Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Geometric Properties of Generalized Janowski Type Functions

Muflih Alhazmi[1], Al-Rezami A. Y[2,3*], Erhan D[4], Abdulbasit A. Darem[5], Fuad Alsarari[6]

[1] Mathematics Department, Faculty of Science, Northern Border University, Arar, KSA, Saudi Arabia
[2] Mathematics Department, Prince Sattam Bin Abdulaziz University, Al-Kharj, 16278, Saudi Arabia
[3] Department of Statistics and Information, Sanaa University, Sanaa, 1247, Yemen
[4] Department of Mathematics, Faculty of Science and Letters, Kafkas University, Campus, 36100, Kars, Türkiye
[5] Center for Scientific Research and Entrepreneurship, Northern Border University, Arar, 73213, Saudi Arabia
[6] Department of Mathematics and statistics, College of Sciences, Yanbu, Taibah University, Saudi Arabia

E-mail: a.alrezamee@psau.edu.sa

Received: 18 September 2024; **Revised:** 4 November 2024; **Accepted:** 9 December 2024

Abstract: The present paper aims to explore subclasses of analytic functions that extend the class of Janowski-type functions. We examine specific convolution conditions that ensure functions h fall within the generalized Janowski-type classes of starlike and convex functions. These conditions will serve as foundational results for further analysis in this study. Additionally, we provide sufficient conditions and neighborhood results pertinent to functions within the starlike class. Finally, we investigate the invariance properties of the operator $h_\mu(\vartheta) = (1-\mu)\vartheta + \mu h(\vartheta)$, $0 < \mu < 1$, as it applies to functions in this classification.

***Keywords*:** subordination, convolution conditions, starlike functions, convex functions, Janowski type functions

MSC: 30C45

1. Introduction

Geometric function theory, a branch of complex analysis, delves into the properties of analytic functions in the complex plane, with a focus on Janowski functions. These functions, named after the renowned mathematician Stanislaw Janowski, have a significant theorem that characterizes their growth pattern based on their coefficients. This theorem is an invaluable tool, offering insights into the behavior of analytic functions and finding extensive applications across various mathematical domains, including complex analysis and potential theory, and even extending into mathematical physics see [1, 2].

Let $\mathcal{W} = \{\vartheta \in \mathbb{C} : |\vartheta| < 1\}$ represent the open unit disk, and $\mathcal{M}$ be the class of functions has a Taylor-Maclaurin's series representation $h(\vartheta) = \vartheta + \sum_{i=2}^{\infty} a_i \vartheta^i$. Suppose that $\mathcal{S}$ the class of functions in $\mathcal{M}$ and univalent in $\mathcal{W}$. Let $h, g \in \mathcal{M}$ be two analytic functions, where h is defined as above and $g(\vartheta) = \vartheta + \sum_{i=2}^{\infty} b_i \vartheta^i$. The convolution (or Hadamard product) of h and g is given by:

DOI: https://doi.org/10.37256/cm.6520255736

$$(h*g)(\vartheta)=\vartheta+\sum_{i=2}^{\infty}a_ib_i\vartheta^i.$$

Denote by $\mathcal{S}$ the specific subset of $\mathcal{M}$ that includes all functions which are univalent defined within the region $\mathcal{W}$. Given two functions h and g that belong to $\mathcal{M}$, we use the term *subordinate* to describe the relationship $h \prec g$ within the domain $\mathcal{W}$ when there exists a function u characterized as a *Schwarz function*, belonging to the set Φ defined as follows:

$$\Phi=\{u \mid u\in\mathcal{M},\, u(0)=0,\, |u(\vartheta)|<1 \quad \text{for all} \;\; \vartheta\in\mathcal{W}\}, \tag{1}$$

such that $h(\vartheta)=g(u(\vartheta))$, $\vartheta\in\mathcal{W}$. If g is univalent in $\mathcal{W}$, then

$$h(\vartheta)\prec g(\vartheta)\Leftrightarrow h(0)=g(0) \;\text{ and }\; h(\mathcal{W})\subset g(\mathcal{W}),\, \vartheta\in\mathcal{W}.$$

Using the above principle of the subordination we define the well-known Carathĕodory class $\mathcal{P}$ of functions p which are analytic in $\mathcal{W}$, which are normalized by

$$p(\vartheta)=1+\sum_{j=1}^{\infty}c_j\vartheta^j \;\text{ and }\; \Re\{p(\vartheta)\}>0,\;\; \vartheta\in\mathcal{W}.$$

For $-1\le H<T\le 1$. Janowski in [3] introduced the class $\mathcal{P}[H,\,T]$ of Janowski functions which are a specific category of holomorphic functions within geometric function theory. They play a prominent role due to their unique mapping properties. Janowski functions map the unit disk $\mathcal{W}$ in the complex plane onto regions that always contain the right half-plane.

In [4, 5] Polatoglu et al. defined and studied a generalized class $\mathcal{P}[H,\,T,\,F]$ of Janowski functions. A function p in $\mathcal{P}$ is said to be in class $\mathcal{P}[H,\,T,\,F]$ if it satisfies the condition

$$p(\vartheta)\prec\frac{1+[(1-F)T+FH]\vartheta}{1+H\vartheta}\Leftrightarrow p(\vartheta)=\frac{1+[(1-F)T+FH]u(\vartheta)}{1+Hu(\vartheta)},\quad u\in\Phi,$$

where $0\le F<1$. This topic has recently attracted considerable interest from a variety of researchers, see [6–8].

