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Research Article

Analyzing Feedback $M/G/1$ Double Retrial Orbit with Two-Phase Optional Service and Repair under Working Vacation Policy

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Abstract: This research presents a comprehensive analysis of the $M/G/1$ double retrial queue model with a two-phase optional service and repair, under a working vacation policy. The model addresses scenarios where patients are willing to pay more for enhanced comfort and care. To accommodate this, an optional service phase is introduced, allowing patients to choose between ordinary and premium service levels. The system is further refined by a two-phase service mechanism: in the first phase, patients receive an initial level of service, and depending on their satisfaction or specific needs, they may opt for an enhanced second phase of service. Additionally, the model incorporates feedback and repair processes, reflecting a more realistic representation of queue dynamics. Patients who are dissatisfied with the service or encounter issues can re-enter the system for further attention. The non-Markovian model equations are solved using the probability-generating function approach, which facilitates the analysis of complex queue behaviors and the calculation of key performance indicators, such as expected system length and queue length. The paper includes detailed numerical results with test data to demonstrate the accuracy of the model's predictions and validate the proposed system design's effectiveness.

Keywords: double orbit, two-phase optional service, repair, feedback, working vacations

MSC: 65L05, 34K06, 34K28

1. Introduction

In the queueing model, Singh and Kaur [1] explored a retrial unreliable server queue with batch arrivals, considering arrival rates, potential breakdowns, repair or vacation possibilities, and governing equations using variables. Nila and Sumitha [2] examined a bulk arrival retrial model with impatient clients, a Bernoulli vacation, feedback, and server failure. Along with a numerical analysis of system properties, the study investigates two stages of heterogeneous service, customer feedback, steady-state solutions, and performance measures. Melikov et al. [3] investigated a multi-server retrial queuing model with feedback based on Markovian Arrival Process (MAP) and exponential service time distributions, presenting steady-state analysis, numerical examples, and economic implications. Saravanan et al. [4] introduced a Markovian retrial queueing system featuring optional service, an unstable server, balking, and feedback. It explores the implications of system performance measurements and factors, with an emphasis on customers' options for immediate

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service, retrying, requesting a second optional service, and immediate repair. Boualem [5] studied a non-Markovian model that allows retrial times, enabling customers to decide not to join the line. Customers can wait or return if service is interrupted, but the server cannot start serving other customers until the interrupted service is completed. Zirem et al. [6] explored a batch arrival queue using a Poisson process and a first-come-first-served retrial policy, applicable to cognitive radio networks and manufacturing systems. They included supplementary variables and numerical examples. Sundararaman et al. [7] investigated working vacation queueing models, analyzing system performance and customer service characteristics in a pandemic context, incorporating regular and retrial waiting queues. Gao et al. [8] introduced a non-Markovian retrial model with busy breakdowns, utilizing the Markov chain approach and probability generating functions (PGF) to examine stable circumstances, performance measures, and reliability indices. Revathi [9] examined a server queuing system within a retrial context, emphasizing optional re-service, customer search, delayed repairs, modified Bernoulli vacations, and various additional strategies. Han et al. [10] investigated server breakdown behavior in an $M/M/1$ retrial queue, examining the implications of idle and busy times, Nash equilibrium, and socially desirable solutions. Kumar [11] examined a single-server queuing approach with positive and negative client types, analyzing performance indices and parameter sensitivity using variable approaches and numerical examples. Rajadurai et al. [12] studied a queueing structure with feedback on a single server trial and several vacations, assessing its performance during vacations and providing steady-state PGF performance measurements. Sundarapandiyan and Nandhini [13] explored a non-Markovian feedback G-queue with delayed repair, which includes Poisson processes, Bernoulli working vacations, delay time, steady-state PGF, system performance, reliability indicators, and a stochastic decomposition feature. Rani et al. [14] presented the Markovian queueing model for service systems, focusing on orbital search and system disaster. It employs a golden section search for optimal parameters as well as an adaptive neuro-fuzzy inference system. Kim and Kim [15] explored retrial queuing systems, concentrating on factors such as queue length, waiting time, tail asymptotics, and stability analysis. Their findings apply to telephone systems, contact centers, telecommunication networks, and daily life situations. Lisovskaya et al. [16] investigated a multi-server retrial model with two orbits, focusing on both positive and negative customers. The service time is exponentially distributed and varies by customer type. If resources are scarce, consumers are assigned to one of two orbits, with negative customers being removed from the system. The study uses asymptotic and numerical analysis to examine the system's performance measures. Baskar and Saravanarajan [17] introduced a double-orbit retrial queue model to address patient dissatisfaction in healthcare. They use a probability-generating function approach and non-Markovian characteristics, assessing system length, expected queue length, and numerical accuracy, with an emphasis on personalized service options. Jain and Metha [18] explored the optimal design of an unstable server retrial queue with two retrial orbits to mitigate customer discontent and improve performance simulation. The study considers how much consumers are willing to pay for greater comfort and improved service. The Adaptive Neuro-Fuzzy Interface System (ANFIS) soft computing approach and probability-generating function method are used to determine the system's performance measures. Ganasekar and Kandaiyan [19] presented a non-Markovian retrial queuing system featuring feedback and delayed repair under a working vacation policy. The study includes an analysis of transition representations, consumer balking, and reneging of the steady-state probability generating function, key performance metrics, and the effects of various system configurations. Arivudainambi and Gowsalya [20] investigated a server retrial queueing system with dual service types and Bernoulli vacation, assessing its size probabilities, parameter impacts, and special circumstances numerically calculated. Murugan and Keerthana [21] examined a waiting server with an exponential distribution, while the retrial and service times follow a generic distribution. Customers are served at a reduced rate during their vacations. The PGF is calculated using the total number of clients and the average number in the invisible waiting area. Kalaiselvi and Saravanarajan [22] studied the dynamics of a retrial queue, analyzing its performance metrics, cost function, and convergence analysis, providing insights for improving queueing system efficiency.

