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Research Article

On Convergence of a Novel Jacobian-Free Parametric Iterative Vectorial Schemes

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Abstract: In this work, we have proposed a Jacobian free iterative vectorial multiparametric family for solving systems of nonlinear equations. The scheme is obtained by replacing the Jacobian matrix by divided difference operator in a family of third and fourth order iterative methods which maintains the convergence order. This way we avoid the expensive inversion of the inverse of the Jacobian which may not even exist. Moreover, the efficiency index of the method is studied in detail. The comparisons of the numerical experiments of the proposed family and other competitive methods corroborate the utility of presented method over the existing ones.

***Keywords*:** nonlinear systems, banach spaces, local convergence, iterative scheme, efficiency index

MSC: 47H99, 46B99, 49M15, 65D99, 65G99

1. Introduction

The most efficient and widely opted scheme for solving vectorial non-linear problems of the form

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq B_1 \to B_2$ be a nonlinear Fréchet differentiable operator, B_1 and B_2 are Banach spaces i.e complete normed spaces and Ω is an open convex subset of B_1 is the quadratically convergent Newton's method [1] given by

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k),\ k = 0,\ 1,\ 2, \ldots \tag{2}$$

provided F' does not vanish in Ω, x_0 is initial point and $F'(x_k)^{-1} \in \mathscr{L}(B_2,\ B_1)$, where $\mathscr{L}(B_2,\ B_1)$ is set of bounded linear operators from B_2 into B_1. In terms of computational cost, two evaluations per iteration are used to attain the second order of convergence of (2). Such a computational effort, especially one involving the inverse is very expensive in general or

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may not even exist rendering the usage of (2) or other methods using inverses of the Fréchet derivative impossible [1–3]. That is why researchers and practitioners have replaced the Fréchet-derivative by divided differences of order one [4–8]. A noticeable refinement of Newton's method has been attempted by Traub (the Traub-Steffensen method) [1]:

$$x_{k+1} = x_k - [u_k,\ x_k;\ F]^{-1} F(x_k),\ k = 0,\ 1,\ 2,\ \ldots, \tag{3}$$

where $u_k = x_k + \beta F(x_k)$ and $[u_k,\ x_k;\ F]$ is first divided difference of F. A variety of modified Newton's method or Newton-like methods of higher order have been developed in literature [2–14]. The family of third and fourth order methods given in [15] in Banach spaces is given as

$$\begin{cases} y_k = x_k - \alpha F'(x_k)^{-1} F(x_k), \\ z_k = x_k - F'(x_k)^{-1}(F(y_k) + \alpha F(x_k)), \\ w_k = x_k - F'(x_k)^{-1}(F(z_k) + F(y_k) + \alpha F(x_k)), \end{cases} \tag{4}$$

where $\alpha \in \mathbb{R}$. The iterative expression of iterative class (4) involves a Jacobian matrix. It is well known that such the iterative methods are highly more stable than Jacobian-free methods but there are many practical situations where the calculations of a Jacobian matrix, if it exists, are computationally expensive, and/or it requires a great deal of time for them to be given or calculated. Therefore, Jacobian-free schemes are quite popular for finding the roots of nonlinear equations and systems of nonlinear equations. The foremost aim of the current study is to design a Jacobian-free Parametric Iterative Vectorial Scheme maintaining the convergence order and without considerable increase in computational cost.

The rest of the article is structured as follows: Two versions of the local convergence one using Taylor series and another generalized continuity are presented in Section 2 and Section 3, respectively. At the end of Section 3 the semilocal analysis can be found. The work on efficiency index appears in Section 4. Numerical examples can be found in Section 5 and the conclusions in Section 6.

2. Design of the Jacobian-free vectorial scheme

The parametric family free from Jacobian matrix $F'(x^*)$ is obtained by simply replacing it with a symmetrical divided difference operator $[x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F(x_k)]$, where $\varpi \in \mathbb{R} \setminus \{0\}$. Thus, the proposed family is modified but has same characteristics and convergence order as that of (4). The Jacobian-free scheme obtained for $\alpha = 1$ has the iterative structure given as:

$$\begin{cases} y_k = x_k - [x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F]^{-1} F(x_k), \\ z_k = y_k - [x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F]^{-1} F(y_k), \\ x_{k+1} = z_k - [x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F]^{-1} F(z_k), \end{cases} \tag{5}$$

where $\varpi \in \mathbb{R} \setminus \{0\}$ and is denoted by $M41$. The next theorem proves that the family $M41$ has fourth convergence order for any value of real non-zero parameter ϖ.

Theorem 1 [16] Let the function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in a convex set Ω containing the zero x^* of $F(x)$. Suppose that $F'(x)$ is continuous and nonsingular at x^*. If an initial approximation x_0 is sufficiently close to x^*, then the local convergence order of the method (5) is four.

Proof. Developing $F(x_k)$ in the neighborhood of x^*, we write

$$F(x_k) = F'(x^*)\left[e_k + A_2 e_k^2 + A_3 e_k^3 + A_4 e_k^4 + O(e_k^5)\right], \tag{6}$$

where $e_k = x_k - x^*$, $A_i = \dfrac{1}{i!}$ Γ\$, $F^{(i)}(x^*)$, $F^{(i)}(x^*) \in \mathscr{L}(\mathbb{R}^n \times, \cdot \overset{i}{\cdot} \cdot, \times \mathbb{R}^n, \mathbb{R}^n)$, $\Gamma \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $(e_k)^i = (e_k, e_k, \overset{i-times}{\cdots\cdots}, e_k)$ with $e_k \in \mathbb{R}^n$, $i = 2, 3, \ldots$ Also,

$$F'(x_k) = F'(x^*)\left[I + 2A_2 e_k + 3A_3 e_k^2 + 4A_4 e_k^3 + O(e_k^4)\right]. \tag{7}$$

$$F''(x_k) = F'(x^*)\left[2A_2 + 6A_3 e_k + 12A_4 e_k^2 + O(e_k^3)\right]. \tag{8}$$

$$F'''(x_k) = F'(x^*)\left[6A_3 + 24A_4 e_k + O(e_k^2)\right]. \tag{9}$$

Using the Genocchi-Hermite formula [17], the following expansion is obtained:

$$[x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F] = F'(x_k) + \frac{1}{6} F'''(x_k)(\varpi F(x_k)^2 + O(\varpi F(x_k))^3). \tag{10}$$

Substituting (7) and (9) in (10) yields

$$\begin{aligned}
[x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F] =& F'(x^*)(1 + 2A_2 e_k + 3A_3 e_k^2 + 4A_4 e_k^3 + O(e_k^4)) \\
& + \frac{1}{6}(6A_3 + 24A_4 e_k + O(e_k^2)) * \left(\varpi^2 F'(x^*)^2 (e_k + A_2 e_k^2 + A_3 e_k^3 + O(e_k^4))^2\right) \\
=& F'(x^*)[I + 2A_2 e_k + A_3(3I + \varpi^2 F'(x^*)^2) e_k^2 \\
& + (4A_4 + 2\varpi^2 F'(x^*)^2 A_3 A_2 + \varpi^2 F'(x^*)^2 4A_4) e_k^3] + O(e_k^4).
\end{aligned}$$

It can be simply checked that the inverse operator $[x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F]^{-1}$ has the following expression

$$
\begin{aligned}
[x_k+\varpi F(x_k),\ x_k-\varpi F(x_k);\ F]^{-1} =&\Big(I-2A_2e_k+\left(4A_2^2-A_3(3I+\varpi^2F'(x^*)^2)\right)e_k^2 \\
&+(4A_2A_3(3I+\varpi^2F'(x^*)^2)-4A_4(I+\varpi^2F'(x^*)^2) \\
&-2\varpi^2F'(x^*)^2A_3A_2)e_k^3\Big)F'(x^*)^{-1}+O(e_k^4).
\end{aligned}
\tag{11}
$$