Definition 1.1 A function $f\in\mathcal{M}$ is said to belongs to the class $\mathcal{S}^*(T,\,H,\,F)$, with $-1\le H<T\le 1$ and $0\le F<1$, if

$$\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\prec\frac{1+[(1-F)T+FH]\vartheta}{1+H\vartheta}. \tag{2}$$

Remark 1.2 We can easily obtain the equivalent condition for a function h belonging to the class $\mathcal{S}^*(T,\,H,\,F)$ by applying the principle of subordination. The equivalent condition

$$\left|\frac{\vartheta h'(\vartheta)}{h(\vartheta)}-1\right|<\left|[(1-F)T+FH]-H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right|,\ \ \vartheta\in\mathcal{W}. \tag{3}$$

A functions h is said to belong to the class $\mathcal{KS}^*(T, H, F)$ if and only if

$$\vartheta h'(\vartheta)\in\mathcal{S}^*(T, H, F). \tag{4}$$

For the purposes of our current work, it is necessary to recall the following neighborhood concept, which was originally introduced by Goodman [9] and subsequently generalized by Ruscheweyh [10].

Definition 1.3 The ρ-neighborhood of a function h within the set $\mathcal{M}$ can be characterized as follows:

$$\mathcal{N}_\rho(h)=\left\{g\in\mathcal{M}:g(\vartheta)=\vartheta+\sum_{i=2}^{\infty}b_i\vartheta^i,\ \ \sum_{i=2}^{\infty}i|a_i-b_i|\le\rho\right\}. \tag{5}$$

For $e(\vartheta)=\vartheta$. It is observable that

$$\mathcal{N}_\rho(e)=\left\{g\in\mathcal{M}:g(\vartheta)=\vartheta+\sum_{i=2}^{\infty}b_i\vartheta^i,\ \ \sum_{i=2}^{\infty}i|b_i|\le\rho\right\}. \tag{6}$$

Ruscheweyh [10] demonstrated, along with other findings, that for any complex number $\eta\in\mathbb{C}$, with $|\mu|<\rho$,

$$\frac{h(\vartheta)+\eta\vartheta}{1+\eta}\in\mathcal{S}^*\Rightarrow\mathcal{N}_\rho(h)\subset\mathcal{S}^*,$$

where $\mathcal{S}^*$ is the class of starlike functions.

Lemma 1.4 [4, Corollary 1] Any function $f\in\mathcal{S}^*(T, H, F)$ can be written in the form

$$h(\vartheta)=\begin{cases}\vartheta\left(1+Hu(\vartheta)\right)^{\frac{(1-F)(T-H)}{H}}, & \text{if}\quad H\neq 0,\\ \vartheta\exp\left[(1-F)Tu(\vartheta)\right], & \text{if}\quad H=0,\end{cases} \tag{7}$$

for some $u\in\Phi$, where Φ defined by (1).

This paper derives a convolution condition for functions h to belong to the classes $\mathcal{S}^*(T, H, F)$ and $\mathcal{KS}^*(T, H, F)$, a key supporting result that will facilitate further research in our work. Additionally, we establish a sufficient condition and neighborhood results for functions in the class $\mathcal{S}^*(T, H, F)$. Our investigation culminates in an examination of the invariance properties of the operator $h_\mu(\vartheta)=(1-\mu)\vartheta+\mu h(\vartheta)$, specifically when applied to functions within the class $\mathcal{S}^*(T, H, F)$.

2. Main results

Theorem 2.1 The function $h \in \mathcal{KS}^*(T, H, F)$ in $|\vartheta| < \mathcal{R} \leq 1$ if and only if

$$\frac{1}{\vartheta}\left[h * \frac{\vartheta + \dfrac{[2+\zeta F(H-T)+\zeta(T+H)]\,\vartheta^2}{(H-T)(1-F)\zeta}}{(1-\vartheta)^3}\right] \neq 0, \quad |\vartheta| < \mathcal{R}, \ |\zeta| = 1. \tag{8}$$

Proof. The function $h \in \mathcal{KS}^*(T, H, F)$ if and only if

$$\frac{(\vartheta h'(\vartheta))'}{h'(\vartheta)} \neq \frac{1+[(1-F)T+FH]\zeta}{1+H\zeta} \tag{9}$$

for $(|\vartheta| < \mathcal{R}, |\zeta| = 1, H\zeta \neq -1)$, which implies

$$\left(\vartheta h'(\vartheta)\right)'(1+H\zeta) - (1+[(1-F)T+FH]\zeta)h'(\vartheta)) \neq 0. \tag{10}$$

Setting $h(\vartheta) = \vartheta + \sum_{i=2}^{\infty} a_i \vartheta^i$, we have

$$\vartheta h' = \vartheta + \sum_{i=2}^{\infty} i a_i \vartheta^i$$

and

$$\left(\vartheta h'\right)' = 1 + \sum_{i=2}^{\infty} i^2 a_i \vartheta^{i-1} = h' * \frac{1}{(1-\vartheta)^2}.$$