Our investigation into the double orbit retrial queue was motivated by a thorough review of the existing literature, which has highlighted the importance of unreliable servers in queuing systems. While much of the research in this area has addressed different facets of retrial queues and server reliability, there has been no comprehensive study examining the $M/G/1$ double orbit retrial queue model in conjunction with several key features in the systems. Specifically, the literature has not yet explored the integration of a working vacation policy, two-phase optional service phases, repair processes, and feedback mechanisms within the context of a double-orbit retrial queue. The structure of the article is as follows: Section

2 provides a detailed description and analysis of the system, including the application of the model. Section 3 offers an overview of the steady-state analysis, focusing on the calculation of the system's patient count, the number of patients in the orbit, and the associated system states. Section 4 discusses the system's performance measurements. Section 5 examines key exceptional scenarios within the model. Finally, Section 6 presents numerical results to demonstrate the different parameters that influence the system's performance metrics.

2. Model description and analysis

This study uses a non-Markovian model of feedback double retrial orbit with two-phase optional service and repair under a working vacation policy to evaluate performance indices. Patients using the retry queuing system can wait in either of the two orbits, premium or ordinary, based on their financial situation and the facility's demands if the server is busy. The foundation for patient care is First Come First Serve (FCFS). Figure 1 shows the structure of the queueing model. The following mathematical formulation of the created retrial queueing system is predicated on the following hypotheses:

The arrival process: A Poisson process with a rate of κ is used to enter patients into the system.

The retrial process: In the given scenario, when a new patient arrives and finds the server available, they will immediately utilize the server's services. However, if the server is occupied or on vacation, the patient can choose to leave the service area and join a group of patients in what is called "orbit". Upon arrival, patients may enter one of two trial orbits: the premium or the ordinary orbit. The server provides service to patients according to general distribution, with service being provided to patients from the premium and ordinary orbits first. If a patient from the ordinary orbit finds the server free, they can retry for service at a rate of κ. Similarly, if a patient from the premium orbit discovers the server is available, they can retry for service at a rate of ζ. Patients join the ordinary orbit with a probability of σ, while the premium orbit patients join with the complementary probability $\bar{\sigma} = 1 - \sigma$. The time between retrial attempts for patients in the ordinary orbit follows an arbitrary distribution $U_1(x)$, with a Laplace-Stieltjes transform (LST) denoted as $U_1^*(\theta)$. Likewise, the inter-retrial time for patients in the premium orbit is governed by a different arbitrary distribution $U_2(x)$, with its LST represented as $U_2^*(\theta)$.

