\beta_{\tilde{A}(a)}$, $\delta_{\tilde{A}(a)}$, and $\gamma_{\tilde{A}(a)}$ denote the membership functions related to truthiness, indeterminacy, and falsity, respectively.

When the neutrosophic set $\tilde{A}$ is continuous, it can be expressed as:

$$\tilde{A} = \int_a < T_{\tilde{A}}(\alpha),\ I_{\tilde{A}}(\alpha),\ F_{\tilde{A}}(\alpha) > \ da \quad \forall\ a \in X.$$

This formulation encompasses the continuous nature of the neutrosophic set, integrating over the independent variable a.

Definition 2.5 Single Valued Linear Pentagonal Neutrosophic Number: A single-valued pentagonal neutrosophic $\tilde{S}$ number is defined and described as $\tilde{S} = \left\langle [(\phi_1,\ \phi_2,\ \phi_3,\ \phi_4,\ \phi_5);\ \rho][(\delta_1,\ \delta_2,\ \delta_3,\ \delta_4,\ \delta_5);\ \sigma][(\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5);\ \omega] \right\rangle$. Where $\beta,\ \gamma,\ \alpha \in [0,\ 1]$. The truth membership function $(\phi_{\tilde{S}}):\ \tilde{\mathbb{R}} \to [0,\ \beta]$, indeterminacy membership function $(\theta_{\tilde{S}}):\ \tilde{\mathbb{R}} \to [\gamma,\ 1]$ and membership function $(\psi_{\tilde{S}})$: $\tilde{\mathbb{R}} \to [\alpha,\ 1]$ are given as:

$$\phi_{\tilde{S}}(x) = \begin{cases} \phi_{\tilde{S}1}(x), & \phi_1 \le x \le \phi_2 \\ \phi_{\tilde{S}2}(x), & \phi_2 \le x \le \phi_3 \\ \beta, & x = \phi_3 \\ \phi_{\tilde{S}3}(x), & \phi_3 \le x \le \phi_4 \\ \phi_{\tilde{S}4}(x), & \phi_4 \le x \le \phi_5 \\ 0, & \text{Otherwise} \end{cases}$$

$$\theta_{\tilde{S}}(x) = \begin{cases} \theta_{\tilde{S}1}(x), & \theta_1 \le x \le \theta_2 \\ \theta_{\tilde{S}2}(x), & \theta_2 \le x \le \theta_3 \\ \gamma, & x = \theta_3 \\ \theta_{\tilde{S}3}(x), & \theta_3 \le x \le \theta_4 \\ \theta_{\tilde{S}4}(x), & \theta_4 \le x \le \theta_5 \\ 1, & \text{Otherwise} \end{cases}$$

$$\psi_{\tilde{S}}(x) = \begin{cases} \psi_{\tilde{S}1}(x), & \psi_1 \le x \le \psi_2 \\ \psi_{\tilde{S}2}(x), & \psi_2 \le x \le \psi_3 \\ \alpha, & x = \psi_3 \\ \psi_{\tilde{S}3}(x), & \psi_3 \le x \le \psi_4 \\ \psi_{\tilde{S}4}(x), & \psi_4 \le x \le \psi_5 \\ 1, & \text{Otherwise} \end{cases}$$

The numerical computation of Single Valued Linear Pentagonal Neutrosophic Numbers is mentioned below. These examples highlight how truth, indeterminacy, and falsity values are structured and applied in decision-making contexts.

Table 1 systematically categorizes these values to reveal interrelations among parameters. The computational approach follows neutrosophic logic principles, enabling detailed analysis of uncertainty and vagueness in practical scenarios.

Table 1. Numerical computation of SVLPNN

S.No	SVLPNN	De-eutrophication value
1	$< 3, 5, 9, 11, 13; 5, 7, 9, 11, 13; 1, 3, 5, 7, 9 >$	12.1966
2	$< 0.8, 0.9, 1.0, 1.1, 1.2; 0.2, 0.3, 0.4, 0.5, 0.6; 1.5, 1.6, 1.7, 1.8, 1.9 >$	1.0333
3	$< 0.15, 0.25, 0.35, 0.45, 0.55; 0.36, 0.46, 0.56, 0.66, 0.76; 0.52, 0.62, 0.72, 0.82, 0.92 >$	0.54333

Definition 2.6 Unique Single Valued Hexagonal Neutrosophic Number: A single-valued unique hexagonal neutrosophic Number $\tilde{N}$ is defined as $\tilde{N}=\left\langle (\kappa_{11}, \kappa_{22}, \kappa_{33}, \kappa_{44}, \kappa_{55}, \kappa_{66}); \Psi, \Phi, \Omega \right\rangle$ on $\tilde{\mathbb{R}}$ where truth, falsity, indeterminacy membership functions are specified as follow below. Where $\eta_{\tilde{S}}(x): X \rightarrow [0, \psi]$, $\vartheta_{\tilde{S}}(x): X \rightarrow [\phi, 1]$, $\zeta_{\tilde{S}}(x): X \rightarrow [\omega, 1]$.

$$\eta_{\tilde{S}}(x) = \begin{cases} \dfrac{(x-\kappa_{11})}{(\kappa_{22}-\kappa_{11})}\Psi & \kappa_{11} \leq x \leq \kappa_{22} \\ \dfrac{(x-\kappa_{22})}{(\kappa_{33}-\kappa_{22})}\Psi & \kappa_{22} \leq x \leq \kappa_{33} \\ \dfrac{(x-\kappa_{33})}{(\kappa_{44}-\kappa_{33})}\Psi & \kappa_{33} \leq x \leq \kappa_{44} \\ \psi & \text{x}=\kappa_{44} \\ \dfrac{(\kappa_{55}-x)}{(\kappa_{55}-\kappa_{44})}\Psi & \kappa_{44} \leq x \leq \kappa_{55} \\ \dfrac{(\kappa_{66}-x)}{(\kappa_{66}-\kappa_{55})}\Psi & \kappa_{55} \leq x \leq \kappa_{66} \\ 0 & \text{otherwise} \end{cases}$$

$$
\vartheta_{\tilde{S}}(x)=\begin{cases}
\dfrac{(x-\kappa_{11})}{(\kappa_{22}-\kappa_{11})}\Phi+(\kappa_{22}-x) & \kappa_{11}\leq x\leq \kappa_{22}\\[2ex]
\dfrac{(x-\kappa_{22})}{(\kappa_{33}-\kappa_{22})}\Phi+(\kappa_{33}-x) & \kappa_{22}\leq x\leq \kappa_{33}\\[2ex]
\dfrac{(x-\kappa_{33})}{(\kappa_{44}-\kappa_{33})}\Phi+(\kappa_{44}-x) & \kappa_{33}\leq x\leq \kappa_{44}\\[2ex]
\phi & \text{x=}\kappa_{44}\\[2ex]
\dfrac{(\kappa_{55}-x)}{(\kappa_{55}-\kappa_{44})}\Phi+(x-\kappa_{55}) & \kappa_{44}\leq x\leq \kappa_{55}\\[2ex]
\dfrac{(\kappa_{66}-x)}{(\kappa_{66}-\kappa_{55})}\Phi+(x-\kappa_{66}) & \kappa_{55}\leq x\leq \kappa_{66}\\[2ex]
1 & \text{otherwise}
\end{cases}
$$

$$
\zeta_{\tilde{S}}(x)=\begin{cases}
\dfrac{(x-\kappa_{11})}{(\kappa_{22}-\kappa_{11})}\omega+(\kappa_{22}-x) & \kappa_{11}\leq x\leq \kappa_{22}\\[2ex]
\dfrac{(x-\kappa_{22})}{(\kappa_{33}-\kappa_{22})}\omega+(\kappa_{33}-x) & \kappa_{22}\leq x\leq \kappa_{33}\\[2ex]
\dfrac{(x-\kappa_{33})}{(\kappa_{44}-\kappa_{33})}\omega+(\kappa_{44}-x) & \kappa_{33}\leq x\leq \kappa_{44}\\[2ex]
\omega & \text{x=}\kappa_{44}\\[2ex]
\dfrac{(\kappa_{55}-x)}{(\kappa_{55}-\kappa_{44})}\omega+(x-\kappa_{55}) & \kappa_{44}\leq x\leq \kappa_{55}\\[2ex]
\dfrac{(\kappa_{66}-x)}{(\kappa_{66}-\kappa_{55})}\omega+(x-\kappa_{66}) & \kappa_{55}\leq x\leq \kappa_{66}\\[2ex]
1 & \text{otherwise}
\end{cases}
$$

3. Single-valued neutrosophic weighted average aggregation operators

This section presents two aggregation operators: the single-valued neutrosophic weighted averaging (SVNWA) operator and the operator for single-valued neutrosophic hexagonal numbers (SVNHNs). These operators have been specifically designed to handle single-valued neutrosophic numbers with precision and effectiveness.

3.1 *Operational laws for SVNNs*

Definition 3.1 The Single-Valued Neutrosophic Weighted Average (SVNWA) operator for a collection of Single-Valued Neutrosophic Numbers (SVNNs) is denoted by $s_j = \langle t_j,\ i_j,\ f_j\rangle$. This set comes with an associated weighting vector $W = (w_1,\ w_2,\ \ldots,\ w_n)$. The SVNWA operator is defined as follows:

$$\mathrm{SVNWA}(s_1,\ s_2,\ \ldots,\ s_n) = \prod_{j=1}^{n} w_j a_j = \left(1 - \prod_{j=1}^{n}(1-t_j)^{w_j},\ \prod_{j=1}^{n}(i_j)^{w_j},\ \prod_{j=1}^{n}(f_j)^{w_j}\right).$$

where: w_j is the element j of the weighting vector, $w_j \in [0,\ 1]$, and $\prod_{j=1}^{n} w_j = 1$.