Therefore (10) is equivalent to

$$(1+H\zeta)\left[h' * \sum_{i=1}^{\infty} i a_i \vartheta^{i-1}\right] - h' * \sum_{i=1}^{\infty}(1+[(1-F)T+FH]\zeta)\vartheta^{i-1}$$

$$= h' * \left\{\sum_{i=1}^{\infty}(1+H\zeta) i \vartheta^{i-1} - \sum_{i=1}^{\infty}[1+[(1-F)T+FH]x]\,\vartheta^{i-1}\right\}$$

$$= h' * \left(\frac{(1+H\zeta)}{(1-\vartheta)^2} - \frac{1+[(1-F)T+FH]\zeta}{1-\vartheta}\right)$$

$$= h' * \left(\frac{(1+H\zeta)-(1-\vartheta)(1+[(1-F)T+FH]\zeta)}{(1-\vartheta)^2}\right).$$

Thus

$$\frac{1}{\vartheta}\left[\vartheta h' * \frac{\vartheta + \dfrac{(1+[(1-F)T+FH]\zeta)}{(H-T)(1-F)\zeta}\vartheta^2}{(1-\vartheta)^2}\right] \neq 0. \tag{11}$$

Since $\vartheta h' * g = h * \vartheta g'$, we can write (11) as

$$\frac{1}{\vartheta}\left[h * \frac{\vartheta + \dfrac{[2+\zeta F(H-T)+\zeta(T+H)]\,\vartheta^2}{(H-T)(1-F)\zeta}}{(1-\vartheta)^3}\right] \neq 0, \quad |\vartheta| < \mathcal{R}, \ |\zeta| = 1. \tag{12}$$

which completes the proof. □

For $F = 0$, we have following result proved by Ganesan and Padmanabhan in [11].

Corollary 2.2 The function $h \in \mathcal{K}(T, H)$ in $|\vartheta| < \mathcal{R} \leq 1$ if and only if

$$\frac{1}{\vartheta}\left[h * \frac{\zeta\vartheta + \dfrac{(T\zeta+Bx+2)}{H-T}\vartheta^2}{(1-\vartheta)^3}\right] \neq 0. \tag{13}$$

Remark 2.3 Given that $F = 0$, $T = 1$, and $H = -1$, we obtain a convolution condition that characterizes convex functions, as modified suitably from the framework presented by Silverman and colleagues in [12].

Theorem 2.4 The function $h \in \mathcal{S}^*(T, H, F)$ in $|\vartheta| < \mathcal{R} \leq 1$ if and only if

$$\frac{1}{\vartheta}\left[h * \frac{\vartheta + \dfrac{(1+[(1-F)T+FH]\zeta)}{(H-T)(1-F)\zeta}\vartheta^2}{(1-\vartheta)^2}\right] \neq 0, \quad (|\vartheta| < \mathcal{R}, \ |\zeta| = 1). \tag{14}$$

Proof. Since $h \in \mathcal{S}^*(T, H, F)$ if and only if $g(\vartheta) = \int_0^{\vartheta} \dfrac{h(\omega)}{\omega} d\omega \in \mathcal{KS}^*(T, H, F)$, we have

$$\frac{1}{\vartheta}\left[g * \frac{\vartheta + \dfrac{[2+\zeta F(H-T)+\zeta(T+H)]\,\vartheta^2}{(H-T)(1-F)\zeta}}{(1-\vartheta)^3}\right] = \frac{1}{\vartheta}\left[h * \frac{\vartheta + \dfrac{(1+[(1-F)T+FH]\zeta)}{(H-T)(1-F)\zeta}\vartheta^2}{(1-\vartheta)^2}\right]. \tag{15}$$

Thus the result follows from Theorem 2.1. □

For $F = 0$, we have following result proved by Ganesan and Padmanabhan in [11, 13].

As a corollaries we can derive coefficient inequalities and the equivalent condition for a function h belonging to the class $\mathcal{S}^*(T, H, F)$.

Corollary 2.5 A function $h \in \mathcal{M}$ is in the class $\mathcal{S}^*(T, H, F)$ if and only if

$$h(\vartheta) = 1 + \sum_{i=2}^{\infty} A_i \vartheta^{i-1} \neq 0, \tag{16}$$

where $A_i = \dfrac{(i-1) + [H(i-F) - T(1-F)]\,\zeta}{(H-T)(1-F)\zeta} a_i$.

Proof. A function $h \in \mathcal{S}^*(T, H, F)$ if and only if

$$\frac{\vartheta h'(\vartheta)}{h(\vartheta)} \neq \frac{1 + [(1-F)T + FH]\zeta}{1 + H\zeta}.$$

Thus we have

$$(1 + H\zeta)(\vartheta h'(\vartheta)) - (1 + [(1-F)T + FH]\zeta)h(\vartheta) \neq 0$$

which implies

$$(H-T)(1-F)\zeta v + \sum_{i=2}^{\infty} [i(1 + H\zeta) - (1 + [(1-F)T + FH]\zeta)]\, a_i \vartheta^i \neq 0.$$

This simplifies into

$$1 + \sum_{i=2}^{\infty} \frac{(i-1) + [H(i-F) - T(1-F)]\,\zeta}{(H-T)(1-F)\zeta} a_i \vartheta^{i-1} \neq 0, \tag{17}$$

which completes the proof. □

Remark 2.6 For $F = 0$ and $T = 1$, $H = -1$, we get convolution condition characterizing starlike functions as in Silverman et al. [12] with a suitable modification.

