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Research Article

On the Maximum Number of Connected Induced Subgraphs of a Graph

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Abstract: We characterize the structure of graphs of a given order that maximize the number of connected induced subgraphs across seven different graph classes, each with specific parameters such as minimum degree, independence number, vertex cover number, vertex connectivity, edge connectivity, chromatic number, and the number of bridges. This work contributes to filling a gap in the existing literature.

***Keywords*:** minimum degree, independence number, vertex cover number, vertex connectivity, edge connectivity, number of bridges, chromatic number, connected induced subgraphs

MSC: 05C30, 05C35, 05C40, 05C69, 05C70

1. Introduction

Let G be a simple graph (i.e., undirected, with no loops or multiple edges) with a finite vertex set $V(G)$ and edge set $E(G)$, so we are considering labeled graphs in the sense that we are not working with *isomorphism* classes of graphs, which corresponds to "unlabeled graphs". The **order** of G is defined as the cardinality $|V(G)|$. The **degree** of a vertex $u \in V(G)$ is the number of vertices adjacent to u in G; we denote by $\delta(G)$ the minimum degree among all vertices of G.

A **subgraph** of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $G-e$ (respectively, $G-u$ or $G-S$) for the subgraph of G obtained by deleting an edge e (respectively, a vertex u, or a set of edges/vertices S). An **induced subgraph** of G is a subgraph obtained by deleting a set of vertices; specifically, we denote by $G-(V(G) \setminus S)$ the subgraph induced by S, which consists of S and all edges whose endpoints are contained within S.

An **independent set** in G is a set of vertices such that no two vertices in the set are adjacent. Thus, a set S of vertices is independent if and only if the subgraph induced by S has no edges. The maximum size of an independent set in G is called its **independence number**, denoted by $\alpha(G)$.

The graph G is said to be **connected** if every pair of vertices in G is part of a path; otherwise, G is **disconnected**. A **bridge** (or **cut-edge**) of G is an edge whose deletion increases the number of connected components of G. It is known (see, e.g., [1]) that an edge of G is a bridge if and only if it does not belong to any cycle in G.

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The **vertex connectivity** of G, denoted by $c(G)$, is the minimum size of a vertex set S such that $G-S$ is either a single-vertex graph or has more connected components than G. The **edge connectivity** of G, denoted by $c'(G)$, is the minimum size of an edge set S such that $G-S$ has more connected components than G.

A **vertex cover** of G is a set $S \subseteq V(G)$ that contains at least one endpoint of every edge in G; in other words, the vertices in S cover the edges of G. The **vertex cover number**, denoted by $\beta(G)$, is the minimum size of a vertex cover of G.

A graph G is **l-colorable** if we can assign one of l colors to each vertex such that adjacent vertices receive different colors. If G is l-colorable but not $(l-1)$-colorable, we say that the **chromatic number** of G is l. In other words, the chromatic number of G is the minimum number of colors needed to color the vertices of G such that adjacent vertices have different colors.

A graph G is said to be **complete** if every pair of distinct vertices in G is adjacent; the complete graph of order n is denoted by K_n.

For all the notation defined on a graph G, the context determines the usage, whether it explicitly references the graph G or not.

An **extremal problem** seeks the minimum or maximum value of a function over a certain class of objects. In graph theory, the term "extremal problem" typically refers to finding an optimum value over a class of graphs. In our context, we are concerned with determining the maximum number of connected induced subgraphs, denoted by (G), for simple graphs G with a given order and other structural parameters.

The problem of counting the number of connected subgraphs in a graph has applications in various fields. In the study of network reliability, connected induced subgraphs are used to assess the resilience of a network, helping to understand how robust a network is to the removal of vertices or edges. In biology, this concept aids in analyzing molecular structures and interactions within protein-protein interaction networks. In computer science, connected induced subgraphs serve as essential substructures for designing efficient algorithms to analyze graph data, among other applications.

Several upper and lower bounds on the number of connected subgraphs or connected induced subgraphs, in terms of other graph parameters, have been investigated (see, for example, [2–10]). Research on extremal problems in this area seems to have originated with Pandey and Patra [10], who studied the number of connected (not necessarily induced) subgraphs in both general graphs and unicyclic graphs, as a natural extension of counting subtrees in trees. By a **unicyclic graph**, we mean a connected graph containing exactly one cycle.