The Service Process: A single server handles both phases of the service process. When a patient arrives and finds the server available, they begin the first phase of service. Upon completing this primary service, the patient has the option to continue with a secondary service. This secondary phase is optional, meaning the patient can either leave the system after the first phase with probability $(1-p)$ (or) choose to proceed to the secondary phase with probability $p(0 \leq p \leq 1)$. If the patient opts for the secondary service, it follows immediately after the primary phase. The duration of both the primary and secondary services are generally distributed, described by the random variables S_1 and S_2, with distribution functions $S_1(\varphi)$ and $S_2(\varphi)$, and their respective Laplace-Stieltjes transforms (LST) $S_1^*(\theta)$ and $S_2^*(\theta)$. While $E(S_2)$ and $E(S_2)^2$ denote the first and second moments of the secondary service, $E(S_1)$ and $E(S_1)^2$ denote the first and second moments of the primary service.

Working vacation procedures: The server takes a vacation, with an exponential distribution with parameter V applied for the duration when the orbits are empty. The server continues to serve patients at a reduced rate when they arrive during this vacation; this situation is referred to as a "working vacation". The service procedure slows down while someone is on a working vacation. During this period, the first and second moments are represented by $E(V)$ and $E(V)^2$, respectively, while the service time is represented by a general random variable ψ_υ. $\psi_\upsilon(x)$ is the distribution function for this service time, and $\psi_\upsilon^*(\theta)$ is the Laplace-Stieltjes transform (LST) for it.

Feedback Rule: When patients obtain their normal services but are unsatisfied, they face two possible actions based on their level of dissatisfaction. These actions are governed by the probability of quitting or rejoining the system as a feedback patient. Specifically, a dissatisfied patient can exit the system altogether with a probability $\bar{\beta} = (1-\beta)$, effectively deciding not to pursue any further services. Alternatively, with probability $\beta(0 \leq \beta \leq 1)$, the patient may choose to stay within the ordinary orbit, but this time as a feedback patient. As a feedback patient, they re-enter the service process, receiving another opportunity to obtain the service in hopes of achieving satisfaction.

Break down and Repair Policy: During regular busy periods, servers may experience short-term failure due to exogenous Poisson processes with rate γ, resulting in a potential disaster. When a server malfunctions, it gets shipped for repair right away. The server refuses to serve incoming patients in this time frame and remains inactive until the repair is finished. The length of time the repair takes is determined by a probability distribution $G(x)$. Its initial moment, or expected value, is represented by $E(G)$, and its Laplace-Stieltjes Transform (LST) is $G^*(\theta)$.

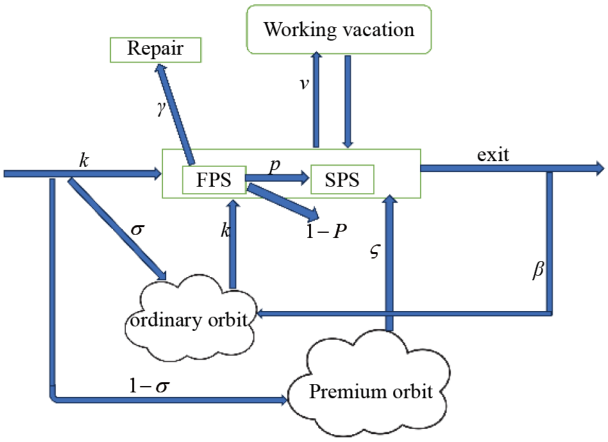

Figure 1. Structure of the queuing model

2.1 *Using the model in practice*

A single server system in a hospital setting uses the same pool of resources doctors, nurses, and facilities to serve both regular and elite patients. Patients may be classified as "ordinary" or "premium" depending on several variables, including the type of care they receive, the extent of their insurance, and their willingness to pay for extra features or services.

Ordinary patients: usually obtain necessary or routine medical care in a hospital. These patients may be covered by minimally comprehensive health insurance policies, insurance provided by the government, or no insurance at all. Without any extra advantages or specialized facilities, their care frequently adheres to conventional operating procedures and regulations. Compared to premium patients, the availability of doctors, treatment options, and facilities is usually restricted to ordinary patients. Because there are many patients and few resources available, patients may have to wait longer for appointments, treatments, or operations.

Premium patients: Conversely, choose better care or amenities at the hospital, frequently by making extra payments or having health insurance that covers premiums. These patients might benefit from accelerated appointments, first-choice scheduling for procedures or therapies, and far shorter wait times. Higher-end patients usually have access to a wider variety of specialists, including elite physicians and consultants. In addition to standard hospital amenities, patients may be provided with concierge services, private rooms, individualized treatment plans, and improved comfort levels. They might also be able to use exclusive hospital amenities like VIP lounges or staff members who are only available to patients. More specialized care, such as expedited appointment scheduling and comprehensive follow-up care, is frequently provided to premium patients. They might also be able to obtain high-end therapy choices that are not accessible to regular patients.