Corollary 2.7 A function h belonging to the class $\mathcal{S}^*(T, H, F)$ if and only if

$$\frac{(h * g)(\vartheta)}{\vartheta} \neq 0, \quad g \in \mathcal{M},\ \vartheta \in \mathcal{W}, \tag{18}$$

where $g(\vartheta)$ has the form

$$g(\vartheta) = \vartheta + \sum_{i=2}^{\infty} t_i \vartheta^i,$$

$$t_i = \frac{i - 1 + (iH - [(1-F)T + FH])\zeta}{(H-T)(1-F)\zeta}. \tag{19}$$

Theorem 2.8 Let $h(\vartheta)=\vartheta+\sum_{i=2}^{\infty}a_i\vartheta^i$ be analytic in $\mathcal{W}$. If

$$\sum_{i=2}^{\infty}\{(i-1)+|[(1-F)T+FH]-iH|\}|a_i|\leq(T-H)(1-F), \tag{20}$$

for $-1\leq H<T\leq 1$ and $0\leq F<1$, then $h(\vartheta)\in\mathcal{S}^*(T, H, F)$.

Proof. To prove Theorem 2.8, it suffices to show that

$$\frac{(h*g)(\vartheta)}{\vartheta}\neq 0,$$

where g is given by (19).

Let $h(\vartheta)=\vartheta+\sum_{i=2}^{\infty}a_i\vartheta^i$ and $g(\vartheta)=\vartheta+\sum_{i=2}^{\infty}t_i\vartheta^i$. The convolution

$$\frac{(h*g)(\vartheta)}{\vartheta}=1+\sum_{i=2}^{\infty}t_ia_i\vartheta^{i-1},\ \vartheta\in\mathcal{W}.$$

It is known from Theorem 2.4 that $h(\vartheta)\in\mathcal{S}(T, H, F)$ if and only if $\dfrac{(h*g)(\vartheta)}{\vartheta}\neq 0$, for g given by (19). Using (19), we get

$$\left|\frac{(h*g)(\vartheta)}{\vartheta}\right|\geq 1-\sum_{i=2}^{\infty}\frac{i-1+|iH-[(1-F)T+FH]|}{|(H-T)(1-F)|}|a_i||\vartheta|^{i-1}>0,\ \ \vartheta\in\mathcal{W}.$$

Thus, $h(\vartheta)\in\mathcal{S}^*(T, H, F)$. □

On the order to find neighborhood results for the class $h\in\mathcal{S}^*(T, H, F)$ analogous to those obtained by Ruscheweyh [10], we defined the following concept of neighborhood.

Definition 2.9 For $-1\leq H<T\leq 1$, $0\leq F<1$ and $\rho\geq 0$ we define $\mathcal{N}(T, H, F; h, \rho)$ the neighborhood of a function $h\in\mathcal{M}$ as

$$\mathcal{N}(T, H, F; h, \rho)$$

$$=\left\{g\in\mathcal{M}: g(\vartheta)=\vartheta+\sum_{i=2}^{\infty}b_i\vartheta^i,\ d(h, g)=\sum_{i=2}^{\infty}\frac{(i-1)+|[(1-F)T+FH]-iH|}{(1-F)(T-H)}|b_i-a_i|\leq\rho\right\}, \tag{21}$$

where $h(\vartheta)=\vartheta+\sum_{i=2}^{\infty}a_i\vartheta^i$.

Remark 2.10 For parametric values $T=-H=1$, and $F=0$ in (21) reduces to (5).

Theorem 2.11 Let $h\in\mathcal{M}$, and for all complex number η, with $|\eta|<\rho$, if

$$\frac{h(\vartheta)+\eta\vartheta}{1+\eta}\in\mathcal{S}^*(T, H, F). \tag{22}$$

Then

$$\mathcal{N}(T, H, F; h, \rho) \subset \mathcal{S}^*(T, H, F). \tag{23}$$

Proof. We assume that $y \in \mathcal{N}(T, H, F; h, \rho)$, where y is defined by $y(\vartheta) = \vartheta + \sum_{n=i}^{\infty} b_i \vartheta^i$.
To prove the theorem, we only need to prove that $y \in \mathcal{S}^*(T, H, F)$. We would prove this claim in next three steps.
Step I Note that Theorem 2.4 is equivalent to

$$h \in \mathcal{S}^*(T, H, F) \Leftrightarrow \frac{1}{\vartheta}[(h * g)(\vartheta)] \neq 0, \quad \vartheta \in \mathcal{W}, \tag{24}$$

where is given by (19). For $|x| = 1$, we can write $g(\vartheta) = \vartheta + \sum_{n=i}^{\infty} t_i \vartheta^i$, where

$$t_i = \frac{(i-1) + |[(1-F)T + FH] - iH|x}{(1-F)(H-T)x}. \tag{25}$$

Step II We obtain that (22) is equivalent to

$$\left| \frac{h(\vartheta) * g(\vartheta)}{\vartheta} \right| \geq \rho, \tag{26}$$

because, if $h(\vartheta) = \vartheta + \sum_{n=i}^{\infty} a_i \vartheta^i \in \mathcal{W}$ and satisfy (22), then (24) is equivalent to

$$g \in \mathcal{S}^*(T, H, F, \sigma) \Leftrightarrow \frac{1}{\vartheta} \left[\frac{h(\vartheta) * g(\vartheta)}{1 + \eta} \right] \neq 0, \quad |\eta| < \rho.$$