Among other results, it is established in [3] that the path uniquely achieves the minimum number of connected induced subgraphs among all connected graphs of a given order, while the maximum is attained only by the complete graph. Moreover, the findings in [3] can be generalized to graphs with a given order and a specified number of components. On the other hand, the work in [5] focuses on the class of all connected graphs with a given order and a specified number of cut vertices, as well as the class of all connected graphs with a given order and a specified number of pendant vertices.

Additionally, in [7], we studied inequalities that relate the sum of a graph invariant to the same invariant in its complement. These inequalities, known as **Nordhaus-Gaddum type inequalities**, were explored in the context of the number of connected induced subgraphs of graphs of a given order.

In this note, we characterize the unique extremal graph of a given order that maximizes the number of connected induced subgraphs in certain graph classes that have not been considered thus far, specifically for each of the following parameters: minimum degree, independence number, vertex cover number, vertex connectivity, edge connectivity, chromatic number, and number of bridges.

Observation: Adding an edge between a pair of non-adjacent vertices in a graph G increases the number of connected induced subgraphs by at least one.

Indeed, let u and v be non-adjacent vertices in G. By adding an edge $e = \{u, v\}$, we create a new graph G' with $E(G') = E(G) \cup \{e\}$. The addition of e does not eliminate any existing connected induced subgraphs and forms at least one new subgraph: the one consisting of u, v, and e. This ensures that the number of connected induced subgraphs increases by at least one.

We will sometimes use this observation implicitly, without further notice.

2. Main results

For a graph G and $u \in V(G)$, we denote by $\eta(G)_u$ the number of connected induced subgraphs of G that contain the vertex u.

Let $n,\ \delta$ be positive integers such that $n-2 \geq \delta$. We construct the graph $G_{n,\ \delta}$ by taking one copy of K_{n-1} and adding another vertex that is adjacent to exactly δ vertices of K_{n-1}. Note that $G_{n,\ \delta}$ has order n and minimum degree δ.

Proposition 2.1 For a graph G with order n and minimum degree δ, we have

$$\eta(G) \leq \eta(G_{n,\ \delta}) = 2^n - 2^{n-1-\delta},$$

with equality if and only if $G \simeq G_{n,\ \delta}$.

Proof. Let G be a graph with order n and minimum degree δ. Fix a vertex u of degree δ in G. For G to achieve the maximum $\eta(\cdot)$, the subgraph $G-u$ must be complete, since otherwise, we could increase $\eta(\cdot)$ by adding an edge between every pair of non-adjacent vertices. Thus, $G_{n,\ \delta}$ uniquely realizes the maximum $\eta(\cdot)$ among all graphs with order n and minimum degree δ.

Let w be the unique vertex of $G_{n,\ \delta}$ that does not belong to the K_{n-1} part. Consider $\eta(G_{n,\ \delta})_w$, the number of those connected induced subgraphs of $G_{n,\ \delta}$ that contain w. Every such subgraph either consists of w only, or w and a nonempty subset of vertices of K_{n-1}. In the latter case, the subgraph must also contain a nonempty subset of those δ vertices of K_{n-1} which are the neighbors of w. Their count is given by $(2^\delta - 1)2^{n-1-\delta}$. Thus, $\eta(G_{n,\ \delta})_w = 1 + (2^\delta - 1)2^{n-1-\delta}$. It follows that

$$\eta(G_{n,\ \delta}) = \eta(G_{n,\ \delta})_w + \eta(G_{n,\ \delta} - w) = \eta(G_{n,\ \delta})_w + \eta(K_{n-1})$$

$$= (1 + (2^\delta - 1)2^{n-1-\delta}) + (2^{n-1} - 1) = 2^n - 2^{n-1-\delta}.$$

□

The graph $H_{n,\ \alpha}$ is obtained by taking disjoint copies of $K_{n-\alpha}$ and $\overline{K_\alpha}$ (i.e., α independent vertices), and then adding an edge between every vertex of $K_{n-\alpha}$ and every vertex of $\overline{K_\alpha}$.

Proposition 2.2 For a graph G with order n and independence number α, we have

$$\eta(G) \leq \eta(H_{n,\ \alpha}) = \alpha + 2^n - 2^\alpha,$$

with equality if and only if $G \simeq H_{n,\ \alpha}$.