Two-Phase Optional Service: In this extended model, after patients are initially attended to by the server (e.g., the chief doctor or a specialist), they may be offered a two-phase optional service. The first phase might involve basic treatment or initial consultation. Following this, patients can opt for a second phase of service, which could involve more detailed or specialized treatment depending on their medical needs. Premium patients, for instance, might more frequently opt for this second phase, benefiting from additional services such as in-depth diagnostic tests, comprehensive health evaluations, or extended care sessions. Ordinary patients may choose this option less often, possibly due to cost considerations or insurance limitations.

Feedback Mechanism: The model also incorporates a feedback mechanism, where patients who have completed their initial service may re-enter the queue if further medical attention is required. This scenario is common in healthcare settings, where a patient may need follow-up treatment or additional consultations after the initial visit. For example, after receiving the first phase of treatment, a patient might return for further investigation if the symptoms persist or if the initial treatment is not fully effective. This feedback loop is particularly relevant for chronic conditions or complex cases requiring multiple visits and continuous monitoring.

Repair Mechanism: Given the possibility of server (doctor or specialist) unavailability due to system failures, such as equipment breakdowns or medical staff becoming unavailable, a repair mechanism is crucial. If the primary server (e.g., the chief doctor) is unavailable, the system may undergo a 'repair' process, where a substitute server, such as a doctor's assistant or another qualified medical professional, steps in to provide care. However, the service rate during this repair phase might be slower, reflecting the assistant's lower efficiency compared to the chief doctor. This dynamic ensures continuity of care but also introduces variability in service quality and speed, which is crucial for modeling real-world healthcare systems.

Working Vacation: The concept of a working vacation is also incorporated into the hospital management system. In this scenario, the chief doctor might take a 'working vacation', during which they are not fully available but still handle a reduced workload. During such periods, the doctor's assistant might take over more responsibilities, albeit at a slower service rate. This situation is reflective of real-world practices where senior medical staff may take reduced duties during certain periods, necessitating a shift in the workload to other healthcare providers. The working vacation scenario ensures that the system continues to function, albeit at a reduced capacity, maintaining patient flow even when key resources are partially unavailable.

3. Analysis of the steady state probabilities

To create the steady-state equations for the retrial system, this division treats the passed times of lower-speed service, passed premium retrial time, passed regular retrial time, and passed times of normal service as supplementary variables. Next, we quickly determine the orbit size generating functions (GFs) for different server states and the PGF of the total number of patients in the system and orbit.

3.1 *The steady state equations*

In terms of steady state methods, we take into $U_1(0) = 0$, $U_1(\infty) = 1$, $U_2(0) = 0$, $U_2(\infty) = 1$, $S_i(0) = 0$, $S_i(\infty) = 1$ $i = 1,\ 2$, $G(0) = 0$, $G(\infty) = 1$ and $V(0) = 0$, $V(\infty) = 1$ at $\varphi = 0$ are continuous. The functions $a_1(\varphi)$, $a_2(\varphi)$, $\mu_i(\varphi)$, $i = 1,\ 2$, $\psi_\upsilon(\varphi)$, and $\xi(\varphi)$, respectively, represent the hazard rates for ordinary retrial, premium retrial, normal both phase service, lower rate service, and repair rate.

$$a_1(\varphi)d\varphi = \frac{dU_1(\varphi)}{1 - U_1(\varphi)}$$

$$a_2(\varphi)d\varphi = \frac{dU_2(\varphi)}{1 - U_2(\varphi)}$$

$$\mu_i(\varphi)d\varphi = \frac{dS_i(\varphi)}{1 - S_i(\varphi)} i = 1,\ 2$$

$$\psi_\upsilon(\varphi)d\varphi = \frac{dV(\varphi)}{1-V(\varphi)}$$

$$\xi(\varphi)d\varphi = \frac{dG(\varphi)}{1-G(\varphi)}$$

Apart from it, let $U_1^0(\iota)$, $U_2^0(\iota)$, $S_1^0(\iota)$, $S_2^0(\iota)$, $V^0(\iota)$ and $G^0(\iota)$ denote the passed retrial times of ordinary orbit, premium orbit, first phase primary service, secondary phase optional service, the lower rate service times, and repair state. At time ι. Additionally, produce the random variable,