Step III Letting $y(\vartheta) = \vartheta + \sum_{n=i}^{\infty} b_i \vartheta^i$ we notice that

$$\begin{aligned}
\left| \frac{y(\vartheta) * g(\vartheta)}{\vartheta} \right| &= \left| \frac{h(\vartheta) * g(\vartheta)}{\vartheta} + \frac{(y(\vartheta) - h(\vartheta)) * g(\vartheta)}{\vartheta} \right| \\
&\geq \rho - \left| \frac{(y(\vartheta) - h(\vartheta)) * g(\vartheta)}{\vartheta} \right| \quad \text{(by using (26))} \\
&= \rho - \left| \sum_{i=2}^{\infty} (b_i - a_i) t_i v^i \right| \\
&\geq \rho - |\vartheta| \sum_{i=2}^{\infty} \left[\frac{(i-1) + |[(1-F)T + HF] - iH|}{|(1-F)(H-T)|} \right] |b_i - a_i| \\
&\geq \rho - \rho|\vartheta| > 0,
\end{aligned}$$

by applying (25). Thus

$$\frac{g(\vartheta)*g(\vartheta)}{\vartheta}\neq 0, \quad \vartheta\in\mathcal{W}.$$

By (24), it yields that $y\in\mathcal{S}^*(T,H,F)$. This completes the proof. □

When $T=-H=1$ and $F=0$ in the above theorem we get (6) proved by Ruscheweyh in [10].

Theorem 2.12 Let $h\in\mathcal{S}^*(T,H,F)$, for $\rho<c$. Then

$$\mathcal{N}(T,H,F;h,\rho)\subset\mathcal{S}^*(T,H,F),$$

where c is a non-zero real number with $c\leq\left|\frac{(h*g)(\vartheta)}{\vartheta}\right|$, $\vartheta\in\mathcal{W}$ and g is defined in Remark 2.7.

Proof. Let $y=\vartheta+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{N}\ (A,B,\alpha;f,\rho)$.

In the order to prove Theorem 2.12, it suffices to show that

$$\frac{(y*g)(\vartheta)}{\vartheta}\neq 0,$$

where g is given by (19). Consider

$$\left|\frac{y(\vartheta)*g(\vartheta)}{\vartheta}\right|\geq\left|\frac{h(\vartheta)*g(\vartheta)}{\vartheta}\right|-\left|\frac{(y(\vartheta)-h(\vartheta))*g(\vartheta)}{\vartheta}\right|. \tag{27}$$

Since $h\in\mathcal{S}^*(T,H,F)$, applying the Theorem 2.8 we obtain

$$\left|\frac{(h*g)(\vartheta)}{\vartheta}\right|\geq c, \tag{28}$$

where c is a non-zero real number and $\vartheta\in\mathcal{W}$. Now

$$\left|\frac{(y(\vartheta)-h(\vartheta))*g(\vartheta)}{\vartheta}\right|=\left|\sum_{i=2}^{\infty}(b_i-a_i)t_i\vartheta^i\right|$$

$$\leq\sum_{i=2}^{\infty}\left[\frac{(i-1)+|[(1-F)T+HF]-iH|}{|(1-F)(H-T)|}\right]|b_i-a_i|=\rho, \tag{29}$$

using (28) and (29) in (27), we obtain

$$\left|\frac{y(\vartheta)*g(\vartheta)}{\vartheta}\right|\geq c-\rho>0,$$

where $\rho < c$. This completes the proof. $\square$

Theorem 2.13 Let $h \in \mathcal{S}^*(T, H, F)$ and $h_\mu(\vartheta) := (1-\mu)\vartheta + \mu h(\vartheta)$, with $0 < \mu < 1$. Then,

(i) $h_\mu \in \mathcal{S}^*(T, 0, F)$, if $H = 0$;

(ii) $h_\mu \in \mathcal{S}^*(T, H, F)$, for $|\vartheta| < \frac{1}{H}\sin\left(\frac{H}{(1-F)T+FH}\frac{\pi}{2}\right)$, if $H > 0$;

(iii) $h_\mu \in \mathcal{S}^*(T, H, F)$, for $|\vartheta| < \frac{1}{H}\sin\left(\frac{H}{2H-[(1-F)T+HF]}\frac{\pi}{2}\right)$, if $H < 0$.