Definition 3.2 Let $\tilde{S}_k = (\mathscr{T}, \mathscr{I}, \mathscr{F})$ and $\tilde{S}_{k11} = (T_{11},\ I_{11},\ F_{11})$ and $\tilde{S}_{k12} = (T_{12},\ I_{12},\ F_{12})$ are three SVNNs over the universe X, and here's how the following operations are defined:

(i) $\tilde{S}_{k11} \oplus \tilde{S}_{k12} = (T_{11} + T_{12} - T_{11} + T_{12},\ I_{11}I_{12},\ F_{11}F_{12})$

(ii) $\tilde{S}_{k11} \odot \tilde{S}_{k12} = (\frac{T_{11}T_{12}}{2},\ I_{11} + I_{12},\ \frac{I_{11}I_{12}}{2},\ F_{11} + F_{12},\ \frac{F_{11}F_{12}}{2})$

(iii) $\gamma\tilde{S}_k = (1-(1-T),\ I,\ F)$

(iv) $\tilde{S}_k^{\gamma} = (T,\ 1-(1-I)^{\gamma},\ 1-(1-F)^{\gamma})$

Definition 3.3 Let, $\tilde{S}_{kuv} = (T_{uv},\ I_{uv},\ F_{uv})$ $(s = 1,\ 2,\ \ldots,\ m;\ t = 1,\ 2,\ \ldots,\ n)$ which represents a definitive collection of weight vectors for the parameters e_t's and expert y_t's. These vectors must satisfy $t \geq 0$, $s \geq 0$, such that $\prod_{v=1}^{n} t = 1$, and $\prod_{u=1}^{m} s = 1$. In this context, we define the single-valued neutrosophic soft weighted averaging (SVNWA) operator as the functionSVNWA : $\tilde{S}^n \to \tilde{S}$, where

$$\mathrm{SVNWA}(\tilde{S}_{k11},\ \tilde{S}_{k12},\ \ldots,\ \tilde{S}_{kmn}) = \prod_{v=1}^{n} \Pi_t \left(\prod_{u=1}^{m} \theta_s \tilde{S}_{kuv}\right).$$

The following theorem is derived regarding the properties of the SVNWA operator.

Theorem 1 Let $\tilde{S}_{kuv} = (T_{uv},\ I_{uv},\ F_{uv})$ $(s = 1,\ 2,\ \ldots,\ m;\ t = 1,\ 2,\ \ldots,\ n)$ be a set of SVNHNs. Then, the aggregated components' values utilizing the SVNWA operator are also an SVNHN and can be depicted as

$$\begin{aligned} &\mathrm{SVNWA}(S_{k11},\ S_{k12},\ \ldots,\ S_{kmn}) \\ &= \left[1 - \prod_{t=1}^{m}\left(\prod_{u=1}^{m}(1-T_{uv})^{\theta_s}\right)^{\Pi_t},\ \prod_{t=1}^{m}\left(\prod_{u=1}^{m} I_{uv}^{\theta_s}\right)^{\Pi_t},\ \prod_{t=1}^{m}\left(\prod_{u=1}^{m} F_{uv}^{\theta_s}\right)^{\Pi_t}\right] \end{aligned} \tag{1}$$

Proof. For $m = 1$, we have $\theta_1 = 1$. According to Definition 7 using operational law,

$$\text{SVNWA}(S_{k11},\, S_{k12},\, \ldots,\, S_{kmn}) = \prod_{n} \Pi_t [S_{k1t}]$$

$$= \left[1 - \prod_{v=1}^{n} (1 - T_{1t})^{\Pi_t},\, \prod_{v=1}^{n} I_{1t}^{\Pi_t},\, \prod_{v=1}^{n} F_{1t}^{\Pi_t}\right] \tag{2}$$

$$= \left[1 - \prod_{v=1}^{n} \left(\prod_{u=1}^{m} (1 - F_{uv})^{\theta_s}\right)^{\Pi_t},\, \prod_{v=1}^{n} \left(\prod_{u=1}^{m} I_{uv}^{\theta_s}\right)^{\Pi_t},\, \prod_{v=1}^{n} \left(\prod_{u=1}^{m} F_{uv}^{\theta_s}\right)^{\Pi_t}\right].$$

Similarly, for $n = 1$ and $\Pi_1 = 1$,

$$\text{SVNHWA}(C_{e11},\, C_{e12},\, \ldots,\, C_{emn}) = \prod_{m} \theta_{s1} C_{kuv}$$

$$= \left[1 - \prod_{v=1}^{n} (1 - T_{s1})^{\theta_s},\, \prod_{u=1}^{m} I_{s1}^{\theta_s},\, \prod_{u=1}^{m} F_{s1}^{\theta_s}\right] \tag{3}$$

$$= \left[1 - \prod_{t=1}^{m} \left(\prod_{u=1}^{m} (1 - T_{uv})^{\theta_s}\right)^{\Pi_t},\, \prod_{t=1}^{m} \left(\prod_{u=1}^{m} I_{uv}^{\theta_s}\right)^{\Pi_t},\, \prod_{t=1}^{m} \left(\prod_{u=1}^{m} F_{uv}^{\theta_s}\right)^{\Pi_t}\right].$$

Assuming (5) is true for $n = p_1 + 1$, $m = p_2$ and $n = p_1$, $m = p_2 + 1$, we can deduce that

$$\left[p \prod_{1}^{p+1} \Pi_t\right] \left[\prod_{p_2} \theta_s C_{kuv}\right] = \left[1 - p_1 + 1 \prod_{t=1}^{p_2} \left(p_2 \prod_{u=1}^{m} (1 - T_{uv})^{\theta_s}\right)^{\Pi_t},\, p_1 + 1 \prod_{t=1}^{p_2}\right.$$
$$\left.\left[\left(p_2 \prod_{u=1}^{m} I_{uv}^{\theta_s}\right)^{\Pi_t},\, p_1 + 1 \prod_{t=1}^{p_2} \left(p_2 \prod_{u=1}^{m} F_{uv}^{\theta_s}\right)^{\Pi_t}\right]\right]. \tag{4}$$

Similarly,

$$\left[\prod_{p_1} \Pi_t\right] \left[p \prod_{2+1}^{s} \theta_s C_{kuv}\right] = \left[1 - p_1 \prod_{t=1}^{p_2+1} \left(p_2 + 1 \prod_{u=1}^{m} (1 - T_{uv})^{\theta_s}\right)^{\Pi_t},\, p_1 \prod_{t=1}^{p_2+1}\right.$$
$$\left.\left[\left(p_2 + 1 \prod_{u=1}^{m} I_{uv}^{\theta_s}\right)^{\Pi_t},\, p_1 \prod_{t=1}^{p_2+1} \left(p_2 + 1 \prod_{u=1}^{m} F_{uv}^{\theta_s}\right)^{\Pi_t}\right]\right] \tag{5}$$

By induction, for $n = p_1 + 1$ and $m = p_2 + 1$, we obtain

$$\left[p\prod_{1}^{p+1}\Pi_t\right]\left[p\prod_{2+1}^{s}\theta_s C_{kuv}\right] = \left[1 - p_1 + 1\prod_{t=1}^{p_2+1}\left(p_2 + 1\prod_{u=1}^{m}(1 - T_{uv})^{\theta_s}\right)^{\Pi_t},\ p_1 + 1\prod_{t=1}^{p_2+1}\right.$$
$$\left.\left[\left(p_2 + 1\prod_{u=1}^{m} I_{uv}^{\theta_s}\right)^{\Pi_t},\ p_1 + 1\prod_{t=1}^{p_2+1}\left(p_2 + 1\prod_{u=1}^{m} F_{uv}^{\theta_s}\right)^{\Pi_t}\right]\right] \tag{6}$$

Thus, the result holds for all $m,\ n \geq 1$. Since $0 \leq T_{uv} \leq 1$ implies $0 \leq \prod_{s=1}(1 - T_{uv})^{\theta_s} \leq 1$, and similarly for I_{uv} and F_{uv}, we have

$$0 \leq 1 - \prod_{t=1}\left(\prod_{u=1}(1 - T_{uv})^{\theta_s}\right)^{\Pi_t} + \prod_{t=1}\left(\prod_{s=1} I_{uv}^{\theta_s}\right)^{\Pi_t} + \prod_{t=1}\left(\prod_{s=1} F_{uv}^{\theta_s}\right)^{\Pi_t}. \tag{7}$$

The proof is completed. □

Corollary When there is only one parameter $m = 1$, the SVNWA operator simplifies to SVNWA.

The SVNWA operation for m SVNHNs, denoted by SVNWA($\tilde{S}_{k11}$, $\tilde{S}_{k21}$, ..., $\tilde{S}_{km1}$), is given by:

$$\text{SVNWA}(\tilde{S}_{k11},\ \tilde{S}_{k21},\ ...,\ \tilde{S}_{km1}) = \left(1 - \prod_{u=1}^{m}(1 - T_s)\right),\ \left(\prod_{u=1}^{m} I_s\right),\ \left(\prod_{u=1}^{m} F_s\right).$$

Consequently, it can be asserted that the aggregation operator defined within the SVNS environment can be viewed as a specific instance of the proposed operator.