$$\mathscr{L}(\iota) = \begin{cases} 0, & \text{Provided that the server is presently idle and on a working vacation} \\ 1, & \text{Provided the server is ordinary orbit patient and the server is free} \\ 2, & \text{Provided the server is premium orbit patient and the server is free} \\ 3, & \text{Provided the server is primary first phase service at time}\iota\text{and the server is busy} \\ 4, & \text{Provided the server is optional secondary phase service at time}\iota\text{and the server is busy} \\ 5, & \text{Provided the server is lower service phase service at time}\iota\text{and the server is busy} \\ 6, & \text{Provided the server is under repair at time}\iota. \end{cases}$$

As a result, the process of creating a bivariate Markov chain$\{\mathscr{L}(\iota),\ \mathscr{J}(\iota);\ \iota \geq 0\}$, the SVs $U_1^0(\iota)$, $U_2^0(\iota)$, $S_i^0(\iota)$, $i = 0,\ 1$, $V^0(\iota)$, and $G^0(\iota)$ inspire the creation of a bivariate Markov process$\{\mathscr{L}(\iota),\ \mathscr{J}(\iota);\ \iota \geq 0\}$. Here, $\mathscr{L}(\iota)$ governs the state of the server as $(0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6)$ based on the server's status as available or occupied during both service phases, the working vacation period, and repair. $\mathscr{J}(\iota)$ indicates the patient number that is currently accessible in the orbit.

For $\mathscr{L}(\iota) > 0$, we supply six extra variables to deal with a Markov process: If $\mathscr{L}(\iota) = 1$ and $\mathscr{J}(\iota) > 0$, the passed ordinary retrial time is indicated by $U_1^0(\iota)$. When $\mathscr{L}(\iota) = 2$ and $\mathscr{J}(\iota) > 0$, the passed premium retrial time is denoted by $U_2^0(\iota)$. The variable $S_i^0(\iota)$, for $i = 1,\ 2$, indicates the passed time for a patient served in the primary and secondary phase of optional service when $\mathscr{L}(\iota) = 3,\ 4$, and $\mathscr{J}(\iota) \geq 0$. If $\mathscr{L}(\iota) = 5$ and $\mathscr{J}(\iota) \geq 0$, the passed time for a patient served during a lower-rate service period is indicated by $V^0(\iota)$. In the case where $\mathscr{L}(\iota) = 6$ and $\mathscr{J}(\iota) \geq 0$, the passed time of the patient being repaired is indicated by $G^0(\iota)$.

Theorem 1 For our system to be stable, the embedded Markov chain $\{V_n;\ n\varepsilon N\}$ is ergodic iff $\rho < 1$, where $\rho = \beta\sigma + (1-\beta)\{1-U_1^*(\kappa)-(\kappa+\zeta)E(S_1)+1\}-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\}$.

Proof. Foster's criteria [23] specifically affirm the necessary ergodicity by stating that the chain $\{V_n;\ n \in N\}$ is irreducible and aperiodic. If there is a non-negative function $e(l)$, $l \in N$, and $\varepsilon > 0$, then the Markov chain is ergodic, except for a finite number of l's, $l \in N$, and $\eta_l \leq -\in$ for all $l \in N$. $\eta_l = E[e(v_{n+1}) - e(v_n)/v_n = l]$ is the average drift. In this example, consider the statement $e(l) = l$. As a result, we get

$$\eta_l = \begin{cases} (1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1, & \text{if } l = 0 \\ \beta\sigma + (1-\beta)\{1-U_1^*(\kappa)-(\kappa+\zeta)E(S_1)+1\}-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\}, & \text{if } l = 1,\ 2,\ \ldots \end{cases}$$

Here $[\beta\sigma + (1-\beta)]\{1-U_1^*(\kappa)-(\kappa+\zeta)E(S_1)+1\}-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\} < 1$ is undoubtedly an essential condition for ergodicity.