Taking $\tilde{e}_k = y_k - x^*$ and using (11), we obtain using Taylor development of $F(x_k)$

$$
\begin{aligned}
\tilde{e}_k =&x_k-x^*-[x_k+\varpi F(x_k),\ x_k-\varpi F(x_k);\ F]^{-1}F(x_k) \\
=&e_k-\Big(I-2A_2e_k+\left(4A_2^2-A_3(3I+\varpi^2F'(x^*)^2)\right)e_k^2 \\
&+(4A_2A_3(3I+\varpi^2F'(x^*)^2)-4A_4(I+\varpi^2F'(x^*)^2)-2\varpi^2F'(x^*)^2A_3A_2)e_k^3\Big) \\
&*\Big(e_k+A_2e_k^2+A_3e_k^3+A_4e_k^4\Big)+O(e_k^5) \\
=&e_k-e_k-A_2e_k^2-A_3e_k^3+2A_2e_k^2+2A_2^2e_k^3+2A_2A_3e_k^4-\left(4A_2^2-A_3(3I+\varpi^2F'(x^*)^2)\right)e_k^3 \\
&-\left(4A_2^2-A_3(3I+\varpi^2F'(x^*)^2)\right)A_2e_k^4-\Big(4A_2A_3(3I+\varpi^2F'(x^*)^2) \\
&-4A_4(I+\varpi^2F'(x^*)^2)-2\varpi^2F'(x^*)^2A_3A_2\Big)e_k^4+O(e_k^5) \\
=&A_2e_k^2+\left(-2A_2^2+A_3(2I+\varpi^2F'(x^*)^2)\right)e_k^3 \\
&+\Big(-A_4+2A_2A_3-4A_2^3+3A_3A_2+\varpi^2A_3F'(x^*)^2A_2-12A_2A_3 \\
&-4A_2A_3\varpi^2F'(x^*)^2+4A_4+4A_4\varpi^2F'(x^*)^2+2\varpi^2A_3F'(x^*)^2A_2\Big)e_k^4+O(e_k^5) \\
=&A_2e_k^2+Y_3e_k^3+Y_4e_k^4+O(e_k^5),
\end{aligned}
$$

where

$$Y_3 = -2A_2^2 + A_3(2I + \varpi^2 F'(x^*)^2)$$

$$Y_4 = 3A_4 - 10A_2A_3 + 3A_3A_2 - 4a_2^3 - 4A_2A_3\varpi^2 F'(x^*)^2 + 3\varpi^2 F'(x^*)^2 A_3A_2.$$

Expanding $F(y_k)$ about x^* Using Taylor's series and using above result, we have

$$\begin{aligned} F(y_k) =& F'(x^*)\left[\tilde{e}_k + A_2\tilde{e}_k^2 + A_3\tilde{e}_k^3 + O(\tilde{e}_k^4)\right] \\ =& F'(x^*)\Big[A_2e_k^2 + Y_3e_k^3 + (Y_4 + A_2^3)e_k^4 + O(e_k^5)\Big]. \end{aligned} \tag{12}$$

Taking $\hat{e}_k = z_k - x^*$ and using (12), we obtain the following

$$\begin{aligned} \hat{e}_k =& y_k - x^* - [x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F]^{-1} F(y_k) \\ =& \tilde{e}_k - \Bigg(I - 2A_2e_k + \left(4A_2^2 - A_3(3I + \varpi^2 F'(x^*)^2)\right)e_k^2 \\ & + (4A_2A_3(3I + \varpi^2 F'(x^*)^2) - 4A_4(I + \varpi^2 F'(x^*)^2) - 2\varpi^2 A_3 F'(x^*)^2 A_2)e_k^3\Bigg) \\ & * \Bigg(A_2e_k^2 + Y_3e_k^3 + (Y_4 + A_2^3)e_k^4 + O(e_k^5)\Bigg) + O(e_k^5) \\ =& A_2e_k^3 + Y_3e_k^3 + Y_4e_k^4 + O(e_k^5) \\ & - \Bigg(A_2e_k^2 + Y_3e_k^3 + (Y_4 + A_2^3)e_k^4 - 2A_2^2e_k^3 - 2A_2Y_3e_k^4 \\ & + (4A_2^2 - A_3(3I + \varpi^2 F'(x^*)^2))A_2e_k^4 + O(e_k^5)\Bigg) \\ =& A_2e_k^2 + Y_3e_k^3 + Y_4e_k^4 - A_2e_k^2 - Y_3e_k^3 - (Y_4 + A_2^3)e_k^4 + 2A_2^2e_k^3 + 2A_2Y_3e_k^4 \\ & - \left(4A_2^2 - A_3(3I + \varpi^2 F'(x^*)^2)\right)A_2e_k^5 + O(e_k^5) \\ =& 2A_2^2e_k^3 + Y_6e_k^4 + O(e_k^5), \end{aligned}$$

where

$$Y_6 = -7A_2^3 + 3A_3A_2 + 2A_2A_3 + A_2A_3\varpi^2F'(x^*)^2 + \varpi^2F'(x^*)^2A_3A_2.$$

Expanding $F(z_k)$ about x^* Using Taylor's series and using above result, we have

$$\begin{aligned} F(z_k) =& F'(x^*)\left[\hat{e}_k + A_2\hat{e}_k^2 + O(\hat{e}_k^3)\right] \\ =& F'(x^*)\left[2A_2^2e_k^3 + Y_6e_k^4 + O(e_k^5)\right]. \end{aligned} \tag{13}$$

Using (11) and (13), the error equation is given as

$$\begin{aligned} e_{k+1} =& z_k - x^* - [x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F]^{-1}F(z_k) \\ =& -2A_2^2e_k^3 + Y_6e_k^4 - \Big(I - 2A_2e_k + \left(4A_2^2 - A_3(3I + \varpi^2F'(x^*)^2)\right)e_k^2 \\ &+ (4A_2A_3(3I + \varpi^2F'(x^*)^2) - 4A_4(I + \varpi^2F'(x^*)^2) - 2\varpi^2F'(x^*)^2A_3A_2)e_k^3\Big) \\ &* (2A_2^2e_k^3 + Y_6e_k^4 + O(e_k^5)) = 4A_2^3e_k^4 + O(e_k^5). \end{aligned} \tag{14}$$

Therefore, (14) proves that the Jacobian-free parametric family (5) has fourth convergence order for $\varpi \in \mathbb{R} \setminus \{0\}$ and hence, the proof is completed. □

Remark 1 It is worth noticing that the proof of Theorem 1 is shown using Taylor series expansions for $B_1 = B_2 = \mathbb{R}^m$, where m is a natural number and by assuming $F^{(5)}$ which is not present on the method exists. Other drawbacks of this approach are the unavailability of upper bounds on $\|x_k - x^*\|$. This is we do not know the number of iterates in advance required to achieve a desired error tolerance (see also the motivational example in (P_1) of Section 3 that follows). Moreover, there is no information on the isolation of the solution. These problems are addressed in Section 3.

3. An extended convergence

The convergence analysis of Section 2 uses Taylor series. There are certain constraints with this technique:

(P_1) The existence of $F^{(5)}$ which is not on the method (5) has been assumed but there exist even scalar equations where this assumption is violated. Indeed, let $\Omega = [-2,\ 2]$. Define the function $F : \Omega \to \mathbb{R}$ by

$$F(t) = \begin{cases} c_1t^2\log t + c_2t^5 + c_3t^4, & \text{for } t \neq 0 \\ 0, \quad t = 0, & \end{cases}$$

where $c_1 \neq 0$ and $c_2 + c_3 = 0$ are constants. It follows by thus definition that $t^* = 1 \in \Omega$ solves the equation $F(t) = 0$. However, $F^{(3)}$ is not continuous on Ω, since the function is not continuous at $t = 0$. Therefore, the results of Section 2 cannot assure that $\lim_{k\to\infty} x_k = 1$. But for $\varpi = 1$, and $x_0 = 1.2$ the method 5 converges to t^*. This observation indicates that the sufficient convergence conditions of the Section 2 can be replaced by weaker ones.

(P_2) There is no natural integer K such that $\|x_k - x^*\| \leq \varepsilon(\varepsilon > 0)$ for each $k \geq K$.

(P_3) There is no uniqueness of the solution domain.

(P_4) The more challenging semilocal analysis of convergence is not presented.

(P_5) The analysis is restricted to finite dimensional Euclidean space.

Therefore, The items $(P_1) - (P_5)$ are the motivation for introducing this Section. We handle these problems as follows:

$(P_1)'$ The local convergence uses conditions only on $\mathscr{T}$, i.e the divided difference on the method (5).

$(P_2)'$ The number K is determined. Thus, the number of iterations to be carried out are known in advance.

$(P_3)'$ A domain is provided that contains only one solution.

$(P_4)'$ The semilocal analysis is provided using majorizing sequences [4, 18].

and

$(P_5)'$ The results are valid on Banach space.

Moreover, the concept of generalized continuity [4, 12, 18] is used to control the divided difference $\mathscr{T}$.

It is convenient for the convergence study to introduce some notations. Set $U = x + \varpi F(x)$, $V = x - \varpi F(x)$ and $\mathscr{T}(x) = [U,\ V;\ F]$. Then, the method (5) becomes

$$\begin{cases} y_k = x_k - \mathscr{T}_k^{-1} F(x_k), \\ z_k = y_k - \mathscr{T}_k^{-1} F(y_k), \\ x_{k+1} = z_k - \mathscr{T}_k^{-1} F(z_k). \end{cases} \tag{15}$$

Let $\mathscr{S}(x^*,\ r) = \{x \in B_1 : \|x - x^*\| < r\}$, $\mathscr{S}[x^*,\ r]$ be closure of $\mathscr{S}(x^*,\ r)$, $b = |\varpi|\|M\|$ and $\mathscr{Q} = [0,\ +\infty)$. Moreover, by (SZP) we mean the smallest zero of a function which is positive. Furthermore, (FCND) is the notation for a function which is continuous on each of each variables and nondecreasing on all variables. Next, we first present.