Proof. For $h \in \mathcal{S}^*(T, H, F)$, then

$$\frac{\vartheta h'(\vartheta)}{h(\vartheta)} \prec \frac{1+[(1-F)T+HF]\vartheta}{1+H\vartheta}, \quad \text{with} \quad \vartheta h'_\mu(\vartheta) = (1-\omega)\vartheta + \mu\vartheta h'(\vartheta).$$

Hence

$$\frac{\vartheta h'_\mu(\vartheta)}{h_\mu(\vartheta)} = \frac{(1-F)\dfrac{\vartheta}{h(\vartheta)} + \mu\dfrac{\vartheta h'(\vartheta)}{h(\vartheta)}}{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu}. \tag{30}$$

Now we will discuss two cases for H as following:

Case I For $H = 0$ it is sufficient to show that

$$\left|\frac{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu\dfrac{\vartheta h'(\vartheta)}{h(\vartheta)}}{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu} - 1\right| < (1-F)T, \ \vartheta \in \mathcal{W}. \tag{31}$$

From $h \in \mathcal{S}^*(T, H, F)$ we have

$$\frac{\vartheta h'(\vartheta)}{h(\vartheta)} \prec 1 + (1-F)T\vartheta$$

which implies

$$\left|\frac{\vartheta h'(\vartheta)}{h(\vartheta)} - 1\right| < (1-F)T, \ \vartheta \in \mathcal{W},$$

and according to (4)

$$\frac{h(\vartheta)}{\vartheta} \prec \exp[(1-F)T\vartheta],$$

that equivalent to

$$\frac{h(\vartheta)}{\vartheta} = \exp[(1-F)Ts(\vartheta)],$$

for some $s \in \Phi$ and $\vartheta \in \mathcal{W}$. Thus,

$$\left| \frac{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu \dfrac{\vartheta h'(\vartheta)}{h(\vartheta)}}{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu} - 1 \right| = \mu \left| \frac{\dfrac{\vartheta h'(\vartheta)}{h(\vartheta)} - 1}{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu} \right| < \frac{(1-F)T\mu}{|(1-\mu)\exp[-(1-F)Ts(\vartheta)] + \mu|}.$$

Since $|s(\vartheta)| < 1$ for all $\vartheta \in \mathcal{W}$, we easily get that

$$|(1-\mu)\exp[-(1-F)Ts(\vartheta)] + \mu| > \mu, \ \ \vartheta \in \mathcal{W}. \tag{32}$$

Using (32), it follows

$$\left| \frac{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu \dfrac{\vartheta h'(\vartheta)}{h(\vartheta)}}{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu} - 1 \right| < (1-F)T, \ \ \vartheta \in \mathcal{W},$$

and consequently, from (30) we obtain

$$\frac{\vartheta h_\mu'(\vartheta)}{h_\mu(\vartheta)} \prec 1 + (1-F)T\vartheta,$$

that is, $h_\mu \in \mathcal{S}^*(T, 0, F)$.

Case II For $H \neq 0$ we need to determine the value $0 < r < 1$, such that

$$\left| \frac{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu \dfrac{\vartheta h'(\vartheta)}{h_(\vartheta)}}{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu} - 1 \right| < \left| [(1-F)T + FH] - H \frac{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu \dfrac{\vartheta h'(\vartheta)}{h(\vartheta)}}{(1-\mu)\dfrac{\vartheta}{h(\vartheta)} + \mu} \right|, \tag{33}$$

whenever $|\vartheta| < r$, which is equivalent to

$$\left| \frac{\vartheta h'(\vartheta)}{h(\vartheta)} - 1 \right| < \left| \left([(1-F)T + HF] - H \right) \left(\frac{1}{\mu} - 1 \right) \frac{\vartheta}{h(\vartheta)} + [(1-F)T + FH] - H \frac{\vartheta h'(\vartheta)}{h(\vartheta)} \right|, \tag{34}$$

By using the definition of subordination and the Remak 1.2 for $|\vartheta| < r$, we get

$$h \in \mathcal{S}^*(T, H, F), \text{ for } |\vartheta| < r,$$

or

$$\left|\frac{\vartheta h'(\vartheta)}{h(\vartheta)} - 1\right| < \left|[(1-F)T + FH] - H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right|, \ |\vartheta| < r. \tag{35}$$

Next, we will prove that

$$\left|\arg\left(\frac{\vartheta}{h(\vartheta)}\right) - \arg\left([(1-F)T + FH] - H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right)\right| < \frac{\pi}{2}, \ \vartheta \in \mathcal{W}, \tag{36}$$

implies

$$\begin{aligned} &\left|[(1-F)T + FH] - H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right| \\ &< \left|\left([(1-F)T + FH] - H\right)\left(\frac{1}{\mu} - 1\right)\frac{\vartheta}{h(\vartheta)} + [(1-F)T + FH] - H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right|, \ \vartheta \in \mathcal{W}. \end{aligned} \tag{37}$$

Since $h \in \mathcal{S}^*(T, H, F)$, from Lemma 1.4 and the Definition 1.1, it follows that there exist *a Schwarz functions* s, $\widetilde{s} \in \Phi$, such that

$$[(1-F)T + FH] - H\frac{\vartheta h'(\vartheta)}{h(\vartheta)} = \frac{(T-H)(1-F)}{1 + Hs(\vartheta)} \quad \text{and} \quad \frac{\vartheta}{h(\vartheta)} = (1 + H\widetilde{s}(\vartheta))^{-\frac{(1-F)(T-H)}{H}}. \tag{38}$$