Theorem 2 (Idempotency Property) Let $\tilde{S}_{kuv} = (\mathscr{T}_{uv},\ \mathscr{I}_{uv},\ \mathscr{F}_{uv})$ $(u = 1,\ 2,\ \ldots,\ m;\ \ v = 1,\ 2,\ \ldots,\ n)$ be a number of SVNHNs that are all equal, i.e., $\tilde{S}_{kuv} = \tilde{S}_e$ for all $s,\ t$, then $\text{SVNHWA}(\tilde{S}_{k11},\ \tilde{S}_{k12},\ \ldots,\ \tilde{S}_{kuv}) = \tilde{S}_k$.

Proof. Since $\tilde{S}_{kuv} = \tilde{S}_k = (\mathscr{T},\ \mathscr{I},\ \mathscr{F})$ for all $u,\ v$, then,

$$\text{SVNHWA}(\tilde{S}_{k11},\ \tilde{S}_{k12},\ \ldots,\ \tilde{S}_{kuv}) = \left(1 - \prod_{v=1}^{n}\left(\prod_{u=1}^{m}(1 - \mathscr{T})^{\theta_s}\right)^{\varphi_t}\right),\ \prod_{v=1}^{n}\left(\prod_{u=1}^{m}\mathscr{I}^{\theta_s}\right)^{\varphi_t},\ \prod_{v=1}^{n}\left(\prod_{u=1}^{m}\mathscr{F}^{\theta_s}\right)^{\varphi_t}$$

$$= \left(1 - (1 - \mathscr{T})^{\prod_{u=1}^{m}\theta_s\prod_{v=1}^{n}\varphi_t}\right),\ \left(\mathscr{I}^{\prod_{u=1}^{m}\theta_s}\right)^{\prod_{v=1}^{n}\varphi_t},\ \left(\mathscr{F}^{\prod_{u=1}^{m}\theta_s}\right)^{\prod_{v=1}^{n}\varphi_t}$$

$$= (1 - (1 - \mathscr{T},\ \mathscr{I},\ \mathscr{F}))$$

$$= (\mathscr{T},\ \mathscr{I},\ \mathscr{F})$$

Hence the proof is completed.

Theorem 3 (Homogeneity Property) For any real number $\lambda > 0$, we have

$$\text{SVNWA}(\lambda\tilde{S}_{k11},\ \lambda\tilde{S}_{k12},\ \ldots,\ \lambda\tilde{S}_{kmn}) = \lambda\text{SVNWA}(\tilde{S}_{k11},\ \tilde{S}_{k12},\ \ldots,\ \tilde{S}_{kuv}).$$

Proof. Let $\tilde{S}_{kuv} = (\mathscr{T}_{uv},\ \mathscr{I}_{uv},\ \mathscr{F}_{uv})$ $(u = 1,\ 2,\ \ldots,\ m;\ \ v = 1,\ 2,\ \ldots,\ n)$ be a number of SVNSNs and $\lambda > 0$ be any real number. Then, $\lambda\tilde{S}_{kuv} = \left(1-(1-\mathscr{T}_{uv})^{\lambda},\ (\mathscr{I}_{uv})^{\lambda},\ (\mathscr{F}_{uv})^{\lambda}\right)$. Thus,

$$\text{SVNWA}(\lambda\tilde{S}_{k11},\ \lambda\tilde{S}_{k12},\ \ldots,\ \lambda\tilde{S}_{kuv}) = \left(1-\prod_{v=1}^{n}\left(\prod_{u=1}^{m}(1-\mathscr{T}_{uv})^{\lambda\theta_s}\right)^{\varphi_t}\right),\ \prod_{v=1}^{n}\left(\prod_{u=1}^{m}(\mathscr{I}_{uv})^{\theta_s\lambda}\right)^{\varphi_t},$$

$$\prod_{v=1}^{n}\left(\prod_{u=1}^{m}(\mathscr{F}_{uv})^{\theta_s\lambda}\right)^{\varphi_t}$$

$$= \left(1-\prod_{v=1}^{n}\prod_{u=1}^{m}(1-\mathscr{T}_{uv})^{\theta_s\varphi_t\lambda}\right)$$

$$= \left(1-\prod_{v=1}^{n}\prod_{u=1}^{m}(1-\mathscr{T}_{uv})^{\theta_s\varphi_t\lambda}\right),\ \prod_{v=1}^{n}\prod_{u=1}^{m}(\mathscr{I}_{uv})^{\theta_s\varphi_t\lambda},\ \prod_{v=1}^{n}\prod_{u=1}^{m}(\mathscr{F}_{uv})^{\theta_s\varphi_t\lambda}$$

$$= \left(1-\prod_{v=1}^{n}\prod_{u=1}^{m}(1-\mathscr{T}_{uv})^{\theta_s\varphi_t\lambda}\right)$$

$$= \lambda\text{SVNWA}(\tilde{S}_{k11},\ \tilde{S}_{k12},\ \ldots,\ \tilde{S}_{kuv})$$

Hence the proof is completed. □

Theorem 4 (Shift-invariance property): If $\tilde{S}_k = h(\mathscr{T},\ \mathscr{I},\ \mathscr{F})$ be another SVNHN, then

$$\text{SVNHWA}(\tilde{S}_{k11}\oplus\tilde{S}_k,\ \tilde{S}_{k12}\oplus\tilde{S}_k,\ \ldots,\ \tilde{S}_{kuv}\oplus\tilde{S}_k) = \text{SVNHWA}(\tilde{S}_{k11},\ \tilde{S}_{k12},\ \ldots,\ \tilde{S}_{kuv})\oplus\tilde{S}_k.$$

Proof. Since $\tilde{S}_e$ and $\tilde{S}_{uv}$ are SVNHNs, we have $\tilde{S}_e\oplus\tilde{S}_{uv} = (1-(1-\mathscr{T})(1-\mathscr{T}_{uv},\ \mathscr{I}_{uv},\ \mathscr{F}_{uv}))$. Hence,

$$\text{SVNHWA}(\tilde{S}_{k11} \oplus \tilde{S}_k,\ \tilde{S}_{k12} \oplus \tilde{S}_k,\ \ldots,\ \tilde{S}_{kuv} \oplus \tilde{S}_k)$$

$$= \prod_{t=1}^{n} \left(\prod_{s=1}^{m} (1 - \mathscr{T}_{uv})^{\theta_s} (1 - \mathscr{T})^{\theta_s} \right)^{\varphi_t} \prod_{t=1}^{n} \left(\prod_{s=1}^{m} (\mathscr{I}_{uv})^{\theta_s} (\mathscr{I})^{\theta_s} \right)^{\varphi_t} \cdot \prod_{t=1}^{n} \left(\prod_{s=1}^{m} (\mathscr{F}_{uv})^{\theta_s} (\mathscr{F})^{\theta_s} \right)^{\varphi_t}$$

$$= \left(1 - \prod_{t=1}^{n} \left(\prod_{s=1}^{m} (1 - \mathscr{T}_{uv})^{\theta_s} \right)^{\varphi_t},\ \mathscr{I}^{\prod_{s=1}^{m} \theta_s \sum_{t=1}^{n} \varphi_t},\ \mathscr{F}^{\prod_{s=1}^{m} \theta_s \sum_{t=1}^{n} \varphi_t} \right)$$

$$= \left(1 - \prod_{t=1}^{n} \left(\prod_{s=1}^{m} (1 - \mathscr{T}_{uv})^{\theta_s} \right)^{\varphi_t},\ \mathscr{I}^{\prod_{s=1}^{m} \theta_s \sum_{t=1}^{n} \varphi_t},\ \mathscr{F}^{\prod_{s=1}^{m} \theta_s \sum_{t=1}^{n} \varphi_t} \right) \oplus (\mathscr{T},\ \mathscr{I},\ \mathscr{F})$$

$$= \text{SVNHWA}(\tilde{S}_{k11},\ \tilde{S}_{k12},\ \ldots,\ \tilde{S}_{kuv}) \oplus \tilde{S}_k.$$

Hence the result.

The proof is completed. □

4. Materials and methods

4.1 *De-eutrophication of a linear neutrosophic hexagonal number*

By developing the de-eutrophication method, researchers have found a way to convert outcomes into a specific numerical value using the hexagonal neutrosophic number and its membership functions [12]. The developers have consistently created convenient methods for converting a fuzzy number into a precise number [13]. Some of these approaches are listed below.

• Defuzzification Basic Distributions (DBAD)
• Area of Bisector (AOB)
• Area of Center (AOC)
• Quality Extended Method (QEM)
• Defuzzification Fuzzy Clustering (DFC)

In hexagonal neutrosophic numbers, there are three distinct membership functions, making it difficult to convert them into crisp numbers. A new method called the "removal area method" has been proposed in recent research to address this issue. This method enables us to delineate crisp values from the hexagonal representation by calculating the effective areas that each sector of the hexagon covers. To derive crisp numbers, we implement an area removal strategy, whereby we successively subtract the areas that do not contribute to the desired truth value. In doing so, we achieve a more precise determination of the neutrosophic number's essences, thus allowing us to convert the complex membership structure into actionable data. Suppose we assume a linear neutrosophic hexagonal number as follows graphical presentation: The hexagonal numbers, represented visually, display a unique geometric formation where concentric hexagons converge to form a harmonious lattice. In this framework, each hexagonal layer contributes to a cumulative sum, with the neutrosophic elements introducing degrees of uncertainty and indeterminacy. The vertices of each hexagon become pivotal points within this multidimensional space, where the interplay of truth, indeterminacy, and falsehood generates a spectrum of potential outcomes.