3.1 *Local analysis*

The main contribution of the local convergence analysis is the demonstration of the degree of difficulty in choosing the initial points. Suppose:

(C_1) There exists FCND $H_1 : \mathscr{Q} \longrightarrow \mathscr{Q}$, $H_2 : \mathscr{Q} \longrightarrow \mathscr{Q}$ and $W_0 : \mathscr{Q} \times \mathscr{Q} \longrightarrow \mathscr{Q}$ such that the equation $W_0(H_1(t),\ H_2(t)) - 1 = 0$. Denote such zero by R_0 and set $\mathscr{Q}_1 = [0,\ R_0)$.

(C_2) There exists FCND $W : \mathscr{Q}_1 \times \mathscr{Q}_1 \longrightarrow \mathscr{Q}$. Then, for $b \geq 0$ and $G_1 : \mathscr{Q}_1 \longrightarrow \mathscr{Q}$ defined by

$$G_1(t) = \frac{W(b(1 + W_0(t,\ 0))t,\ H_2(t))}{1 - W_0(H_1(t),\ H_2(t))}$$

the equation $G_1(t) - 1 = 0$ has SZP in the interval $\mathscr{Q}_1$. Denote such zero by R_1.

(C_3) For $G_2 : \mathscr{Q}_1 \longrightarrow \mathscr{Q}$ defined by

$$G_2(t) = \frac{W(b(1+W_0(G_1(t)t,\ 0)),\ H_2(t))G_1(t)}{1-W_0(H_1(t),\ H_2(t))}$$

the equation $G_2(t)-1=0$ has SZP. Denote such zero by R_2.

(C_4) For $G_3 : \mathscr{Q}_1 \longrightarrow \mathscr{Q}$ defined by

$$G_3(t) = \frac{W(b(1+W_0(G_2(t)t,\ 0)),\ H_2(t))G_2(t)}{1-W_0(H_1(t),\ H_2(t))}$$

the equation $G_3(t)-1=0$ has SZP. Denote such zero by R_3.

Set

$$R = \min\{R_j\},\ j = 1,\ 2,\ 3. \tag{16}$$

The number R is shown in the Theorem 2 to be a radius of convergence for the method (15). Next, the radius R and the scalar functions H_1, H_2, W_0 and W relate to the divided differences appearing on the method (15).

(C_5) There exists a solution $x^* \in B_1$ of the equation $F(x) = 0$ and $M \in \mathscr{L}(B_1,\ B_2)$ such that $M^{-1} \in \mathscr{L}(B_2,\ B_1)$, and for each $\bar{x},\ \bar{y} \in \Omega$

$$\|M^{-1}(\mathscr{T}(U,\ V)-M)\| \leq W_0(\|U-x^*\|,\ \|V-x^*\|),$$

$$\|U-x^*\| \leq H_1(\|x-x^*\|) \leq \|x-x^*\|$$

and

$$\|V-x^*\| \leq H_2(\|x-x^*\|) \leq \|x-x^*\|.$$

Set $A_0 = \Omega \cap \mathscr{S}(x^*,\ R_0)$.

(C_6)

$$\|M^{-1}(\mathscr{T}-[x,\ x^*;\ F])\| \leq W(\|U-x\|,\ \|V-x^*\|)$$

for each $x \in A_0$. and

(C_7)

$$\mathscr{S}[x^*,\ R] \subset \Omega. \text{ and } b \geq |\varpi|\|M\|.$$

Remark 2 (i) Some candidates for $M = I$ the identity operator on B_1 or $M = F'(x^*)$ or $M = [\tilde{x},\ \tilde{y};\ F]$, where $\tilde{x},\ \tilde{y} \in \Omega$ are some auxiliary points. If we avoid the popular choice $M = F'(x^*)$ which implies that x^* is a simple solution of the equation $F(x) = 0$ by the condition (C_5), then the method (15) can be used to find solutions x^* of multiplicity greater than one.

Moreover, it is worth noticing that the conditions (C_1)-(C_7) do not necessarily imply that x^* is a simple solution of the equation $F(x) = 0$.

(ii) We can provide some possible choices of the functions H_1 and H_2 as follows:

$$\begin{aligned} U - x^* &= x + \varpi F - x^* = x - x^* + \varpi F \\ &= x - x^* + \varpi[x,\ x^*;\ F](x - x^*) \\ &= (I + \varpi[x,\ x^*;\ F])(x - x^*) \left(I + \varpi M M^{-1}([x,\ x^*;\ F] - M + M)\right)(x - x^*) \\ &= \left((I + \varpi M) + \varpi M M^{-1}([x,\ x^*;\ F] - M)\right)(x - x^*) \end{aligned}$$

leading to

$$\|U - x^*\| \leq \left(\|I + \varpi M\| + bW_0(\|x - x^*\|,\ 0)\right)\|x - x^*\|.$$

So, we can choose

$$H_1(t) = (\|I + \varpi M\| + bW_0(t,\ 0))\,t.$$

Similarly, we can set

$$H_2(t) = (\|I - \varpi M\| + bW_0(t,\ t))\,t.$$

Next, the conditions (C_1)-(C_7) and the proceeding terminology assist us to show the local convergence for the method (15). Set $A_1 = \mathscr{S}(x^*,\ R) - \{x^*\}$.

Theorem 2 Suppose that the conditions (C_1)-(C_7) hold and $x_0 \in A_1$. Then, the sequence $\{x_k\}$ produced by (15) is well defined in $\mathscr{S}(x^*,\ R)$, stays in $\mathscr{S}(x^*,\ R)$ for each $k = 0,\ 1,\ 2,\ \ldots$ and is convergent to x^*. Moreover, for $\mathscr{P}_k = \|x_k - x^*\|$ and each $k = 0,\ 1,\ 2,\ \ldots$, the following assertions hold:

$$\|y_k - x^*\| \leq G_1(\mathscr{P}_k)\mathscr{P}_k \leq \mathscr{P}_k < R, \tag{17}$$

$$\|z_k - x^*\| \leq G_2(\mathscr{P}_k)\mathscr{P}_k \leq \mathscr{P}_k \tag{18}$$

and

$$\|x_{k+1}-x^*\| \leq G_3(\mathscr{P}_k)\mathscr{P}_k \leq \mathscr{P}_k. \tag{19}$$

Proof. The invertibility of the divided difference $\mathscr{T}$ is first established. Indeed, by (16) and the conditions (C_1)-(C_5), we have in turn that

$$\begin{aligned}\|M^{-1}(\mathscr{T}-M)\| \leq & W_0(\|U-x^*\|,\ \|V-x^*\|)\\ \leq & W_0(H_1(\|x-x^*\|),\ H_2(\|x-x^*\|)) \leq W_0(R,\ R) < 1,\end{aligned} \tag{20}$$

It follows by (20) and the standard Lemma on invertible operators attributed to Banach [4, 5, 18, 19] that $\mathscr{T}(x_0)$ is invertible, since $x_0 \in A_1$ and

$$\|\mathscr{T}^{-1}M\| \leq \frac{1}{1-W_0(\|U-x^*\|,\ \|V-x^*\|)}. \tag{21}$$

Thus, the iterate y_0 is well defined by the first substep of method (15). Then, we can also write

$$\begin{aligned}y_0-x^* &= x_0-x^*-A_0^{-1}F(x_0)\\ &= A_0^{-1}(A_0-[x_0,\ x^*;\ F])(x_0-x^*).\end{aligned} \tag{22}$$

In view of (16), the conditions (C_5), (C_6) and (22)

$$\begin{aligned}\|y_0-x^*\| &\leq \frac{W(\|U_0-x_0\|,\ \|V_0-x^*\|)\|x_0-x^*\|}{1-W_0(\|U_0-x^*\|,\ \|V_0-x^*\|)}\\ &\leq G_1(\mathscr{P}_0)\mathscr{P}_0 \leq \mathscr{P}_0 < R,\end{aligned} \tag{23}$$

where we used the calculation

$$\begin{aligned}U-x_0 &= \varpi F(x_0) = \varpi[x_0,\ x^*;\ F](x_0-x^*)\\ &= \varpi MM^{-1}[x_0,\ x^*;\ F](x_0-x^*)\\ &= \varpi MM^{-1}([x_0,\ x^*;\ F]-M+M)(x_0-x^*),\end{aligned}$$

so

$$\|U - x_0\| \le b(1 - W_0(\mathscr{P}_0,\ 0))\mathscr{P}_0.$$

Moreover, it follows by (23) that the item (17) holds for $k = 0$ and the iterate $y_0 \in A_0$. If the role of x_0 is replaced by y_0 and z_0, respectively in the second and third substep of the method (15), we get in turn that

$$\begin{aligned} \|z_0 - x^*\| &\le \frac{W(\|U_0 - y_0\|,\ \|V_0 - x^*\|)\|y_0 - x^*\|}{1 - W_0(\|U_0 - x^*\|,\ \|V_0 - x^*\|)} \\ &\le G_2(\mathscr{P}_0)\mathscr{P}_0 \le \mathscr{P}_0 \end{aligned} \tag{24}$$

$$\begin{aligned} \|x_1 - x^*\| &= \le \frac{W(\|U_0 - z_0\|,\ \|V_0 - x^*\|)\|z_0 - x^*\|}{1 - W_0(\|U_0 - x^*\|,\ \|V_0 - x^*\|)} \\ &\le G_3(\mathscr{P}_0)\mathscr{P}_0 \le \mathscr{P}_0. \end{aligned} \tag{25}$$

Thus, the assertions (18), (19) hold if $k = 0$ and the iterates z_0 and $x_1 \in A_0$. Simply switch the iterates $x_0,\ y_0,\ z_0,\ x_1$ by $x_i,\ y_i,\ z_i,\ x_{i+1}$, respectively in the preceding calculations to terminate the induction for the items (17)-(19). Then, from (19), we can have

$$\mathscr{P}_{i+1} \le c\mathscr{P}_i \le c^{i+1}\mathscr{P}_0 < R, \tag{26}$$

where

$$c = G_3(\mathscr{P}_0) \in [0,\ 1).$$

Therefore, we conclude that $\lim\limits_{i \to +\infty} x_i = x^*$. □

Next, a domain is specified where the only solution is x^*.