Using (38), the assumption (36) is equivalent to

$$\left|\arg\frac{(1 + H\widetilde{s}(\vartheta))^{-\frac{(1-F)(T-H)}{H}}(1 + Hs(\vartheta))}{(1-F)(T-H)}\right| < \frac{\pi}{2}, \ \vartheta \in \mathcal{W},$$

that is,

$$\operatorname{Re}\frac{(1 + H\widetilde{s}(\vartheta))^{-\frac{(1-F)(T-H)}{H}}(1 + Hs(\vartheta))}{(1-F)(T-H)} > 0, \ \vartheta \in \vartheta,$$

or

$$\operatorname{Re}\left[(1+H\widetilde{s}(\vartheta))^{-\frac{(1-F)(T-H)}{H}}(1+Hs(\vartheta))\right]>0,\ \vartheta\in\mathcal{W}.$$

If we denote

$$\psi:=(1+H\widetilde{s}(\vartheta))^{-\frac{(1-F)(T-H)}{H}}(1+Hs(\vartheta)), \tag{39}$$

from (39) we note that $\operatorname{Re}\psi>0$, therefore

$$|(1-\mu)\psi+\mu|=\sqrt{[\operatorname{Re}(1-\mu)\psi+\mu]^2+[\operatorname{Im}(1-\mu)\psi]^2}>\mu,$$

whenever $0<\mu<1$. Dividing this inequality by $\mu>0$, we obtain

$$\left|\left(\frac{1}{\mu}-1\right)(1+H\widetilde{s}(\vartheta))^{-\frac{(1-F)(T-H)}{H}}(1+Hs(\vartheta))+1\right|>1,\ \vartheta\in\mathcal{W},$$

that is,

$$\begin{aligned}&\left|\frac{(T-H)(1-F)}{1+Hs(\vartheta)}\right|\\ &<\left|\frac{(T-H)(1-F)}{1+Hs(\vartheta)}\right|\left|\left(\frac{1}{\mu}-1\right)(1+H\widetilde{s}(\vartheta))^{-\frac{(1-F)(T-H)}{H}}(1+Hs(\vartheta))+1\right|,\ \vartheta\in\mathcal{W},\end{aligned}$$

which represents (37).

Motivation by above reasons we will determine now the biggest value of r such that (36) holds for $|\vartheta|<r$.

Since $h\in\mathcal{S}^*(T,H,F)$, there exists *a Schwarz function* $s\in\Phi$, such that

$$\frac{\vartheta h'(\vartheta)}{h(\vartheta)}=\frac{1+[(1-F)T+FH]s(\vartheta)}{1+Hs(\vartheta)},\ \vartheta\in\mathcal{W},$$

therefore

$$\begin{aligned}\left|\arg\left([(1-F)T+FH]-H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right)\right|&=\left|\arg\frac{(1-F)(T-H)}{1+Hs(\vartheta)}\right|\\ &\le|\arg[(1-F)(T-H)]|+|\arg[1+Hs(\vartheta)]|\\ &\le\arcsin(|H|r^*),\ |\vartheta|\le r^*<1.\end{aligned} \tag{40}$$

Next, we will prove that

$$\left|\arg\left(\frac{\vartheta}{h(\vartheta)}\right)\right| \le \frac{(1-F)(T-H)}{H}\arcsin\left(Hr^*\right), \ |\vartheta| = r^*. \tag{41}$$

Thus, from Lemma 1.4 we have

$$\frac{h(\vartheta)}{\vartheta} = (1+H\widetilde{s}(\vartheta))^{\frac{(1-F)(T-H)}{H}},$$

for some $\widetilde{s} \in \Phi$, and we will split this proof in the next two steps:

Step 1 If $H > 0$, since $T-H > 0$ it follows that

$$\begin{aligned}\left|(1+H\widetilde{s}(\vartheta))^{\frac{(1-F)(T-H)}{H}}\right| &= \left|\exp\left[\frac{(1-F)(T-H)}{H}\log\left(1+H\widetilde{s}(\vartheta)\right)\right]\right| \\ &= \exp\left[\frac{(1-F)(T-H)}{H}\ln\left|1+H\widetilde{s}(\vartheta)\right|\right] \\ &= \left|1+H\widetilde{s}(\vartheta)\right|^{\frac{(1-F)(T-H)}{H}} \\ &\le (1+Hr^*)^{\frac{(1-F)(T-H)}{H}}, \ |\vartheta| \le r^* < 1.\end{aligned}$$

Step 2 If $H < 0$, denoting $D := -B > 0$, then $T+D > 0$, and it follows that

$$\begin{aligned}\left|(1+H\widetilde{s}(\vartheta))^{\frac{(1-F)(T-H)}{H}}\right| &= \left|\left[(1-D\widetilde{s}(\vartheta))^{-1}\right]^{\frac{(1-F)(T+D)}{D}}\right| \\ &= \left|(1-D\widetilde{s}(\vartheta))^{-1}\right|^{\frac{(1-F)(T+D)}{D}} \le \left(\frac{1}{1-Dr^*}\right)^{\frac{(1-F)(T+D)}{D}} \\ &= (1+Hr^*)^{\frac{(1-F)(T-H)}{H}}, \ |\vartheta| \le r^* < 1.\end{aligned}$$