The score function value of the single-valued unique hexagonal neutrosophic number is represented with a graphical presentation is given below.

$$\tilde{A}_{SVUHNN}(\tilde{E},\ 0) = (\kappa_{11},\ \kappa_{22},\ \kappa_{33},\ \kappa_{44},\ \kappa_{55},\ \kappa_{66});\ \Psi_T,\ \Phi_I,\ \Omega_F$$

Figure 1 visually represents a single-valued hexagonal neutrosophic number. The red outline represents truthfulness, the green outline represents falsity, and the purple outline indicates the indeterminacy of membership functions. For the given value of δ, the relation value falls within the range $0 \leq \delta \leq 1$. The hexagonal number can be transformed into a triangular neutrosophic number when δ is either 0 or 1. We then conducted a brief analysis of the de-eutrophication value of the linear hexagonal numbers, which is cited from the data source available at [19]. Furthermore, our exploration delves into various mathematical properties, such as the stability of the membership function across different values of δ, which is vital for ensuring consistent decision-making outcomes. It is essential to highlight that as δ transitions between 0 and 1, the influences of truth, falsity, and indeterminacy interlace, providing a comprehensive framework to evaluate complex problems. As we proceed, further investigations into the practical implications of de-eutrophication elucidate how these methodologies can be applied to optimize decision-making processes across diverse domains.

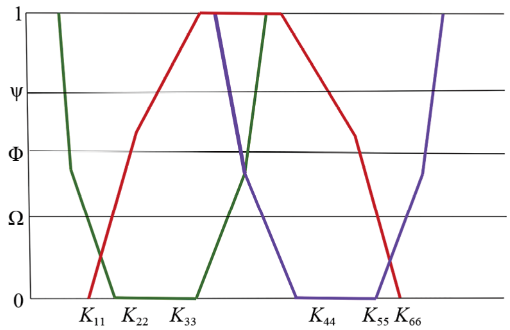

Figure 1. A pictorial presentation of single-valued unique hexagonal neutrosophic number

4.2 *Proposed method*

A score function has been proposed for Single-Valued Hexagonal Neutrosophic Numbers utilizing the 'removal area method'. Additionally, the de-eutrophication value of a Single-Valued Hexagonal Neutrosophic Number has been formulated, and its graphical representation is presented below.

The truth-membership depicted in Figure 2 illustrates the relationship between the varying degrees of truth, indeterminacy, and falsity, allowing for a comprehensive representation of complex systems where traditional binary logic falls short. In this graphical representation, we observe the interplay of these components, reinforcing the significance of a balanced perspective in evaluating data. The Indeterminacy membership function shown in Figure 3 captures the nuances of uncertainty, highlighting the spectrum of indeterminate values that exist between truth and falsity. This representation emphasizes the complexity inherent in situations where information is incomplete or ambiguous. By visualizing indeterminacy, we gain a clearer understanding of its pivotal role in assessing data and making informed decisions in multifaceted scenarios.

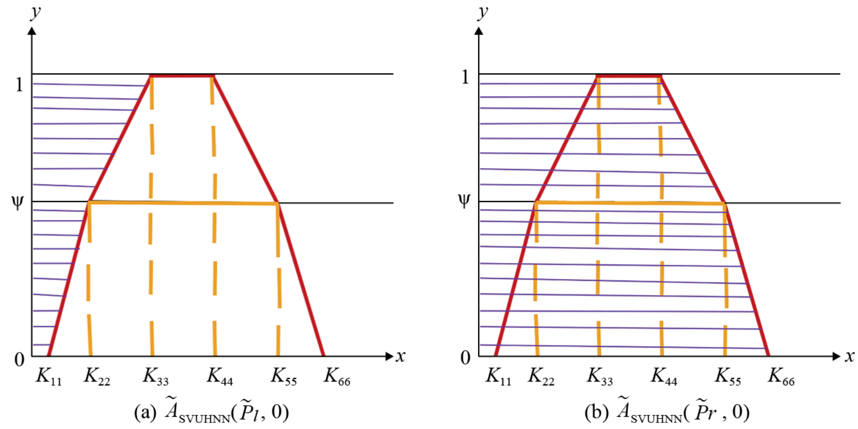

Figure 2. Truth-Membership function of unique hexagonal number

$$\tilde{A}_{SVUHNN}(\tilde{P}_l,\ 0) = \frac{(\kappa_{11}+\kappa_{22})}{2}\Psi_T + \frac{(\kappa_{22}+\kappa_{33})}{2}\Psi_T + \frac{(\kappa_{33}+\kappa_{44})}{2}(1-\Psi_T)$$

$$\tilde{A}_{SVUHNN}(\tilde{P}_r,\ 0) = \frac{(\kappa_{44}+\kappa_{55})}{2}(1-\Psi_T) + \frac{(\kappa_{55}+\kappa_{66})}{2}\Psi_T$$

$$\tilde{A}_{SVUHNN}(\tilde{P},\ 0) = \frac{(\kappa_{11}+2\kappa_{22}+\kappa_{33}+\kappa_{55}+\kappa_{66})\Psi_T + (\kappa_{33}+2\kappa_{44}+\kappa_{55})(1-\Psi_T)}{4}$$

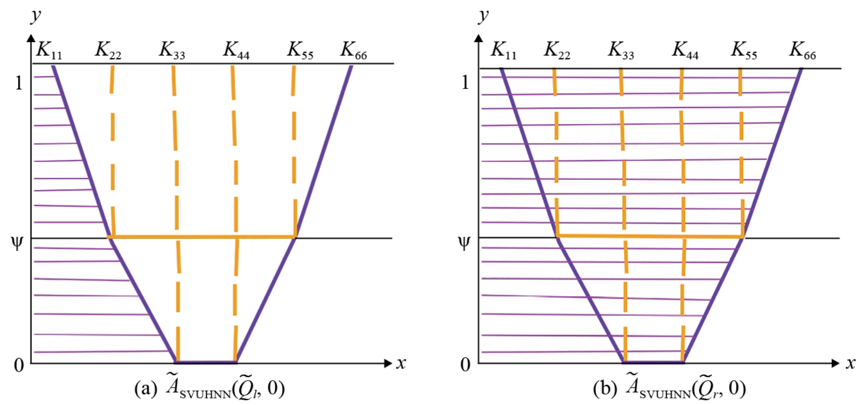

Figure 3. Indeterminacy-Membership function of unique hexagonal number

The falsity membership function in Figure 4 illustrates the spectrum of falseness, from somewhat false to entirely false, concerning truth and indeterminacy. This highlights the importance of considering falsity along with truth and indeterminacy for a more nuanced evaluation of complex datasets, avoiding oversimplification.

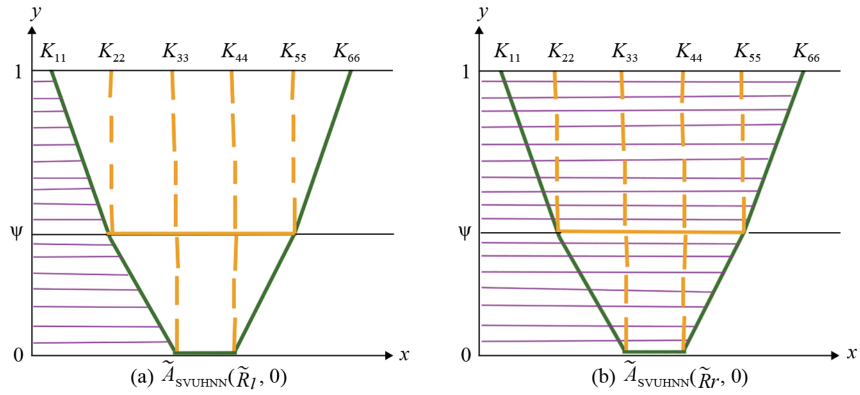

Figure 4. Falsity-Membership function of unique hexagonal number

Similarly calculating for the indeterminacy and falsity from Figure 3 and Figure 4, we obtain

$$\tilde{A}_{SVUHNN}(\tilde{Q},\, 0) = \frac{(\kappa_{11} + 2\kappa_{22} + \kappa_{33} + \kappa_{55} + \kappa_{66})(1 - \Phi_I) + (\kappa_{33} + 2\kappa_{44} + \kappa_{55})\Phi_I}{4}$$

$$\tilde{A}_{SVUHNN}(\tilde{R},\, 0) = \frac{(\kappa_{11} + 2\kappa_{22} + \kappa_{33} + \kappa_{55} + \kappa_{66})(1 - \Omega_F) + (\kappa_{33} + 2\kappa_{44} + \kappa_{55})\Omega_F}{4}$$

$$\tilde{A}_{SVUHNN}(\tilde{E},\, 0) = \frac{\tilde{A}_{SVUHNN}(\tilde{P},\, 0) + \tilde{A}_{SVUHNN}(\tilde{Q},\, 0) + \tilde{A}_{SVUHNN}(\tilde{R},\, 0)}{3}$$

The scoring function's estimated value for a unique hexagonal neutrosophic single-valued number is provided below.

$$\tilde{A}_{SVUHNN}(\tilde{E},\, 0) = (\kappa_{11},\ \kappa_{22},\ \kappa_{33},\ \kappa_{44},\ \kappa_{55},\ \kappa_{66});\ \Psi_T,\ \Phi_I,\ \Omega_F$$

$$\tilde{A}_{SVUHNN}(\tilde{E},\, 0) = \frac{1}{12}\{(\kappa_{11} + 2\kappa_{22}\kappa_{33} + 2\kappa_{44} + \kappa_{66})\Psi_T - (\kappa_{11} + 2\kappa_{22} - 2\kappa_{44} + \kappa_{66})\Phi_I$$

$$- (\kappa_{11} + 2\kappa_{22} - 2\kappa_{44} + \kappa_{66})\Omega_F + (2\kappa_{11} + 4\kappa_{22} + \kappa_{11} + 3\kappa_{33} + 2\kappa_{44} + 3\kappa_{55} + 2\kappa_{66})\} \tag{8}$$

The implementation of a unique hexagonal neutrosophic single-valued number within scoring functions facilitates advanced evaluations in multidimensional decision-making scenarios, yielding insights that reflect the true complexities of the evaluated alternatives. Future research will undoubtedly expand upon these foundational principles, culminating in more refined methodologies that harness the strengths of neutrosophic logic in real-world applications.