Proposition 1 Suppose: The condition (C_5) holds in $\mathscr{S}(x^*,\ q_0)$ for some $q_0 \ge 0$ and there exists $q \ge q_0$ such that

$$W_0(q,\ 0) < 1. \tag{27}$$

Set $A_2 = \Omega \cap \mathscr{S}[x^*,\ q]$. Then, there is no solution other than x^* in the domain A_2.

Proof. Suppose that there exists $y^* \in A_2$ solving the equation $F(x) = 0$ such that $y^* \neq x^*$. Define the divided difference $L_1 = [x^*,\ y^*;\ F]$. By using the condition (C_5) and (27), we obtain in turn that

$$\|M^{-1}(L_1 - M)\| \le W_0(\|x - x^*\|,\ 0) \le W_0(q,\ 0) < 1. \tag{28}$$

Hence, by (29), $L_1^{-1} \in \mathscr{L}(B_2,\ B_1)$. Moreover, by the identity

$$y^* - x^* = L_1^{-1}(F(y^*) - F(x^*)) = L_1^{-1}(0) = 0,$$

we deduce that $y^* = x^*$. □

Remark 3 Clearly, one can take $q_0 = R$ provided that all the conditions (C_1)-(C_7) hold in the Proposition 1.

3.2 *Semilocal analysis*

The calculations are similar to the local case. However, the role of x^*, W_0 and W is exchanged by $x_0 \in \Omega$, ω_0 and ω, respectively. Suppose:

(H_1) There exist FCND $H_3 : \mathscr{Q} \to \mathscr{Q}$, $H_4 : \mathscr{Q} \to \mathscr{Q}$ and $\omega_0 : \mathscr{Q} \times \mathscr{Q} \to \mathscr{Q}$ such that the equation $\omega_0(H_3(t), H_4(t)) - 1 = 0$ has SZP. Denote such zero by s_0 and set $\mathscr{Q}_2 = [0, s_0)$.

(M_2) There exist FCND $\omega : \mathscr{Q}_2 \times \mathscr{Q}_2 \to \mathscr{Q}$ and $M \in \mathscr{L}(B_1, B_2)$. Define the sequence $\{\alpha_k\}$ for some $\alpha_0 = 0$, some $\beta_0 \geq 0$ and each $k = 0, 1, 2, \ldots$ by

$$\gamma_k = \beta_k + \frac{\omega(\beta_k + H_3(\alpha_k), b(1+\omega_0(\alpha_k, 0))\alpha_k)(\beta_k - \alpha_k)}{1 - \omega_0(H_3(\alpha_k), H_4(\alpha_k))},$$

$$\alpha_{k+1} = \gamma_k + \frac{\omega(\gamma_k + H_3(\alpha_k), b(1+\omega_0(\beta_k, 0))\beta_k)(\gamma_k - \beta_k)}{1 - \omega_0(H_3(\alpha_k), H_4(\alpha_k))}, \tag{29}$$

$$\delta_{k+1} = (1+\omega_0(\alpha_k, \alpha_{k+1}))(\alpha_{k+1} - \alpha_k) + (1+\omega_0(H_3(\alpha_k)H_4(\alpha_k))(\beta_k - \alpha_k)$$

and

$$\beta_{k+1} = \alpha_{k+1} + \frac{\delta_{k+1}}{1 - \omega_0(H_3(\alpha_{k+1}), H_4(\alpha_{k+1}))}.$$

The sequence $\{\alpha_k\}$ is shown to be majorizing for $\{x_k\}$ (see Theorem 3). But let us first present a general convergence for it.

(H_3) There exist $s \in [0, s_0)$ such that for each $k = 0, 1, 2, \ldots$

$$\omega_0(H_3(\alpha_k), H_4(\alpha_k)) < 1 \text{ and } \alpha_k \leq s.$$

This condition and (29) imply that for each $k = 0, 1, 2, \ldots$

$$0 \leq \alpha_k \leq \beta_k \leq \gamma_k \leq \alpha_{k+1}$$

and there exists $s^* \in [0, s]$ such that $\lim_{k \to +\infty} \alpha_k = s^*$.

It is well known that s^* is the unique least upper bound of the sequence $\{\alpha_k\}$. As in the local case the functions ω_0 and ω relate to the divided differences on the method.

(H_4) There exist $x_0 \in \Omega$ and $M \in \mathscr{L}(B_1, B_2)$ such that $M^{-1} \in \mathscr{L}(B_2, B_1)$ and for each $x \in \Omega$

$$\|M^{-1}(\mathscr{T} - M)\| \leq \omega_0(\|U - x_0\|,\ \|V - x_0\|).$$

Set $A_3 = \Omega \cap \mathscr{S}(x_0,\ s_0)$.

It follows that $\mathscr{T}_0^{-1} \in \mathscr{L}(B_2,\ B_1)$, since

$$\|M^{-1}(\mathscr{T}_0 - M)\| \leq \omega_0(0,\ 0) < 1.$$

Hence, we can choose $\beta_0 \geq \|\mathscr{T}_0^{-1}F(x_0)\|$.

(H_5)

$$\|M^{-1}(\mathscr{T} - [y,\ x;\ F])\| \leq \omega(\|U - y\|,\ \|V - x\|)$$

$$\|U - x_0\| \leq H_3(\|x - x_0\|) \leq \|x - x_0\|$$

and

$$\|V - x_0\| \leq H_4(\|x - x_0\|) \leq \|x - x_0\|$$

for each $x,\ y \in A_3$, and

(H_6) $\mathscr{S}[x_0,\ s^*] \subset \Omega$.

Remark 4 (i) As in the local case some possible selections for $M = I$ or $M = F'(x_0)$ or $M = [\bar{x},\ \bar{y};\ F]$, where $\bar{x},\ \bar{y} \in \Omega$ are some auxiliary points.

(ii) Let us look at the motivational estimations for the selection of the functions H_3 and H_4:

$$U - x_0 = x + \varpi F(x) - x_0 = x - x_0 + \varpi(F(x) - F(x_0)) + \varpi F(x_0)$$

$$= (I + \varpi[x,\ x_0;\ F])(x - x_0) + \varpi F(x_0)$$

$$= \left(I + \varpi M M^{-1}([x,\ x_0;\ F] - M + M)\right)(x - x_0) + \varpi F(x_0)$$

$$= \left((I + \varpi M) + \varpi M M^{-1}([x,\ x_0;\ F] - F)\right)(x - x_0) + \varpi F(x_0).$$

So, we have

$$\|U - x_0\| \leq \left(\|I + \varpi M\| + b\omega_0(\|x - x_0\|,\ 0)\right)\|x - x_0\| + |\varpi|\|F(x_0)\|.$$

Thus, we can choose

$$H_3(t) = \big(\|I + \varpi M\| + b\omega_0(t,\ 0)\big)t + |\varpi|\|F(x_0)\|.$$

Similarly, we can choose for an upper bound on $\|V - x_0\|$:

$$H_4(t) = \big(\|I - \varpi M\| + b\omega_0(t,\ 0)\big)t + |\varpi|\|F(x_0)\|.$$

The semi-local analysis uses the conditions (H_1)-(H_6).