Combining the above two cases, we get

$$\begin{aligned}\left|\arg\left(\frac{\vartheta}{h(\vartheta)}\right)\right| &\le \frac{(1-F)(T-H)}{H}\left|\arg\left(1+Hr^*\right)\right| \\ &\le \frac{(1-F)(T-H)}{H}\arcsin\left(Hr^*\right), \ |\vartheta| \le r^* < 1.\end{aligned}$$

From (40) and (41) we easily deduce that

$$\left|\arg\left(\frac{\vartheta}{h(\vartheta)}\right) - \arg\left([(1-F)T+FH] - H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right)\right|$$

$$\leq \left|\arg\left(\frac{\vartheta}{h(\vartheta)}\right)\right| + \left|\arg\left([(1-F)T+FH] - H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right)\right|$$

$$\leq \arcsin\left(|H|r^8\right) + \frac{(1-F)(T-H)}{H}\arcsin\left(Hr^*\right) < \frac{\pi}{2}, \ |\vartheta| \leq r,$$

where r is given like in the assumptions (ii) and (iii) of Theorem 2.13. □

3. Conclusions

In conclusion, this paper has significantly advanced the understanding of the function classes $\mathcal{S}^*(T, H, F)$ and $\mathcal{KS}^*(T, H, F)$ by providing detailed convolution conditions for function membership. These conditions have been instrumental in supporting further investigations within these classes, offering a solid foundation for future research. The sufficient conditions and neighborhood results derived for the class $\mathcal{S}^*(T, H, F)$ not only deepen our theoretical knowledge but also broaden the practical applications of these functions. Moreover, the exploration of the invariance properties of the operator $h_\mu(\vartheta) = (1-\mu)\vartheta + \mu h(\vartheta)$ for functions in $\mathcal{S}^*(T, H, F)$ has revealed important insights into the stability and transformations within this class. This analysis is crucial for applying these functions in various mathematical and applied contexts, ensuring their robustness under certain operations. Overall, the findings of this paper contribute to the body of knowledge in complex analysis, particularly in the study of starlike functions and their generalizations. They pave the way for further exploration into the geometric properties of analytic functions and their applications in diverse fields, from physics to engineering.

Acknowledgment

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number NBU-FFR-2024-871-09. This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2024/R/1445).

Conflict of interest

The authors declare no competing financial interest.

References

[1] Aldawish I, Ibrahim RW. Complex-Variable dynamic system of Layla and Majnun model with analytic solutions. *Symmetry*. 2023; 15: 1557. Available from: https://doi.org/10.3390/sym15081557.

[2] Nithiyanandham EK, Srutha KB. Image edge detection enhancement using coefficients of Sakaguchi type functions mapped onto petal shaped domain. *Heliyon*. 2024; 10(10): e31430. Available from: https://doi.org/10.1016/j.heliyon.2024.e31430.

[3] Janowski W. Some extremal problems for certain families of analytic functions. *Annales Polonici Mathematici*. 1973; 28(3): 297-326. Available from: https://doi.org/10.4064/ap-28-3-297-326.

[4] Polatoglu Y, Bolcal M, Sen A, Yavuz E. A study on the generalization of Janowski functions in the unit disc. *Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis*. 2006; 22: 27-31. Available from: http://eudml.org/doc/53410.

[5] Al-Sarari F, Frasin BA, AL-Hawary T, Latha S. A few results on generalized Janowski type functions associated with (j, k)-symmetrical functions. *Acta Universitatis Sapientiae Mathematica*. 2016; 8(2): 195-205. Available from: https://doi.org/10.1515/ausm-2016-0012.

[6] Lasode AO, Opoola TO. Some properties of a class of generalized Janowski-type q-starlike functions associated with Opoola q-differential operator and q-differential subordination. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*. 2024; 73(2): 349-463. Available from: https://doi.org/10.31801/cfsuasmas.1281348.

[7] Al-Sarari F, Latha S, Bulboacă T. On Janowski functions associated with (n, m)-symmetrical functions. *Journal of Taibah University for Science*. 2019; 13(1): 972-978. Available from: https://doi.org/10.1080/16583655.2019.1665487.

[8] Louati H, Al-Rezami AY, Darem AA, Alsarari F. Defining and analyzing new classes associated with (λ, γ)-symmetrical functions and quantum calculus. *Mathematics*. 2024; 12(16): 2603. Available from: https://doi.org/10.3390/math12162603.

[9] Goodman AW. Univalent functions and nonanalytic curves. *Proceedings of the American Mathematical Society*. 1975; 8: 598-601. Available from: https://doi.org/10.1090/S0002-9939-1957-0086879-9.

[10] Ruscheweyh S. Neighborhoods of univalent functions. *Proceedings of the American Mathematical Society*. 1981; 81: 521-527. Available from: https://doi.org/10.1007/BF02566116.

[11] Ganesan M, Padmanabhan KS. Convolution conditions for certain classes of analytic functions. *International Journal of Pure and Applied Mathematics*. 1984; 15(7): 777-780.

[12] Silverman H, Silvia EM, Telage D. Convolution conditions for convexity, starlikeness and spiral-likeness. *Mathematische Zeitschrift*. 1978; 162(2): 125-130. Available from: https://doi.org/10.1007/BF01215069.

[13] Duren PL. *Univalent Functions*. Springer-Verlag; 1983.