5. Replacement model formulation

Our goal is to determine the optimum replacement time for an item whose running or maintenance cost rises with time, while the value of money remains constant throughout the period. For calculation purposes, the machine's purchase cost is $\daleth_i$ and maintenance cost is κ_{ii}, as well as its scarp value κ_{iii}. The average cost over n years is represented by a single-valued unique hexagonal neutrosophic number.

The procedures to resolve the replacement problem are as follows:

Step 1 Consider the problem and create a group of expert decisions to determine the possibilities and criteria.

Step 2 Use the linguistic variable given in Table 2 to convert the problem.

Step 3 The machine's cost value is calculated using the aggregated operation of a single-valued hexagonal neutrosophic weighted average method using Theorem 1 and 2 the cost values are expressed as a single-valued hexagonal neutrosophic number.

Step 4 Applying equation (8) which clarifies the process of converting a neutrosophic number into a precise number.

Step 5 Utilizing Matlab code for converting the neutrosophic number into a precise number, and solving the problem with standard methods.

Step 6 Analyze the data to determine the optimal year for replacing an item before it fails.

The flowchart presentation of the replacement formulation is shown in Figure 5. This visual representation outlines the key steps involved in identifying potential replacement formulations, evaluating their effectiveness, and implementing the most viable options.

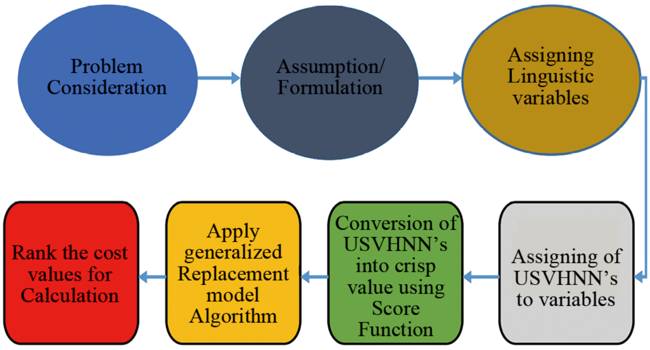

Figure 5. Flowchart process of replacement model

6. Illustrations

6.1 *Example 1*

A steel manufacturing company, operating at full scale, recently integrated a new type of loader machine into its operations to enhance efficiency and productivity. The purchase cost of the machine, represented by the neutrosophic number $\tilde{C}$, encapsulates the inherent uncertainties associated with market fluctuations, additional costs, and potential discounts or surcharges. This cost is given by:

$$\left[< (51{,}000,\ 51{,}300,\ 51{,}700,\ 62{,}000,\ 62{,}300,\ 62{,}700);\ 0.83,\ 0.44,\ 0.22 > \right].$$

Here, the values represent the potential ranges of the machine's cost, accompanied by degrees of truth, indeterminacy, and falsity, reflecting the company's confidence, uncertainty, and skepticism about each range.

Table 2. Failure severity ratings

S. No	Linguistic description	Description
1	Very small problem	[0.1-0.2]
2	Small problem	[0.2-0.3]
3	Slight problem	[0.3-0.4]
4	Noticeable problem	[0.4-0.5]
5	Average problem	[0.5-0.6]
6	Serious problem	[0.6-0.7]
7	Quite serious problem	[0.7-0.8]
8	Major problem	[0.8-0.9]
9	Very major problem	[0.9-1.0]

Table 2 illustrates the cost-effective rate or failure time cost, enabling the conversion of linguistic variables into codes that optimize the SVUHNN for improved decision-making across various scenarios. Utilizing the SVUHNN model allows for a systematic analysis of cost implications related to different failure times, enhancing our understanding of the associated trade-offs. The conversion of linguistic variables into codes facilitates a more nuanced interpretation of the data, enabling clearer comparisons and more informed decisions.

Table 3. Maintenance cost of neutrosophic numbers

Years (n)	Maintenance cost of loader for six years
1	$< (750,\ 800,\ 850,\ 900,\ 950,\ 1{,}000;\ 0.76,\ 0.53,\ 0.24) >$
2	$< (1{,}050,\ 1{,}100,\ 1{,}150,\ 1{,}200,\ 1{,}250,\ 1{,}300;\ 0.68,\ 0.4,\ 0.32) >$
3	$< (1{,}350,\ 1{,}400,\ 1{,}450,\ 1{,}500,\ 1{,}550,\ 1{,}600;\ 0.53,\ 0.44,\ 0.43) >$
4	$< (1{,}650,\ 1{,}700,\ 1{,}750,\ 1{,}800,\ 1{,}850,\ 1{,}900;\ 0.57,\ 0.3,\ 0.4) >$
5	$< (2{,}000,\ 2{,}100,\ 2{,}250,\ 2{,}300,\ 2{,}600,\ 2{,}680;\ 0.43,\ 0.5,\ 0.53) >$
6	$< (3{,}800,\ 4{,}100,\ 4{,}440,\ 4{,}600,\ 4{,}750,\ 5{,}000;\ 0.2,\ 0.3,\ 0.5) >$

Table 4. Scrap cost of neutrosophic numbers

Years (n)	Scrap cost of loader for six years
1	$< (4{,}200,\ 4{,}000,\ 3{,}800,\ 3{,}000,\ 2{,}400,\ 2{,}200;\ 0.83,\ 0.63,\ 0.37) >$
2	$< (1{,}850,\ 2{,}000,\ 1{,}800,\ 1{,}550,\ 1{,}320,\ 1{,}270;\ 0.74,\ 0.64,\ 0.22) >$
3	$< (1{,}340,\ 1{,}200,\ 1{,}100,\ 1{,}000,\ 680,\ 620;\ 0.67,\ 0.43,\ 0.47) >$
4	$< (600,\ 650,\ 580,\ 540,\ 520,\ 500;\ 0.57,\ 0.43,\ 0.63) >$
5	$< (500,\ 550,\ 480,\ 440,\ 420,\ 400;\ 0.42,\ 0.32,\ 0.67) >$
6	$< (400,\ 350,\ 380,\ 340,\ 320,\ 300;\ 0.37,\ 0.25,\ 0.72) >$

To calculate the maintenance and salvage costs, we will use Theorem 1 and Theorem 2, along with the linguistic variables listed in Table 2. Next, we will employ a score function to convert the USVHNN numbers from Tables 3 and 4 into a precise numerical value using Equation (8). This conversion process will transform ambiguous neutrosophic data into exact numerical information, which will then be computed and presented in Table 5. As a result, the machine cost is determined to be 282,449.

Table 5. The crisp value of maintenance and scrap cost

Years (m)	Maintenance cost	Scrap cost
1	$< 3,876 >$	$< 16,932 >$
2	$< 4,726 >$	$< 7,562.8 >$
3	$< 4,637.5 >$	$< 4,261.2 >$
4	$< 6,013.5 >$	$< 1,983.6 >$
5	$< 5,796.4 >$	$< 1,209.6 >$
6	$< 5,240 >$	$< 769.6 >$

In the previous section, we discussed the generalized replacement approach and now we are tackling the machine replacement issues. The numerical results presented in Table 5 will help in making strategic decisions and improving efficiency. By using MATLAB software, we can determine the best year to replace the machine, that the average annual total cost is at its lowest during the fourth year. To achieve this, we will first define the total cost function, which includes the fixed and variable costs associated with the machine, along with any expected maintenance and operational expenses over time. We denote the fixed costs as C_f, the variable costs per year as C_v, and the expected maintenance costs as C_m. The total cost $C_{\text{total}}(n)$ for owning the machine over n years can be expressed as.

$$C_{\text{total}}(n) = C_f + n \cdot C_v + \sum_{i=1}^{n} C_m(i)$$

where $C_m(i)$ represents the maintenance costs incurred in year i. For simplicity, we can assume that maintenance costs increase linearly with the age of the machine, such that:

$$C_m(i) = C_{m0} + k \cdot (i-1)$$

Here, C_{m0} is the initial maintenance cost in the first year, and k is the incremental cost added each subsequent year. Substituting this into our total cost function gives:

$$C_{\text{total}}(n) = C_f + n \cdot C_v + n \cdot C_{m0} + k \cdot \frac{(n-1)n}{2}$$

The term $k \cdot \frac{(n-1)n}{2}$ accounts for the cumulative increase in maintenance costs over n years.