Theorem 3 Suppose that the conditions (H_1)-(H_6) hold. Then, the sequence $\{x_k\}$ generated by the method (15) is well defined in $\mathscr{S}[x_0,\ s^*]$ remains in $\mathscr{S}[x_0,\ s^*]$ for each $k = 0,\ 1,\ 2,\ \ldots$ and converges to a solution $x^* \in \mathscr{S}[x_0,\ s^*]$ of the equation $F(x) = 0$ such that

$$\|x_k - x^*\| \leq s^* - \alpha_k \quad \text{for each} \quad k = 0,\ 1,\ 2,\ \ldots. \tag{30}$$

Proof. The proof is based on induction to first show the estimates:

$$\|y_k - x_k\| \leq \beta_k - \alpha_k, \tag{31}$$

$$\|z_k - y_k\| \leq \gamma_k - \beta_k \tag{32}$$

and

$$\|x_{k+1} - z_k\| \leq \alpha_{k+1} - \gamma_k. \tag{33}$$

Item (31) holds if $k = 0$ by (29) and the choice of β_0 since $\|y_0 - x_0\| = \|\mathscr{T}_0^{-1}F(x_0)\| \leq \beta_0 = \beta_0 - \alpha_0 < s^*$. We also have that the iterate $y_0 \in \mathscr{S}[x_0,\ s^*]$. As in local case but using (H_4)

$$\|M^{-1}(\mathscr{T} - M)\| \leq \omega_0(H_3(\|x_i - x_0\|),\ H_4(\|x_i - x_0\|))$$

$$\leq \omega_0(H_3(\alpha_i),\ H_4(\alpha_i)) < 1 \quad (\text{by}(H_3)),$$

so $\mathscr{T}^{-1} \in \mathscr{L}(B_2,\ B_1)$ and

$$\|\mathscr{T}^{-1}M\| \leq \frac{1}{1 - \omega_0(H_3(\alpha_i),\ H_4(\alpha_i))}. \tag{34}$$

Then, we can write by first substep of the method (15)

$$F(y_i) = F(y_i) - F(x_i) - \mathscr{T}_i(y_i - x_i)$$

$$= ([y_i,\ x_i;\ F] - \mathscr{T}_i)(y_i - x_i)$$

leading by (H_5) and the induction hypotheses that

$$\|M^{-1}F(y_i)\| \leq \omega(\beta_i + H_3(\alpha_i),\ b(1 + \omega_0(\alpha_i,\ 0)\alpha_i)(\beta_i - \alpha_i), \tag{35}$$

where we also used the calculations

$$y_i - U_i = (y_i - x_0) + (x_0 - U_i),$$

$$\|y_i - U_i\| \leq \|y_i - x_0\| + \|x_0 - U_i\| \leq \beta_i + H_3(\alpha_i),$$

and

$$x_i - V_i = -\varpi F(x_i) = -\varpi M M^{-1}\left([x_i,\ x_0;\ F] - M + M\right)(x_i - x_0),$$

so

$$\|x_i - V_i\| \leq b\left(1 + \omega_0(\alpha_i,\ 0)\right)\alpha_i.$$

Thus, we get

$$\|z_i - y_i\| = \|\mathscr{T}_i^{-1}F(y_i)\| \leq \|\mathscr{T}_i^{-1}M\|\|M^{-1}F(y_i)\|$$

$$\leq \frac{\omega(\beta_i + H_3(\alpha_i),\ b(1 + \omega_0(\alpha_i,\ 0)\alpha_i))(\beta_i - \alpha_i)}{1 - \omega_0(H_3(\alpha_i),\ H_4(\alpha_i))} \tag{36}$$

$$= \gamma_i - \beta_i$$

and

$$\|z_i - x_0\| \leq \|z_i - y_i\| + \|y_i - x_0\|$$

$$= \gamma_i - \beta_i + \beta_i - \alpha_0 = \gamma_i < s^*.$$

Hence, the item (32) holds and the iterate $z_i \in \mathscr{S}[x_0,\ s^*]$.

Then, we can write

$$F(z_i) = F(z_i) - F(y_i) - \mathscr{T}_i(z_i - y_i)$$

as in (35) by exchanging $y_i,\ x_i$ by $z_i,\ y_i$, respectively, we get

$$\|M^{-1}F(z_i)\| \leq \omega(\gamma_i + H_3(\alpha_i),\ b(1+\omega_0(\beta_i,\ 0)\beta_i))(\gamma_i - \beta_i).$$

So, by the third substep of the method (15)

$$x_{i+1} - z_i = -\mathscr{T}_i^{-1}F(z_i),$$

so

$$\|x_{i+1} - z_i\| \leq \frac{\omega(\gamma_i + H_3(\alpha_i),\ b(1+\omega_0(\beta_i,\ 0)\beta_i))(\gamma_i - \beta_i)}{1 - \omega_0(H_3(\alpha_i),\ H_4(\alpha_i))} \tag{37}$$

$$\alpha_{i+1} - \gamma_i$$

and

$$\begin{aligned}\|x_{i+1} - x_0\| \leq & \|x_{i+1} - z_i\| + \|z_i - x_0\| \\ = & \alpha_{i+1} - \gamma_i + \gamma_i - \alpha_0 = \alpha_i < s^*.\end{aligned}$$

Thus, the estimate (33) holds and the iterate $x_{i+1} \in \mathscr{S}[x_0,\ s^*]$.

Next, we can write

$$\begin{aligned}F(x_{i+1}) &= F(x_{i+1}) - F(x_i) - \mathscr{T}_i(y_i - x_i) \\ &= ([x_{i+1},\ x_i;\ F] - M + M)(x_{i+1} - x_i) - (\mathscr{T}_i - M + M)(y_i - x_i)\end{aligned}$$

leading to

$$\begin{aligned} \|M^{-1}F(x_{i+1})\| \leq & (1+\omega_0(\|x_i-x_0\|,\ \|x_{i+1}-x_0\|))\|x_{i+1}-x_i\| \\ & +(1+\omega_0(H_3(\|x_i-x_0\|),\ H_4(\|x_i-x_0\|)))\|y_i-x_i\| \\ \leq & (1+\omega_0(\alpha_i,\ \alpha_{i+1}))\|\alpha_{i+1}-\alpha_i\| \\ & +(1+\omega_0(H_3(\alpha_i),\ H_4(\alpha_i)))\|\beta_i-\alpha_i\| = \delta_{i+1}. \end{aligned} \tag{38}$$

The estimate (38) and

$$y_{i+1}-x_{i+1} = -\mathscr{T}_{i+1}^{-1}F(x_{i+1}) = -\mathscr{T}_{i+1}^{-1}MM^{-1}F(x_{i+1})$$

leading to

$$\begin{aligned} \|y_{i+1}-x_{i+1}\| & \leq \|\mathscr{T}_{i+1}^{-1}M\|\|M^{-1}F(x_{i+1})\| \\ & \leq \frac{\delta_{i+1}}{1-\omega_0(H_3(\alpha_{i+1}),\ H_4(\alpha_{i+1}))} = \beta_{i+1}-\alpha_{i+1} \end{aligned} \tag{39}$$

and

$$\begin{aligned} \|y_{i+1}-x_0\| \leq & \|y_{i+1}-x_{i+1}\|+\|x_{i+1}-x_0\| \\ \leq & \beta_{i+1}-\alpha_{i+1}+\alpha_{i+1}-\alpha_0 = \beta_{i+1} < s^*. \end{aligned}$$

Hence, the item (31) holds for $i+1$ replacing i and the iterate $y_{i+1} \in \mathscr{S}[x_0,\ s^*]$.

The induction for items (31)-(33) is terminated and all the iterates $z_i,\ y_i,\ x_i \in \mathscr{S}[x_0,\ s^*]$. It also follows by (36), (37) and (39) that the sequence $\{x_i\}$ is fundamental in Banach space B_1. Hence, there exists $x^* \in \mathscr{S}[x_0,\ s^*]$ such that $\lim_{i\to+\infty} x_i = x^*$. By letting $i \to +\infty$ in (38) and the continuity of the operator F, we deduce that $F(x^*) = 0$. Moreover by the estimate for $k = 0,\ 1,\ 2,\ \ldots$

$$\|x_{i+k}-x_i\| \leq \alpha_{i+k}-\alpha_i \tag{40}$$

and $k \to +\infty$, we conclude that (30) holds. □

Next, a domain is specified in Ω with any one solution.

Proposition 2 Suppose: There exists a solution $t_0 \in \mathscr{S}(x_0,\ s_1)$ for some $s_1 > 0$; The condition (H_4) holds in $\mathscr{S}(x_0,\ s_1)$ and there exists $s_2 \geq s_1$ such that

$$\omega_0(s_2,\ 0) < 1. \tag{41}$$

Set $A_4 = \Omega \cap \mathscr{S}[x_0,\ s_2]$. Then, t_0 is the only solution of the equation $F(x) = 0$ in the domain A_4.

Proof. Suppose there exists $w \in A_4$ such that $F(w) = 0$ and $w \neq t_0$. Then, the divided difference $L = [t_0,\ w;\ F]$ is well defined. By using the condition (H_4) and (41), we get in turn

$$\|M^{-1}(L-M)\| \leq \omega_0(\|w-t_0\|,\ 0) \leq \omega_0(s_2,\ 0) < 1.$$

Thus, $L^{-1} \in \mathscr{L}(B_2,\ B_1)$. Then, from the identity

$$w - t_0 = L^{-1}(F(w) - F(t_0)) = L^{-1}(0) = 0.$$

Therefore, we conclude that $w = t_0$. □

Remark 5 (i) The limit point s^* can be replaced by s_0 (see (H_1)) in the condition (H_6).

(ii) Suppose that all the conditions (H_1)-(H_6) in the Proposition 2 are validated. Then, we can set $t_0 = x^*$ and $s_1 = s^*$.

4. Efficiency index

For making comparison between different iterative methods, classical measure of efficiency, the efficiency index, proposed by Ostrowski is most widely used with the expression

$$I_E = p^{\frac{1}{d}},$$

where p is the convergence order of the method and d represents the number of functional evaluations needed to perform the method per iteration.