For a graph G with order n and vertex cover number β, we have

$$\eta(G) \leq \eta(H_{n,\ n-\beta}),$$

with equality if and only if $G \simeq H_{n,\ n-\beta}$.

Proof. Let G be a graph with order n and independence number α. Fix a set S of α independent vertices in G. For G to achieve the maximum $\eta(\cdot)$, the subgraph $G-S$ must be complete, and every vertex in S must be adjacent to all vertices in $G-S$. Thus, $H_{n,\ \alpha}$ is the unique graph that realizes the maximum $\eta(\cdot)$ among all graphs with order n and independence number α.

The union of any nonempty subsets of vertices of $K_{n-\alpha}$ and $\overline{K_\alpha}$ induces a connected subgraph of $H_{n,\ \alpha}$. Their count is given by $(2^{n-\alpha}-1)(2^\alpha-1)$. Thus, we have

$$\eta(H_{n,\ \alpha}) = \eta(K_{n-\alpha}) + \eta(\overline{K_\alpha}) + (2^{n-\alpha}-1)(2^\alpha-1)$$

$$= (2^{n-\alpha}-1) + \alpha + (2^{n-\alpha}-1)(2^\alpha-1) = \alpha + 2^n - 2^\alpha.$$

For the second statement of the proposition, it is known (see, e.g., [1] Lemma 3.1.21) that $\alpha + \beta = n$, which completes the proof. □

Our next theorem concerns vertex connectivity and edge connectivity.

Theorem 2.3 For a graph G with order n and vertex connectivity c, we have

$$\eta(G) \leq \eta(G_{n,\ c}),$$

with equality if and only if $G \simeq G_{n,\ c}$.

For a graph G with order n and edge connectivity $c' < n-1$, we have

$$\eta(G) \leq \eta(G_{n,\ c'}),$$

with equality if and only if $G \simeq G_{n,\ c'}$.

Proof. Let G be a graph with order n and vertex connectivity c. Fix a set S of c vertices in G such that $G-S$ is disconnected. For G to achieve the maximum $\eta(\cdot)$, the subgraph $G-S$ must have precisely two components, say G_1 and G_2, each of which is a complete graph, and every vertex in S must be adjacent to all vertices in $G-S = G_1 \cup G_2$.

Now, we need to determine the orders n_1 and n_2 of $G_1 = K_{n_1}$ and $G_2 = K_{n_2}$, respectively. Without loss of generality, assume $n_1 \leq n_2$. Suppose that $n_1 > 1$. Fix a vertex $u_1 \in V(G_1)$ and construct a new graph G' by deleting u_1 and adding a new vertex u_2 adjacent to all vertices in $S \cup V(G_2)$. Note that $G-u_1$ and $G'-u_2$ are isomorphic graphs.

By construction, every subset of $V(G)$ that contains u_1 and at least one vertex in S, as well as every subset of $V(G')$ that contains u_2 and at least one vertex in S, induces a connected graph. The number of such subgraphs of G and G' is given by:

$$(2^c-1)2^{n_1-1}\cdot 2^{n_2} \quad \text{and} \quad (2^c-1)2^{n_2}\cdot 2^{n_1-1},$$

respectively. Here, there are 2^c-1 non-empty subsets of elements in S, 2^{n_1-1} subsets of elements in $V(G_1)\setminus\{u_1\}$, and 2^{n_2} subsets of elements in $V(G_2)$.

Thus, we have:

$$\eta(G)_{u_1} = \eta(K_{n_1})_{u_1} + (2^c-1)2^{n_1-1}\cdot 2^{n_2} = 2^{n_1-1} + (2^c-1)2^{n_1-1+n_2},$$

and

$$\eta(G')_{u_2} = \eta(K_{n_2+1})_{u_2} + (2^c - 1)2^{n_2} \cdot 2^{n_1-1} = 2^{n_2} + (2^c - 1)2^{n_2+n_1-1}.$$

Therefore, we obtain:

$$\eta(G') - \eta(G) = \eta(G')_{u_2} - \eta(G)_{u_1} = 2^{n_2} - 2^{n_1-1} > 0,$$

which shows that $\eta(G') > \eta(G)$. Hence, for G to have the maximum $\eta(\cdot)$, we must have $n_1 = 1$. Additionally, S must induce a complete graph, implying that the graph realizing the maximum $\eta(\cdot)$ is indeed $G_{n,\, c}$.