Next, we can implement this function in MATLAB to analyze the total cost for each year of operation. We will calculate the total cost for a range of years, say from 1 to 10, and then plot the results to visually identify the year with the lowest average annual total cost. The MATLAB code for this analysis would look like this. Using MATLAB, we can set up a cost analysis model that takes into account the initial purchase cost, annual operational costs, and any anticipated repairs or replacements needed as the machine ages. By iterating over a range of years, we can calculate the average annual total cost for each year. Therefore, it is recommended to replace the machine after the 4th year. The bar plot in Figure 6 illustrates that replacing the machine at the end of the fourth year is optimal.

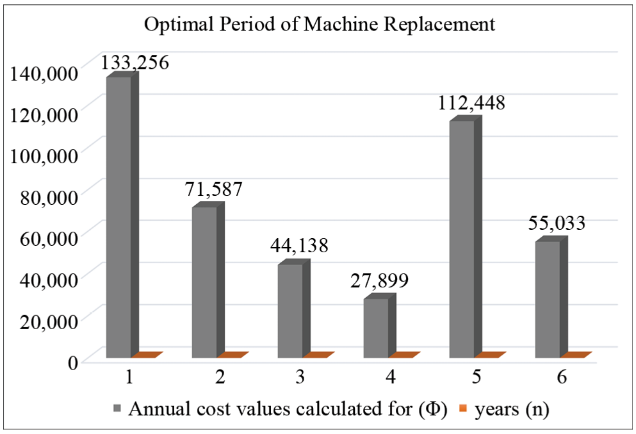

Figure 6. Bar plot visualization of the most effective period for machine replacement

The data shows that efficiency decreases significantly beyond this point, highlighting the cost-effectiveness of upgrading the machinery at this time. When compared to intuitionistic fuzzy and fuzzy numbers applied with the standard replacement method, we obtain the closest solution to the problem. This method has not been explored in the neutrosophic field.

6.2 *Example 2*

The electronic equipment company has 1,000 resistors in stock. In case of a resistor failure, it is replaced for Rs. 7 per resistor. At the same time, replacing all transistors at once would cost Rs. 4 per transistor. The percentage of surviving components, denoted as $S(i)$, is now tabulated in Table 6 at the end of each month i. The task is to find the optimal replacement policy, i.e., whether individual replacement policy or group replacement policy is better. To determine the optimal replacement policy, we will analyze the costs associated with both individual and group replacement strategies over a defined period. The individual replacement policy involves replacing each failed resistor at a cost of Rs. 7, while the group replacement policy entails replacing all resistors at once, incurring a cost of Rs. 4 per transistor.

Table 6. The percent of surviving each month

Month (i)	Percent of surviving $S(i)$
1	$< 90 >$
2	$< 85 >$
3	$< 65 >$
4	$< 35 >$
5	$< 15 >$
6	$< 0 >$

To analyze the effectiveness of the replacement strategies, we need to consider the costs associated with each approach in conjunction with the data provided in Table 6. Using MATLAB code snippet analyze the effectiveness of the strategies by calculating the total costs and visualizing them through a bar graph.

The output displayed in Figure 7 will provide a clear overview of the financial implications of each approach, aiding decision-makers in selecting the most viable replacement strategy. From Table 6, we demonstrate the group replacement issues faced by the electronic company. The graphical representation in Figure 7 was generated using MATLAB software, based on the reproducibility code referenced in Table 9 found in the Appendix.

6.3 *Explication*

Based on Figure 7 above, it is reasonable to adopt the group replacement policy after the third month of resistor installation. When comparing the individual cost, it is found to be more cost-effective than the group replacement cost of 5,607. Therefore, the unique hexagonal single-valued neutrosophic number can be utilized to assess the replacement item's reliability and uncertainty. The output is generated through MATLAB code, which saves time, and the time complexity of the MATLAB program is calculated to be 0.564 seconds. In the future, this can be expanded to include bipolar and hesitant neutrosophic numbers. Additionally, incorporating new number systems allows for the accurate representation of uncertain information in various applications and fields.

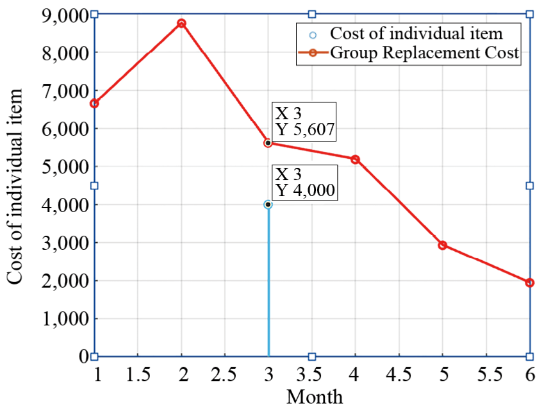

Figure 7. Graphical presentation of group and individual replacement cost or failure rates over time

7. Comparison analysis

A comparative assessment was carried out to compare the recommended approaches for dealing with the replacement policy problem. The study compared a single-valued neutrosophic number with a newly developed unique single-valued hexagonal neutrosophic number. After analyzing the data, it was determined that the newly developed single-valued hexagonal neutrosophic number offers a more precise estimation compared to the single-valued neutrosophic number. As a result, the score function value transforms the single-valued neutrosophic number into a precise one.

$$\tilde{S} = \frac{1}{3}\{(T - I - F + 2)\} \tag{9}$$

The above-derived formula (9) is used to convert SVUHNN into a clear numerical value. The replacement policy is determined based on the surviving probability shown in Table 7, which is a key factor in the comparison process. In our work, we compare our results with an established algorithm. The data in Table 8 is sourced from [15]. By analyzing the best possible outcome achieved in comparison to the work performed, we can determine the most effective replacement method. Figure 8 illustrates the comparison between SVNNs and single-valued unique hexagonal neutrosophic numbers, along with their respective survival probabilities.

Table 7. The surviving probability of each month

Month (x)	surviving probability $S(n)$
1	$< 0.06 >$
2	$< 0.17 >$
3	$< 0.31 >$
4	$< 0.48 >$
5	$< 0.68 >$
6	$< 1.00 >$

Table 8. Comparison of neutrosophic number

SVNNs	Precise number
$< (0.123, 0.81, 0.1) >$	$< 0.1027 >$
$< (0.33, 0.82, 0.7) >$	$< 0.2566 >$
$< (0.46, 0.74, 0.39) >$	$< 0.3311 >$
$< (0.5, 0.41, 0.12) >$	$< 0.5937 >$
$< (0.6, 0.2, 0.1) >$	$< 0.7330 >$
$< (0.8, 0.23, 1) >$	$< 0.84 >$

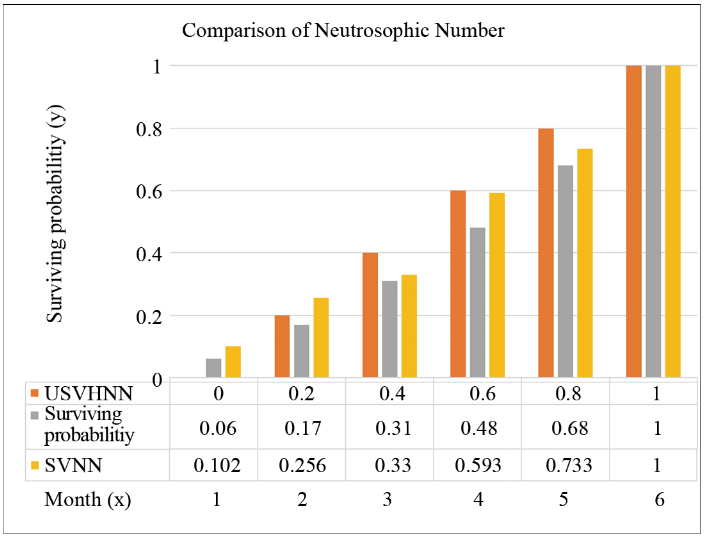

Figure 8. Comparison of neutrosophic numbers with other types of specialized neutrosophic numbers

Utilizing a score equation (9), the MATLAB code function converts a single-valued neutrosophic number into a crisp number. Comparisons with results in Tables 7 and 8 show that the proposed method offers superior approximation, demonstrating the robustness of the approach. While MATLAB code has been employed for replacement models in previous research, our developed score function analytically assesses the time required for optimal machine replacement, proving to be more efficient than other studies in this field.

7.1 *Real life applications*

The Replacement Problem (RP) is a crucial aspect of economic decision-making, as it determines the optimal replacement period for assets to prevent production disruptions. The replacement technique is versatile and can be applied in various domains, including infrastructure projects and equipment lifespan estimation, as shown in Figure 9. The deterioration of machinery and increasing operational costs emphasize the need for efficient replacement policies to maintain system performance. In operations research, the replacement theory concept is used to determine when to replace old equipment with new ones, especially when the existing equipment is no longer viable due to wear and tear or failure. It is also applicable in situations where the item has been damaged or destroyed.