Operational efficiency index, proposed by Traub, is the another measure of efficiency of iterative methods whose formula is the following:

$$I_O = p^{\frac{1}{oe}},$$

where oe represents the number of operations needed to calculate each iteration and is expressed in units of product.

In numerous instances, combination of the efficiency index and the operational efficiency is used, called the computational efficiency index with expression

$$I_{ce} = p^{\frac{1}{d+oe}}.$$

In this section, different indices are studied for the proposed parametric family and results are compared with similar existing methods in literature, which are $M42$ given by Cordero et al. [16], $M51$ given by Cordero et al. [20], $M61$ given by Wang and Fan [7] and $M71$ given by Wang and Zhang [8].

While working on $m \times m$ systems, for calculating $F(W)$ and the divided difference operator, the number of functional evaluations used are m and $m^2 - m$, respectively.

The number of products and quotients needed to perform the operations are as given below:

• The cost of one scalar or vector product and one transpose vector/vector product is m each.

• The cost of matrix/vector product is m^2 whereas that of matrix/matrix product is m^3.

• The number of quotients of divided difference operator is m^2.

• The cost of each LU decomposition is $\frac{1}{3}(m^3 - m)$.

• The cost of each system resolution is m^2.

In case of proposed parametric family for $\varpi = 1$, which is denoted by $M41$, the number of functional evaluations in computing $F(x_k)$, $F(y_k)$, $F(z_k)$, $F(x_k + rF(x_k))$, $F(x_k) - rF(x_k)$ and one divided difference operator is:

$$5m + m^2 - m = m^2 + 4m.$$

For the calculation of a scalar/vector product, a divided difference operator, one LU decomposition and solving three systems, the number of operations required is:

$$m + m^2 + \frac{1}{3}(m^3 - m) + m + 3m^2.$$

The sum of evaluations and operations is:

$$m + m^2 + \frac{1}{3}(m^3 - m) + 3m^2 + m^2 + 4m = \frac{1}{3}(m^3 - m) + 5m^2 + 5m.$$

The the iterative expression for $M42$ [16]:

$$y_k = x_k - [x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F]^{-1} F(x_k),$$

$$x_{k+1} = y_k - [x_k + \varpi F(x_k),\ x_k - \varpi F(x_k);\ F]^{-1} (p_k F(y_k) + q_k F(x_k)),$$

where $\varpi \in \mathbb{R} \setminus \{0\}$ and $v_k = \dfrac{F(y_k)^{tr} F(y_k)}{F(x_k)^{tr} F(x_k)}$, $K_k = \dfrac{1}{1 + \lambda v_k}$, $p_k = K_k(1 + \psi v_k)$ and $q_k = 2K_k v_k$.

It requires 4 functional evaluations and one divided difference operator. The number of evaluations is:

$$4m + m^2 - m = m^2 + 3m.$$

It performs three scalar/vector products, two transpose vector/vector products, a divided difference operator, a single LU decomposition and two systems are solved. The number of operations is:

$$3m+2m+m^2+\frac{1}{3}(m^3-m)+2m^2=\frac{1}{3}(m^3-m)+3m^2+5m.$$

The evaluations and operations adds to give:

$$\frac{1}{3}(m^3-m)+3m^2+5m+m^2+3m=\frac{1}{3}(m^3-m)+4m^2+8m.$$

The following is the iterative expression for $M51$ [20]:

$$y_k=x_k-[a_k,\ b_k;\ F]^{-1}F(x_k),$$

$$z_k=y_k-\alpha[a_k,\ b_k;\ F]^{-1}F(y_k),$$

$$t_k=z_k-\beta[a_k,\ b_k;\ F]^{-1}F(z_k),$$

$$x_{k+1}=z_k-\gamma[a_k,\ b_k;\ F]^{-1}F(t_k),$$

where $a_k=x_k+F(x_k)$ and $b_k=x_k-F(x_k)$. It requires 6 functional evaluations and one divided difference operator. The number of evaluations is:

$$6m+m^2-m=m^2+5m.$$

It performs three scalar/vector products, a divided difference operator, a single LU decomposition and four systems with same matrix are solved. The number of operations is:

$$3m+m^2+\frac{1}{3}(m^3-m)+4m^2=\frac{1}{3}(m^3-m)+5m^2+3m.$$

The addition of evaluations and operations yields:

$$m^2+5m+\frac{1}{3}(m^3-m)+5m^2+3m=\frac{1}{3}(m^3-m)+6m^2+8m.$$

The following is the iterative expression for $M61$ [7]:

$$y_k = x_k - [w_k,\ s_k;\ F]^{-1}F(x_k),$$

$$\mu_1 = (3I - 2[w_k,\ s_k;\ F]^{-1}[y_k,\ x_k;\ F(x_k)])[w_k,\ s_k;\ F]^{-1},$$

$$z_k = y_k - \mu_1 F(y_k),$$

$$x_{k+1} = z_k - \mu_1 F(z_k),$$

where $w_k = x_k + F(x_k)$ and $s_k = x_k - F(x_k)$. It requires five functional evaluations and two divided difference operator. The number of evaluations is:

$$5m + 2m^2 - 2m = 2m^2 + 3m.$$

It performs a matrix multiplication, a scalar matrix multiplication, a vector matrix multiplication, two divided difference operator, and three systems with same coefficient matrix are solved. The number of operations is:

$$2m^2 + \frac{1}{3}(m^3 - m) + 3m^2 + m^2 + m^2 + m^3 = \frac{1}{3}(m^3 - m) + m^3 + 7m^2.$$

The addition of evaluations and operations yields:

$$2m^2 + 3m + \frac{1}{3}(m^3 - m) + m^3 + 7m^2 = \frac{1}{3}(m^3 - m) + m^3 + 9m^2 + 3m.$$

The following is the iterative expression for $M71$ [8]:

$$y_k = x_k - [w_k,\ x_k;\ F]^{-1}F(x_k),$$

$$z_k = y_k - ([y_k,\ x_k;\ F] + [y_k,\ w_k;\ F] - [w_k,\ x_k;\ F])^{-1} F(y_k),$$

$$x_{k+1} = z_k - ([z_k,\ x_k;\ F] + [z_k,\ y_k;\ F] - [y_k,\ x_k;\ F])^{-1} F(z_k),$$

where $w_k = x_k + F(x_k)$. It requires four functional evaluations and five divided difference operator. The number of evaluations is:

$$4m + 5m^2 - 5m = 5m^2 - m.$$

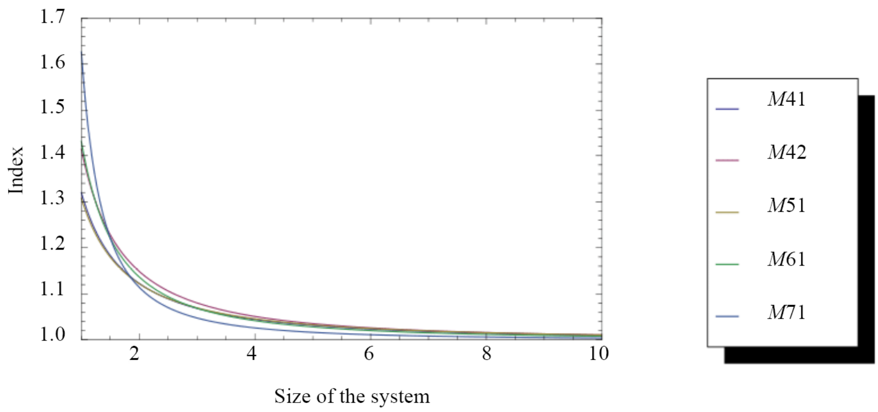
It performs calculations of five divided difference operator, two LU decompositions and one system is solved with each of decomposition. The number of operations is:

$$5m^2+\frac{2}{3}(m^3-m)+2m^2=\frac{2}{3}(m^3-m)+7m^2.$$

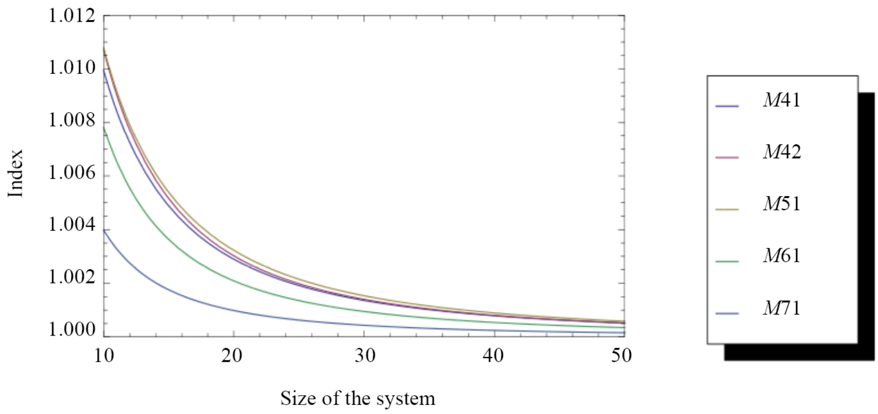
The addition of evaluations and operations yields:

$$5m^2-m+\frac{2}{3}(m^3-m)+7m^2=\frac{2}{3}(m^3-m)+12m^2-m.$$

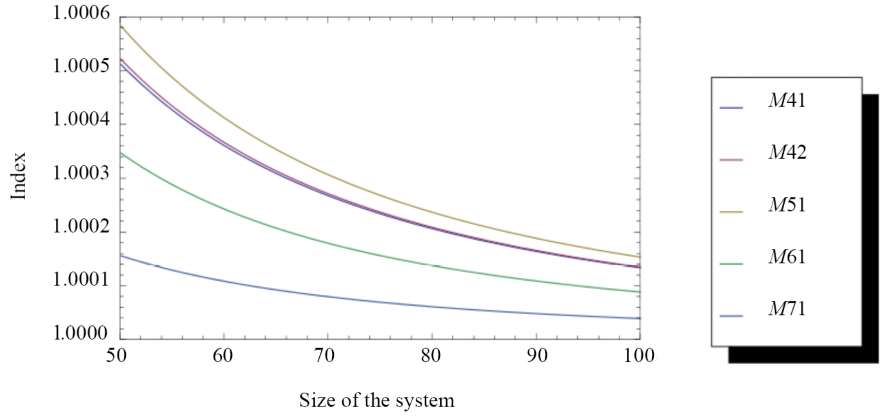
The figures below represent the comparison of efficiency index, operational index and the computational efficiency of methods $M41$, $M42$, $M51$, $M61$ and $M71$.



(a) For $m=1$ to $m=10$

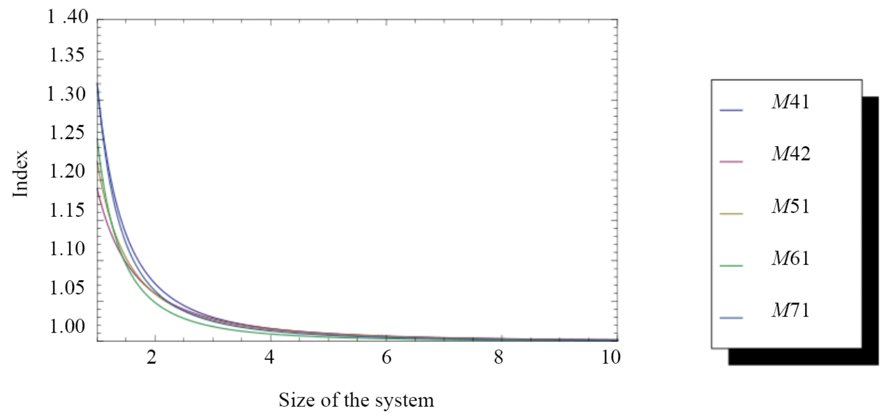


(b) For $m=10$ to $m=50$

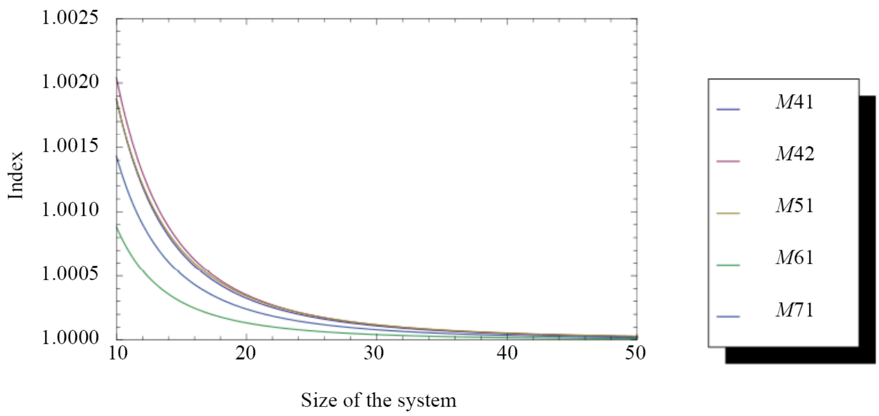


(c) For $m = 50$ to $m = 100$

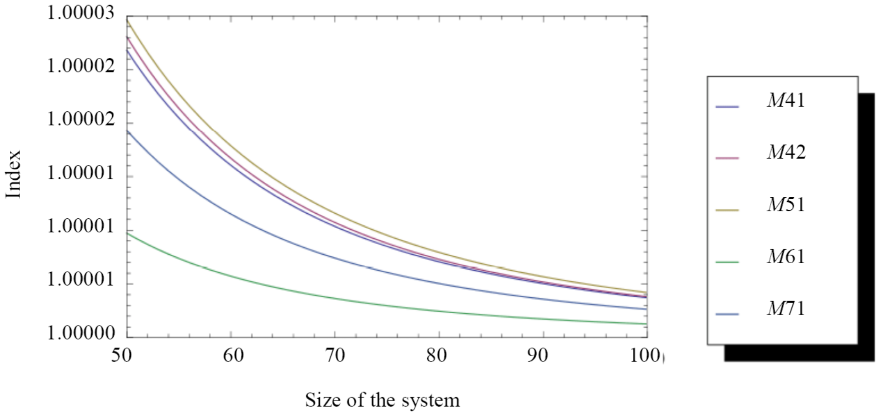
Figure 1. Efficiency index (a) For $m = 1$ to $m = 10$, (b) For $m = 10$ to $m = 50$, (c) For $m = 50$ to $m = 100$



(a) For $m = 1$ to $m = 10$

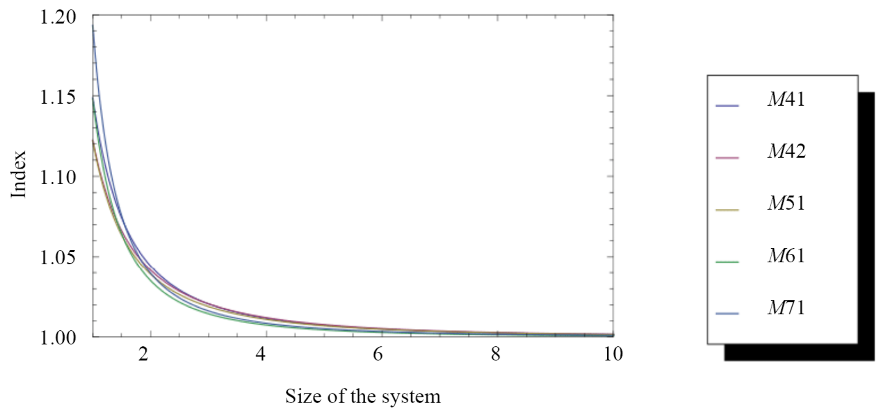


(b) For $m = 10$ to $m = 50$

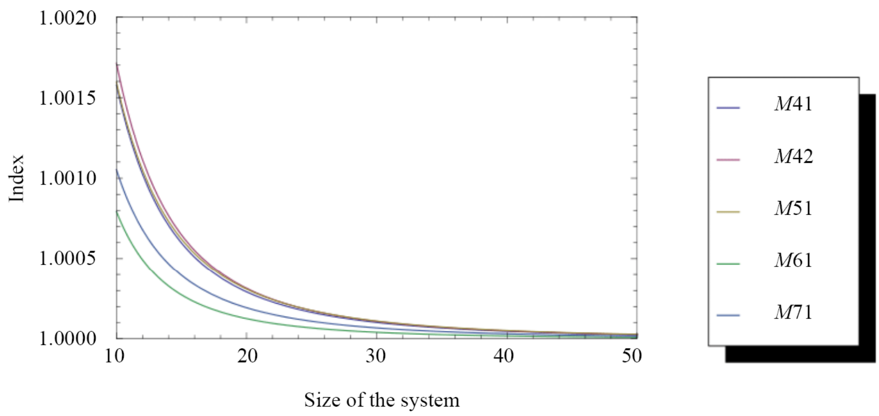


(c) For $m = 50$ to $m = 100$

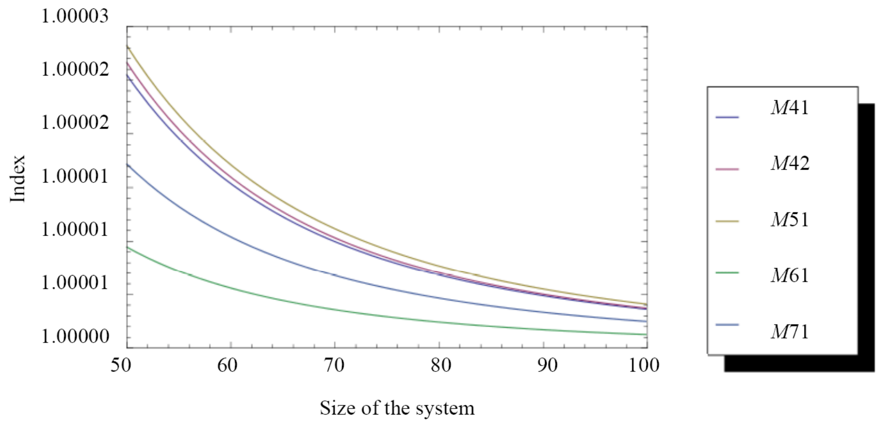
Figure 2. Operational efficiency index (a) For $m = 1$ to $m = 10$, (b) For $m = 10$ to $m = 50$, (c) For $m = 50$ to $m = 100$



(a) For $m = 1$ to $m = 10$



(b) For $m = 10$ to $m = 50$



(c) For $m = 50$ to $m = 100$

Figure 3. Computational efficiency index (a) For $m = 1$ to $m = 10$, (b) For $m = 10$ to $m = 50$, (c) For $m = 50$ to $m = 100$

In Figure 1, the efficiency index of various methods is illustrated for different system sizes. One can see from these figures that $M41$ is better for small size and is competitive with $M42$ and $M51$ for size greater than or equal to 10.