For the second statement of the theorem, consider a graph $G_{n,\, x}$ where u is the unique vertex of degree $x < n-2$. Note that $G_{n,\, x} - u = K_{n-1}$, and

$$\eta(G_{n,\, x})_u = 1 + (2^x - 1)2^{n-1-x} = 1 + 2^{n-1} - 2^{n-1-x},$$

counts the number of u-containing connected induced subgraphs of $G_{n,\, x}$. Thus,

$$\eta(G_{n,\, x}) = \eta(K_{n-1}) + \eta(G_{n,\, x})_u,$$

is a strictly increasing function in x.

Now, let G be a graph with order n and edge connectivity c'. Let c denote the vertex connectivity of G. Then, we have:

$$\eta(G) \leq \eta(G_{n,\, c}) \leq \eta(G_{n,\, c'}),$$

where the first inequality follows from the first statement of the theorem, and the second inequality holds by Whitney's theorem [11], which states that $c \leq c'$. This completes the proof of the theorem. □

The Turàn graph $T_{n,\, l}$ is a complete l-partite graph of order n in which any two partition sets differ in cardinality by at most one [12, 13]. This famous graph appears in many extremal graph theory problems. For instance, it is known that $T_{n,\, l}$ has more edges than any other simple l-partite graph on n vertices, and that $T_{n,\, l-1}$ has more edges than any other simple graph on n vertices containing no K_l.

Theorem 2.4 For a graph G with order n and chromatic number l, we have

$$\eta(G) \leq \eta(T_{n,\, l}),$$

with equality if and only if $G \simeq T_{n,\, l}$.

Denote by $n_1,\, n_2,\, \ldots,\, n_l$ the respective sizes of the partite sets of $T_{n,\, l}$. We have

$$\eta(T_{n,\, l}) = n + \sum_{\substack{S \subseteq \{1,\, 2,\, \ldots,\, l\} \\ |S| \geq 2}} \prod_{j \in S} (2^{n_j} - 1).$$

In particular, for $n_1 = n_2 = \cdots = n_l = n/l$, we have

$$\eta(T_{n,\,l}) = n + 2^n - 1 - l \cdot 2^{n/l} + l\,.$$

Proof. Let G be a graph with order n and chromatic number l. Partition the vertices of G according to their colors. Each of the l partition sets (color classes) forms an independent set with sizes $n_1, n_2, \ldots, n_l$, such that $n_1 + n_2 + \cdots + n_l = n$. Thus, for G to achieve the maximum $\eta(\cdot)$, it must be a complete l-partite graph with these partition sets. In particular, only $K_n = T_{n,\,n}$ realizes the maximum when $l = n$.

Assuming $l < n$, let A and B be two partite sets with the greatest and smallest cardinalities, respectively, among all the l partite sets of G. Thus, we have $|A| > 1$. We want to show that $|A| - 1 \le |B| \le |A|$.

Suppose this is not the case. Fix a vertex $a \in A$. Construct a new graph G' by deleting vertex a, introducing a new vertex b with the same color as those in B, and adding edges between b and every vertex in $V(G) \setminus (B \cup \{a\})$. Note that:

$$\eta(G') - \eta(G) = \eta(G')_b - \eta(G)_a,$$

since $G - a$ and $G' - b$ are isomorphic by construction.

Additionally, every subset of vertices in G that contains both a and an element of $V(G) \setminus A$, as well as every subset in G' containing both b and an element of $V(G') \setminus (B \cup \{b\})$, always induces a connected graph. Moreover, A and $B \cup \{b\}$ are independent sets in G and G', respectively. Thus, we have:

$$\eta(G)_a = 1 + (2^{|V(G)| - |A|} - 1)2^{|A| - 1},$$

and

$$\eta(G')_b = 1 + (2^{|V(G')| - |B| - 1} - 1)2^{|B|}.$$

Indeed, $V(G) \setminus A$ is the neighborhood of every element in A in G, and $V(G') \setminus (B \cup \{b\})$ is the neighborhood of every element in $B \cup \{b\}$ in G'. Thus, we obtain:

$$\eta(G') - \eta(G) = \eta(G')_b - \eta(G)_a = 2^{|V(G')| - 1} - 2^{|B|} + 2^{|A| - 1} - 2^{|V(G)| - 1},$$

which simplifies to:

$$\eta(G') - \eta(G) = 2^{|A| - 1} - 2^{|B|} > 0,$$

given the inequality $|B| < |A| - 1$. This contradicts the assumption that G maximizes $\eta(\cdot)$.