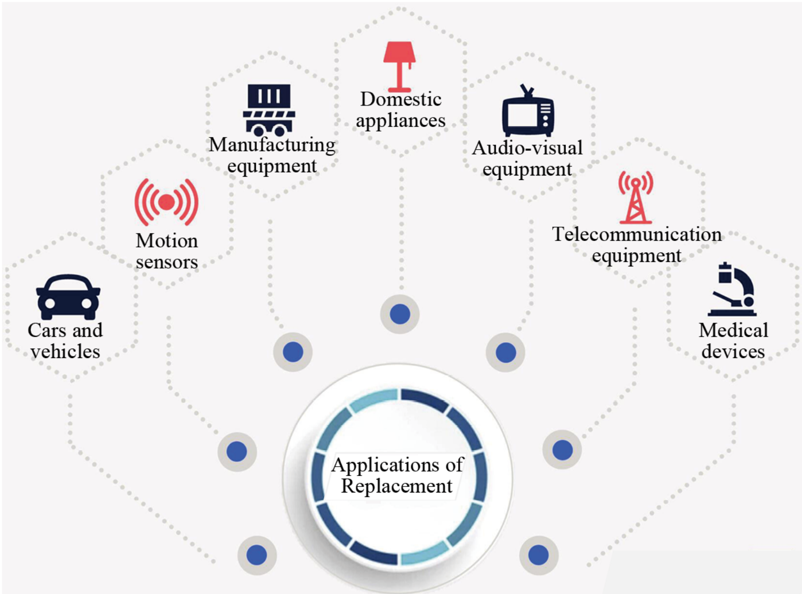

Figure 9. Real-life application usage of replacement management

8. Conclusion

This paper introduces the innovative concept of hexagonal neutrosophic numbers (SVUHNN) and demonstrates their potential in solving complex replacement policy problems, particularly in industrial settings. The study highlights the de-eutrophication process, which effectively converts hexagonal neutrosophic numbers into precise real values through the removal area method, ensuring enhanced clarity and usability in practical applications. By employing the newly developed score function $\tilde{S}$, the method achieves more accurate estimations than traditional single-valued neutrosophic numbers (SVNN). Through comparative analysis using established methods, this study underscores the advantages of the SVUHNN approach. The results, validated through MATLAB implementation, reveal significant improvements in computational efficiency, reduced complexity, and time savings. The proposed approach is particularly robust, as evidenced by its superior approximation capabilities, illustrated in tables and graphical representations of survival probabilities and replacement scenarios. Moreover, the integration of SVUHNN into decision-making processes empowers organizations to simulate various operational scenarios. Decision-makers can evaluate the effects of differing maintenance schedules, operational conditions, and resource constraints on equipment longevity. This deeper understanding facilitates the development of customized maintenance strategies that align with budgetary and operational demands, ultimately driving cost-effective and reliable system management. In conclusion, this work represents a significant advancement in the application of neutrosophic theories to industrial replacement problems, offering precise, computationally efficient, and practically applicable solutions.

8.1 *Drawbacks and limitations*

While this paper introduces the innovative concept of hexagonal neutrosophic numbers (SVUHNN) and demonstrates their effectiveness in addressing replacement policy problems, some limitations should be noted. The reliance on the de-eutrophication process, although effective, may lead to a loss of inherent neutrosophic uncertainty, potentially oversimplifying certain complex scenarios. Additionally, the approach's robustness and accuracy heavily depend on the defined score function and removal area method, which may not generalize well to all problem contexts. Another drawback is the computational dependency on MATLAB, which could limit accessibility for practitioners without access to proprietary software. Future studies could explore more generalized frameworks and extend the method's applicability to a broader range of decision-making environments.

8.2 *Future scope*

In the future, the research could expand to include inventory and replacement issues using bipolar neutrosophic numbers and bipolar soft sets, addressing the two-sided nature of problems and enhancing precision through inclination sequencing. This approach shows promise for tackling complex reliability issues, such as dual component failures, by evaluating component integrity and optimizing system availability considering downtime and uptime. Integrating single-valued hexagonal neutrosophic numbers with machine learning models can enhance decision-making by processing uncertain and incomplete data, improving predictions in interrelated failure scenarios. Further exploration may involve machine learning models for selective maintenance schedules and random mission intervals, providing adaptive strategies for optimizing reliability under uncertainty.

Data availability

The authors confirm that the data supporting the findings of this study are available in the article.

Conflict of interest

The authors declare no competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Riaz M, Hashmi MR. Linear diophantine fuzzy set and its applications towards multi-attribute decision-making problems. *Journal of Intelligent and Fuzzy Systems*. 2019; 37(4): 5417-5439.

[2] Saqlain M, Saeed M, Ahmad R, Smarandache F. Generalization of TOPSIS for neutrosophic hypersoft set using accuracy function and its application. *Neutrosophic Sets and Systems*. 2019; 27: 131-137.

[3] Palanikumar M, Broumi S. Square root Diophantine neutrosophic normal interval-valued sets and their aggregated operators in application to multiple attribute decision-making. *International Journal of Neutrosophic Science*. 2022; 19(3): 63-84.

[4] El Mokrini A, Aouam T. A decision-support tool for policy makers in healthcare supply chains to balance between perceived risk in logistics outsourcing and cost-efficiency. *Expert Systems with Applications*. 2022; 20(1): 116-129.

[5] Chakraborty A, Mondal SP, Alam S, Ahmadian A, Senu N, De D, et al. Disjunctive representation of triangular bipolar neutrosophic numbers de-bipolarization technique and application in multi-criteria decision-making problems. *Symmetry*. 2019; 11(7): 932.

[6] Wang J, Wang JQ, Ma YX. Possibility degree and power aggregation operators of singlevalued trapezoidal neutrosophic numbers and applications to multi-criteria group decision-making. *Cognitive Computation*. 2021; 13(6): 657-672.

[7] Zadeh LA. Fuzzy sets: Information and control. *The Journal of Symbolic Logic*. 1965; 38(4): 338-353.
[8] Zimmermann HJ. *Fuzzy Set Theory and Its Applications*. USA: Springer Science and Business; 1991.
[9] Nivatha G, Varadharajan R. A study on fuzzy replacement model using octagonal fuzzy numbers. *AIP Conference Proceedings*. 2019; 2112(1): 020099.
[10] Sundari S, Saranya V. A novel method to solve replacement problem under fuzzy environment. *AIP Conference Proceedings*. 2020; 2277(1): 090015.
[11] Chakraborty A, Mondal SP, Alam S, Ahmadian A, Senu N, De D, et al. Disjunctive representation of triangular bipolar neutrosophic numbers de-bipolarization technique and application in multi-criteria decision-making problems. *Symmetry*. 2019; 11(7): 932.
[12] Chakraborty A, Mondal S, Said B. De-eutrophication technique of pentagonal neutrosophic number and application in minimal spanning tree. *Neutrosophic Sets and Systems*. 2019; 29: 1-18. Available from: https://doi.org/10.5281/zenodo.3514383.
[13] Richard AS, Rajkumar A. De-neutrophication technique of single-valued linear heptagonal neutrosophic number. *Advances in Mathematics: Scientific Journal*. 2020; 2020(10): 7811-7818.
[14] Richard AS, Rajkumar A. A novel approach to solve the replacement problem using MATLAB programming in the neutrosophic environment. *AIP Conference Proceedings*. 2020; 2529(1): 020015.
[15] Richard AS, Praveena N, Rajkumar A. Special single-valued octagonal neutrosophic number and its applications using MATLAB programming. *Journal of Intelligent and Fuzzy Systems*. 2023; 45(1): 687-698.
[16] Khalifa HAEW, Kumar P. A novel method for neutrosophic assignment problem by using the interval-valued trapezoidal neutrosophic number. *Neutrosophic Sets and Systems*. 2020; 36(3): 1-14.
[17] Chakraborty A, Mondal SP, Alam S, Ahmadian A, Senu N, De D, et al. The pentagonal fuzzy number: Its different representations properties ranking defuzzification and application in game problems. *Symmetry*. 2019; 11(2): 248.
[18] Chakraborty A, Said B, Singh PK. Some properties of pentagonal neutrosophic numbers and its applications in transportation problem environment. *Neutrosophic Sets and Systems*. 2019; 28(16): 1-17.
[19] Chakraborty A. A new score function of pentagonal neutrosophic number and its application in networking problem. *International Journal of Neutrosophic Science*. 2020; 1(1): 40-51.
[20] Chakraborty A. Application of pentagonal neutrosophic number in shortest path problem. *International Journal of Neutrosophic Science*. 2020; 3(1): 21-28.
[21] Chakraborty A, Mondal SP, Alam S, Mahata A. Cylindrical neutrosophic single-valued number and its application in networking problem multi-criterion group decision-making problem and graph theory. *CAAI Transactions on Intelligence Technology*. 2020; 5(2): 68-77.
[22] Chakraborty A, Mondal SP, Alam S, Maity S, Jain S. Hexagonal fuzzy number and its distinctive representation ranking defuzzification technique and application in production inventory management problem. *Granular Computing*. 2021; 6: 507-521. Available from: https://doi.org/10.1007/s41066-020-00212-8.
[23] Broumi S, Nagarajan D, Bakali A, Talea M, Smarandache F, Lathamaheswari M, et al. Implementation of neutrosophic function memberships using MATLAB program. *Neutrosophic Sets and Systems*. 2019; 27(5): 43-52.
[24] Broumi S, Smarandache F. Single-valued neutrosophic trapezoid linguistic aggregation operators based multi-attribute decision making. *Bulletin of Pure and Applied Sciences-Mathematics and Statistics*. 2014; 33e(2): 135.
[25] Broumi S, Nagarajan D, Bakali A, Talea M, Smarandache F, Venkateswara RV. Single-valued neutrosophic techniques for analysis of WIFI connection. In: *Advanced Intelligent Systems for Sustainable Development*. USA: Springer International Publishing; 2019. p.405-412.
[26] Mullai M, Broumi S. Neutrosophic inventory model without shortages. *Asian Journal of Mathematics and Computer Research*. 2018; 23(4): 214-219.
[27] Haddad M, Sanders D. Selection of discrete multiple criteria decision-making methods in the presence of risk and uncertainty. *Operations Research Perspectives*. 2018; 5: 357-370. Available from: https://doi.org/10.1016/j.orp.2018.10.003.
[28] Lee CY, Lee D. An efficient method for solving a correlated multi-item inventory system. *Operations Research Perspectives*. 2018; 5: 13-21. Available from: https://doi.org/10.1016/j.orp.2017.11.002.
[29] Kamal M, Modibbo UM, AlArjani A, Ali I. Neutrosophic fuzzy goal programming approach in selective maintenance allocation of system reliability. *Complex and Intelligent Systems*. 2020; 7(2): 1045-1059.