In Figure 2, the operational efficiency index of different methods is illustrated for different system sizes. It is clear from these figures that $M41$ takes lead for small size and is competitive with $M42$ and $M51$ for systems of greater size.

In Figure 3, the computational efficiency index of various methods is demonstrated for different system sizes. $M41$, $M42$ and $M51$ stand out for system sizes greater than or equal to 1.

5. Numerical performance

This section is devoted to providing some numerical applications. The comparison of performance of $M41$, taking $\varpi = 3.1$, some methods of $M42$ class, namely $M421$, $M422$ and $M423$ with values of parameters $\lambda = \{-4, -5, 0\}$, respectively and $r = 1$, $\psi = 0$ in all cases, $M51$, $M61$ and $M71$ is made. The stopping criterion is $\|x_{k+1} - x_k\| < 10^{-100}$ with maximum 50 iterations. Each calculation is performed using Mathematica 8 on Intel(R) Core(TM) $i5-8250U$ CPU @ $1.60GHz$ $1.80GHz$ with 8GB of RAM running on the Windows 10 Pro version 2017 using multiple-precision arithmetic with 4096 digits. The approximate computational order of convergence is given by formula [10]:

$$\rho = \frac{\ln \|x_{k+1} - x_k\| / \|x_k - x_{k-1}\|}{\ln \|x_k - x_{k-1} / x_{k-1} - x_{k-2}\|}$$

Example 1 [16] Consider the system of 200 equations:

$$\sum_{l=1}^{n} x_l - x_j - e^{x_j} + 4\cos(2\ln(|x_j + 1|)) - 3 = 0, \quad j = 1, 2, \ldots, l,$$

where l is taken as 200 and the initial approximation as $x_0 = \left(\frac{1}{100}, \ldots, \frac{1}{100}\right)^{tr}$ to obtain the solution $x^* = (0, 0, \ldots, 0)^{tr}$. Table 1 displays the results established by different methods and clearly shows that the proposed method $M41$ takes the least CPU time to reach the required tolerance, being significantly less than other methods that double their approximate computational order of convergence. Moreover, it requires the same iterations as the other methods with the higher order of convergence and thus performs better than all other existing methods.

Table 1. Comparison of the performances of methods for Example 1

Method	Iter	$\|F(x_k)\|$	$\|x_{k+1}-x_k\|$	ρ	CPU Time
$M41$	3	$6.27e-110$	$9.03e-028$	4.53	1.04
$M421$	3	$6.97e-106$	$7.99e-027$	4.14	1.91
$M422$	3	$6.51e-106$	$7.85e-027$	4.14	1.94
$M423$	3	$9.13e-106$	$8.55e-027$	4.14	1.66
$M51$	3	$1.93e-253$	$4.10e-050$	5.57	2.39
$M61$	3	$7.90e-320$	$1.08e-053$	6.07	4.14
$M71$	3	$9.45e-507$	$5.38e-097$	8.81	5.00

Example 2 Let $B_1 = B_2 = \mathbb{R}^2$, equipped with norm $\|.\|_\infty$. We shall solve the two by two nonlinear system with absolute values given as

$$3u^2v+v^2-1+|u-1| = 0$$

$$u^4+uv^3-1+|v| = 0.$$

Set $\|u\|_\infty = \|(u_1,\ u_2)\|_\infty = \max\{|u_1|,\ |u_2|\},\ u_1,\ u_2 \in \mathbb{R}$, $F = (F_1+G_1,\ F_2+G_2)$, where

$$F_1(u_1,\ u_2) = 3u_1^2u_2+u_2^2-1,$$

$$F_2(u_1,\ u_2) = u_1^4+u_1u_2^3-1,$$

$$G_1(u_1,\ u_2) = |u_1-1| \text{ and } G_2(u_1,\ u_2) = |u_2|.$$

The standard matrix $M_{2\times 2}(\mathbb{R})$ replaces $[x,\ y;\ G]$ as

$$[x,\ y;\ G]_{i,\ 1} = \frac{G_i(y_1,\ y_2)-G_i(x_1,\ y_2)}{y_1-x_1},$$

$$[x,\ y;\ G]_{i,\ 2} = \frac{G_i(x_1,\ y_2)-G_i(x_1,\ x_2)}{y_2-x_2},\ i = 1,\ 2,\ x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}.$$

Using Newton's method (2), with $x_0 = (1,\ 0)$ we obtain the Table 2.

Table 2. Performance of Newton's method (2) for example 2

k	$x_k^{(1)}$	$x_k^{(2)}$	$\|x_k - x_{k-1}\|$
0	1	0	
1	1	0.333333333333333	$3.333e-1$
2	0.906550218340611	0.354002911208151	$9.344e-2$
3	0.885328400663412	0.338027276361322	$2.122e-2$
4	0.891329556832800	0.326613976593566	$1.141e-2$
5	0.895238815463844	0.326406852843625	$3.909e-3$
6	0.895154671372635	0.327730334045043	$1.323e-3$
7	0.894673743471137	0.327979154372032	$4.809e-4$
8	0.894598908977448	0.327865059348755	$1.140e-4$
9	0.894643228355865	0.327815039208286	$5.002e-5$
10	0.894659993615645	0.327819889264891	$1.676e-5$
11	0.894657640195329	0.327826728208560	$6.838e-6$
12	0.894655219565091	0.327827351826856	$2.420e-6$
13	0.894655074977661	0.327826643198819	$7.086e-7$
...			
39	0.894655373334687	0.327826511746298	$5.149e-19$

Next, using the Secant method

$$x_{k+1} = x_k - [x_{k-1},\ x_k;\ F]^{-1} F(x_k) \tag{42}$$

with $x_{-1} = (1,\ 0)$ and $x_0 = (5,\ 5)$, we obtain the Table 3.

Table 3. Performance of Secant method (42) for example 2

k	$x_k^{(1)}$	$x_k^{(2)}$	$\|x_k - x_{k-1}\|$
0	5	5	
1	1	0	$5.000e-00$
2	0.989800874210782	0.012627489072365	$1.262e-02$
3	0.921814765493287	0.307939916152262	$2.953e-01$
4	0.900073765669214	0.325927010697792	$2.174e-02$
5	0.894939851625105	0.327725437396226	$5.133e-03$
6	0.894658420586013	0.327825363500783	$2.814e-04$
7	0.894655375077418	0.327826521051833	$3.045e-04$
8	0.894655373334698	0.327826521746293	$1.742e-09$
9	0.894655373334687	0.327826521746298	$1.076e-14$
10	0.894655373334687	0.327826521746298	$5.421e-20$

Finally, using our method (5) with $\varpi = 1$, and $x_0 = (1,\ 5)$, we get the Table 4.

Table 4. Performance of proposed method (5) for example 2

k	$x_k^{(1)}$	$x_k^{(2)}$	$\|x_k - x_{k-1}\|$
0	5	5	
1	1	0	5
2	0.909090909090909	0.363636363636364	$3.363e-01$
3	0.894886945874111	0.329098638203090	$3.453e-02$
4	0.894655531991499	0.327827544745569	$1.271e-03$
5	0.894655373334793	0.327826521746906	$1.022e-06$
6	0.894655373334687	0.327826521746298	$6.089e-13$
7	0.894655373334687	0.327826421746298	$2.710e-20$

It follows that the solution of the system is $x^* = (0.894655373334687,\ 0.327826421746298)^{tr}$. Notice that the new method (5) is faster and cheaper than the competing ones.

6. Conclusion

In the foregoing study, a parametric family of iterative methods of fourth-order convergence order free from Jacobian has been presented in this work. This class has been developed from the family defined in [15] so as to maintain the properties by modification in the Jacobian matrix by a divided difference operator that also makes it suitable for non-differentiable problems. The extended convergence, including local and semilocal, has been discussed using weaker convergence conditions which extends its applicability. A comparison of computational efficiencies of the new scheme with existing schemes is shown. It is proved that the proposed scheme is competitive as compared to other known methods in the literature with similar characteristics [6–8, 11]. In the future, we plan to use the ideas of this article to extend the applicability of the work in [7, 8, 10].

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Conflict of interest

The authors declare that they do not have conflict of interests.

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