Therefore, for G to maximize $\eta(\cdot)$, it must be a complete l-partite graph with $|A| - 1 \le |B| \le |A|$. By the choice of sets A and B, we conclude that the graph realizing the maximum $\eta(\cdot)$ is indeed the Turán graph $T_{n,\,l}$.

Denote by $V_1, V_2, \ldots, V_l$ the partite sets of $T_{n,\,l}$, and by $n_1, n_2, \ldots, n_l$ their respective sizes such that $n_1 \le n_2 \le \cdots \le n_l$. Thus $n_1 = \lfloor n/l \rfloor$ and $n_l = \lceil n/l \rceil$. We enumerate the connected induced subgraphs of $T_{n,\,l}$ according to the number of partite sets that are involved. Let S be a subset of $\{1, 2, \ldots, l\}$ such that $|S| \geq 2$. Then $\prod_{j\in S}(2^{n_j}-1)$ counts precisely the number of those connected induced subgraphs of $T_{n,\,l}$ that contain at least an element from V_j for all $j \in S$. On the other hand, there are precisely n_i number of those connected induced subgraphs of $T_{n,\,l}$ that contain elements of V_i only. Therefore, we obtain

$$\eta(T_{n,\,l}) = \sum_{j=1}^{l} n_j + \sum_{\substack{S\subseteq\{1,\,2,\,\ldots,\,l\}\\ |S|\geq 2}} \prod_{j\in S}(2^{n_j}-1) = n + \sum_{\substack{S\subseteq\{1,\,2,\,\ldots,\,l\}\\ |S|\geq 2}} \prod_{j\in S}(2^{n_j}-1)\,.$$

The formula in Theorem 2.4 can be simplified for equal partite sets, i.e.

$$n_1 = n_2 = \cdots = n_l = n/l\,.$$

In this case, we have

$$\eta(T_{n,\,l}) = n + \sum_{k=2}^{l} \sum_{\substack{S\subseteq\{1,\,2,\,\ldots,\,l\}\\ |S|=k}} \prod_{j\in S}(2^{n/l}-1) = n + \sum_{k=2}^{l} \sum_{\substack{S\subseteq\{1,\,2,\,\ldots,\,l\}\\ |S|=k}} (2^{n/l}-1)^k$$

$$= n + \sum_{k=2}^{l} \binom{l}{k}(2^{n/l}-1)^k = n + \sum_{k=0}^{l} \binom{l}{k}(2^{n/l}-1)^k \; - 1 - l(2^{n/l}-1)$$

$$= n + 2^n - 1 - l\cdot 2^{n/l} + l\,.$$

□

For our next theorem, we begin with a lemma, which can also be found in [4, 6].

Lemma 1 (Lemma 2.1 [4], Lemma 1 [6]) Let $L,\ M,\ R$ be three non-trivial (i.e., each has at least two vertices) connected graphs whose vertex sets are pairwise disjoint. Let $l \in V(L)$, $r \in V(R)$, and $u,\ v \in V(M)$ be fixed vertices such that $u \neq v$. Denote by G the graph obtained from $L,\ M,\ R$ by identifying l with u, and r with v. Similarly, let G' (respectively, G'') be the graph obtained by identifying both $l,\ r$ with u (respectively, both $l,\ r$ with v); see Figure 1 for a diagram of these graphs. Then, it holds that:

$$\eta(G') > \eta(G) \quad \text{or} \quad \eta(G'') > \eta(G).$$

An edge of a graph G with an end vertex of degree 1 is called a **pendant edge** of G. The star of order n is denoted by S_n.

For positive integers $b,\ n$ such that $b < n-2$, define $J_{n,\,b}$ to be the graph obtained by identifying one vertex of K_{n-b} with the central vertex of S_{b+1}.