[30] Tu A, Ye J, Wang B. Neutrosophic number optimization models and their application in the practical production process. *Journal of Mathematics*. 2021; 2021: 1-8. Available from: https://doi.org/10.1155/2021/6668711.
[31] Pai SP, Gaonkar RSP. Safety modeling of marine systems using neutrosophic logic. *Proceedings of the Institution of Mechanical Engineers Part M Journal of Engineering for the Maritime Environment*. 2021; 235(1): 225-235.
[32] Galante GM, La Fata CM, Lupo T, Passannanti G. Handling the epistemic uncertainty in the selective maintenance problem. *Computers and Industrial Engineering*. 2020; 141(4): 106293.
[33] Modibbo UM, Arshad M, Abdalghani O, Ali I. Optimization and estimation in system reliability allocation problem. *Reliability Engineering and System Safety*. 2021; 212(2): 107620.
[34] Karamaşa Ç, Demir E, Memiş S, Korucuk S. Weighting the factors affecting logistics outsourcing. *Decision Making Applications in Management and Engineering*. 2020; 4(1): 19-33.
[35] Anthvanet MLJ, Rajkumar A. Multi-criteria decision making in cricket using generalized dodecagonal intuitionistic fuzzy number. *AIP Conference Proceedings*. 2020; 2282(1): 020008.
[36] Abdel-Basset M, Mohamed R, Zaied AENH, Smarandache F. A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. *Symmetry*. 2019; 11(7): 903.
[37] Abdel-Basset M, Mohamed R, Zaied AENH, Smarandache F. A refined approach for forecasting based on neutrosophic time series. *Symmetry*. 2019; 11(4): 457.
[38] Abdel-Basset M, Nabeeh NA, El-Ghareeb HA, Abzuhelfen A. Utilizing neutrosophic theory to solve transition difficulties of iot-based enterprises. *Enterprise Information Systems*. 2020; 14(9-10): 1304-1324.
[39] Nabeeh NA, Abdel-Basset M, El-Ghareeb HA, Aboelfetouh A. Neutrosophic multi-criteria decision approaching IoT-based enterprises. *IEEE Access*. 2019; 7: 59559-59574. Available from: https://doi.org/10.1109/ACCESS.2019.2908919.
[40] Mahata A, Mondal SP, Alam S, Chakraborty A, De SK, Goswami A. Mathematical model for diabetes in a fuzzy environment with stability analysis. *Journal of Intelligent and Fuzzy Systems*. 2019; 36(3): 2923-2932.
[41] Ali J, Naeem M, Mahmood W. Generalized *q*-rung picture linguistic aggregation operators and their application in decision making. *Journal of Intelligent and Fuzzy Systems*. 2023; 44(3): 4419-4443.
[42] Ashraf S, Mahmood T, Abdullah S, Khan Q. Different approaches to multi-criteria group decision making problems for picture fuzzy environment. *Bulletin of the Brazilian Mathematical Society New Series*. 2019; 50(2): 373-397.
[43] Pai SP, Gaonkar RSP. Safety modeling of marine systems using neutrosophic logic. *Proceedings of the Institution of Mechanical Engineers Part M: Journal of Engineering for the Maritime Environment*. 2021; 235(1): 225-235.
[44] Kara K, Yalçın GC, Simic V, Önden İ, Edinsel S, Bacanin N. A single-valued neutrosophicbased methodology for selecting warehouse management software in sustainable logistics systems. *Engineering Applications of Artificial Intelligence*. 2024; 12(9): 107-126.
[45] Tehrim ST, Riaz M. A novel extension of TOPSIS to MCGDM with bipolar neutrosophic soft topology. *Journal of Intelligent and Fuzzy Systems*. 2019; 37(4): 5531-5549.
[46] Singh A, Bhat S. *A Commentary on "An Improved Score Function for Ranking Neutrosophic Sets and Its Application to Decision-Making Process"*. USA: Authorea Preprints; 2021.
[47] Roy SK, Pervin M, Weber GW. Imperfection with inspection policy and variable demand under trade-credit: A deteriorating inventory model. *Numerical Algebra Control and Optimization*. 2020; 10(1): 45-74.
[48] Mosca A, Vidyarthi N, Satir A. Integrated transportation-inventory models: A review. *Operations Research Perspectives*. 2019; 6: 100101. Available from: https://doi.org/10.1016/j.orp.2019.100101.
[49] Taleizadeh A, Yadegari M, Sana SS. Production models of multiple products using a single machine under quality screening and reworking policies. *Journal of Modelling in Management*. 2019; 14(1): 232-259.
[50] Hosseini A, Sahlin T. An optimization model for management of empty containers in the distribution network of a logistics company under uncertainty. *Journal of Industrial Engineering International*. 2019; 15(4): 585-602.
[51] Goçen MY, Çaglar O, Ercan E, Kizilay D. Optimization of costs in empty container repositioning. In: *Proceedings of the International Symposium for Production Research*. USA: Springer International Publishing; 2020. p.732-747.
[52] Jdid M, Alhabib R, Khalid HE, Salama AA. The neutrosophic treatment of the static model for inventory management with safety reserve. *International Journal of Neutrosophic Science*. 2022; 18(2): 262-271.
[53] Jdid M, Salama AA, Alhabib R, Khalid HE, Al Suleiman F. Neutrosophic treatment of the static model of inventory management with deficit. *International Journal of Neutrosophic Science*. 2022; 18(1): 20-29.
[54] Jdid M, Alhabib R, Salama AA. The static model of inventory management without a deficit with Neutrosophic logic. *International Journal of Neutrosophic Science*. 2021; 16(1): 42-48.

[55] Priyadharshini S, Deepa G. Time cost trade-off problem using intuitionistic fuzzy with real-time application in the field of construction. *Contemporary Mathematics*. 2024; 5(3): 2961-2982.
[56] Palanivel M, Vetriselvi S. Optimization of a two-warehouse eoq model for non-instantaneous deteriorating items with polynomial demand advance payment and shortages. *Contemporary Mathematics*. 2024; 5(3): 2770-2781.

Appendix

Table 9. Reproducibility MATLAB code

MATLAB Code II
function [k K Z U Iv Gv] = group replacement(Mc Q11 Q22 Q33 Q44 Q55 Q66 iv gv)
q11 = Q11 / 100; K11 = Q11 / 100; Z11 = Mc * K11;
q22 = Q22 / 100; K22 = (Q22 - Q11) / 100; Z22 = (Mc * K22) + (Z11 * K11);
q33 = Q33 / 100; K33 = (Q33 - Q22) / 100; Z33 = (Mc * K33) + (Z22 * K11) + (Z11 * K22);
q44 = Q44 / 100; K44 = (Q44 - Q33) / 100; Z44 = (Mc * K44) + (Z33 * K22) + (Z22 * K33) + (Z11 * K44);
q55 = Q55 / 100; K55 = (Q55 - Q44) / 100; Z55 = (Mc * q55) + (Z44 * K33) + (Z33 * K44) + (Z22 * K55) + (Z11 * K55);
q66 = Q66 / 100; K66 = (Q66 - Q55) / 100; Z66 = (Mc * q66) + (Z55 * K44) + (Z44 * K55) + (Z33 * K66) + (Z22 * K66) + (Z11 * K66);
k = [q11 q22 q33 q44 q55 q66];
K = [K11 K22 K33 K44 K55 K66];
Z = [Z11 Z22 Z33 Z44 Z55 Z66];
Z111 = Z11 + 0;
Z222 = Z22 + Z111;
Z333 = Z33 + Z222;
Z444 = Z44 + Z333;
Z555 = Z55 + Z444;
Z666 = Z66 + Z555;
Z = [Z111 Z222 Z333Z444 Z555 Z666];
Disp (‘INDIVIDUAL VALUE (iv = 7)’);
i11 = Z111 * iv;
i22 = Z222 * iv;
i33 = Z333 * iv;
i44 = Z444 * iv;
i55 = Z555 * iv;
i66 = Z666 * iv;
I = [i11 i22 i33 i44 i55 i66];
Disp (‘GROUP REPLACEMENT VALUE (gv = 4)’);
Disp (‘k K Z U Iv Gv’);
G = Mc * gv;
hold on;
X = [1 2 3 4 5 6];
Y = [i11 i22 i33 i44 i55 I66];
Scatter(x y ‘o’);
Line(x y);
Plot(x y ‘-o’ ‘linewidth’ 3);
hold on;
X = n ; Y = Gv;
Stem (X Y);
hold off;
Xlabel (‘Month’);
Ylabel (‘Cost of Individual Item’);
Legend (‘GROUP REPLACEMENT COST’ ‘Cost of Individual Item’);
end