Theorem 2.5 For a graph G with order n and $b < n-2$ bridges, we have

$$\eta(G) \leq \eta(J_{n,\,b}) = 2^{n-1} + b + 2^{n-b-1} - 1,$$

with equality if and only if $G \simeq J_{n,\,b}$.

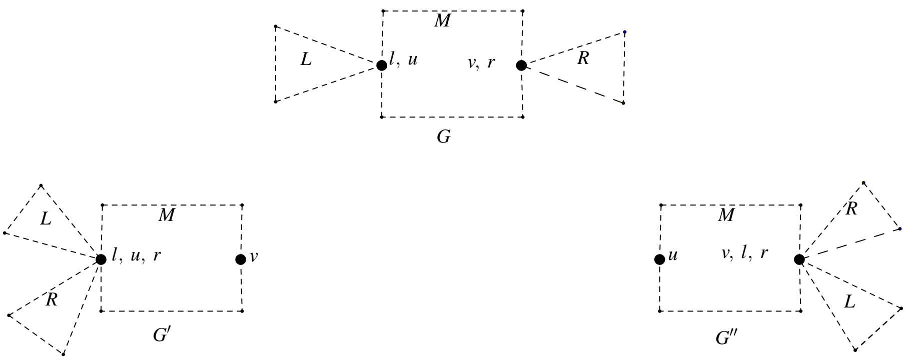

Figure 1. The graphs $G,\ G',\ G''$ described in Lemma 1

Proof. Let G be a graph with order $n > 2$ and b bridges. Fix a bridge with end vertices $u,\ v$ in G. Then, specializing $M = uv$ (i.e., taking M to be the edge uv) in Lemma 1 shows that all edges of G preserve their status (bridge/non-bridge) in both G' and G''. Moreover, the bridge uv becomes a pendant edge in both G' and G''.

Thus, it suffices to prove the theorem for G in the class of graphs with order n and b pendant edges (or pendant vertices).

Fortunately, it was determined in [5, Theorem 3] and [7] that the graph structure which maximizes the number of connected induced subgraphs, given both order and the number of pendant vertices, is as follows: If $b < n-2$, the extremal graph is obtained by identifying one vertex of K_{n-b} with the central vertex of S_{b+1}.

The first part of the theorem follows from these results.

First note that for any $u \in V(K_m)$, we have $\eta(K_m)_u = 2^{m-1}$. Let w be the unique vertex common to K_{n-b} and S_{b+1} in the graph $J_{n,\,b}$. Since w is the central vertex of S_{b+1}, it holds that

$$\eta(J_{n,\,b})_w = \eta(K_{n-b})_w \cdot \eta(S_{b+1})_w = 2^{n-b-1} \cdot 2^b = 2^{n-1}\,.$$

The graph $J_{n,\,b} - w$ consists of b copies of K_1 and one copy of K_{n-b-1}. Thus.

$$\eta(J_{n,\,b} - w) = b + \eta(K_{n-b-1}) = b + 2^{n-b-1} - 1\,.$$

Altogether, we obtain

$$\eta(J_{n,\,b}) = \eta(J_{n,\,b})_w + \eta(J_{n,\,b} - w) = 2^{n-1} + b + 2^{n-b-1} - 1\,.$$

□

3. Concluding comments

Among all connected graphs with order n and minimum degree δ, the minimum number of connected induced subgraphs is attained by the path if $\delta = 1$ [3] and by the cycle if $\delta = 2$ [3, 5]. This motivates the following problem: What is the minimum number of connected induced subgraphs among all graphs with a given order and minimum degree $\delta > 2$? Recall that Proposition 2.1 addresses the maximization counterpart of this problem.

Theorem 2.3 establishes that for a graph with a given order and connectivity, the graph that maximizes the number of connected induced subgraphs is the same for both vertex-connectivity and edge-connectivity. As noted in [3, 5], the path (respectively, the cycle) also attains the minimum number of connected induced subgraphs among all graphs with order n and connectivity 1 (respectively, connectivity 2).

However, other relationships between the number of connected induced subgraphs and graph connectivity have not yet been established. Therefore, a natural question that arises is to determine the general case where the connectivity is greater than 2. It seems to us that even the case of connectivity 3 could be very challenging.

Statements and declarations

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Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest

The authors declare no competing financial interest.

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