If the Markov chain $\{W_n;\ n\varepsilon N\}$ satisfies Kaplan's status, that is, if $\eta_l < \infty$ for all $l \geq 0$ and $\exists\ l_0 \in N$ s.t $\eta_l \geq 0$ for $l \geq l_0$, then the necessary condition is satisfied, as stated by Humblett et al. [24]. The one-step transition matrix of

$\{V_n;\ n \in N\}$ is $M = (m_{kl})$ for $l < k - i$ and $k > 0$. In this instance, the pertinent one-step transition matrix is $M = (m_{kl})$. The non-ergodicity of the Markov chain is implied by $\rho \geq 1$. □

Consider the sequence of epochs $\{\iota_n;\ n = 1,\ 2,\ ...\}$ when a shorter service period or the conclusion of the service period happens. The RQ system incorporates the random vector sequence $V_n = \{\zeta(\iota_n+),\ S(\iota_n+)\}$, which forms a Markov chain. According to Theorem (3.1), $V_n;\ n \in N$ is ergodic if $\rho < 1$ where$[\beta\sigma + (1-\beta)]\{1 - U_1^*(\kappa) - (\kappa+\zeta)E(S_1) + 1\} - (1-\sigma)\{1 - U_2^*(\zeta) - (\kappa+\zeta)E(S_2) + 1\} < 1$. This is essential to the stability of our system.

The probabilities for the approach $\{\mathscr{J}(\iota),\ \iota \geq 0\}$ are given as follows: $U_0(\iota) = P\{\mathscr{L}(\iota) = 0,\ \mathscr{J}(\iota) = 0\}$; the prob. densities are

$$U_{1,\ n}(\varphi,\ \iota)d\varphi = P\{\mathscr{L}(\iota) = 1,\ \mathscr{J}(\iota) = n,\ \varphi \leq F_1^0(\iota) < \varphi + d\varphi\},$$

for $\iota \geq 0,\ \varphi \geq 0$ and $n \geq 1$.

$$U_{2,\ n}(\varphi,\ \iota)d\varphi = P\{\mathscr{L}(\iota) = 2,\ \mathscr{J}(\iota) = n,\ \varphi \leq F_2^0(\iota) < \varphi + d\varphi\},$$

for $\iota \geq 0,\ \varphi \geq 0$ and $n \geq 1$.

$$S_{i,\ n}(\varphi,\ \iota)d\varphi = P\{\mathscr{L}(\iota) = 3,\ \mathscr{J}(\iota) = n,\ \varphi \leq A^0(\iota) < \varphi + d\varphi\},$$

for $\iota \geq 0,\ \varphi \geq 0$ and $n \geq 0. i = 1,\ 2$

$$V_n(\varphi,\ \iota)d\varphi = P\{\mathscr{L}(\iota) = 4,\ \mathscr{J}(\iota) = n,\ \varphi \leq A_w^0(\iota) < \varphi + d\varphi\},$$

for $\iota \geq 0,\ \varphi \geq 0$ and $n \geq 0$.

$$G_n(\varphi,\ \iota)d\varphi = P\{\mathscr{L}(\iota) = 5,\ \mathscr{J}(\iota) = n,\ \varphi \leq A_r^0(\iota) < \varphi + d\varphi\},$$

for $\iota \geq 0,\ \varphi \geq 0$ and $n \geq 0$.

If $U_0 = lim_{\iota\to\infty}U_0(\iota)$, the limiting densities are satisfied by the sequel.

$$U_{1,\ n}(\varphi) = lim_{\iota\to\infty}U_{1,\ n}(\varphi,\ \iota);\ U_{2,\ n}(\varphi) = lim_{\iota\to\infty}U_{2,\ n}(\varphi,\ \iota);$$

$$S_{i,\ n}(\varphi) = lim_{\iota\to\infty}S_{i,\ n}(\varphi,\ \iota)i = 1,\ 2;$$

$$V_n(\varphi) = lim_{\iota\to\infty}V_n(\varphi,\ \iota);$$

$$G_n(\varphi) = lim_{\iota\to\infty}G_n(\varphi,\ \iota);$$

We construct the following system of equations by using the supplementary variable approach.

$$(\kappa+\upsilon)U_0(\iota)=(1-\beta)\int_0^\infty S_{i,\,0}(\varphi)\mu_i(\varphi)d\varphi+(1-\beta)\int_0^\infty V_0(\varphi)\psi(\varphi)d\varphi+\int_0^\infty G_0(\varphi)\xi(\varphi)d\varphi \tag{1}$$

$$\frac{d}{d\varphi}U_{1,\,n}(\varphi)+(\kappa+a_1(\varphi))U_{1,\,n}(\varphi)=0,\ n\geq 1 \tag{2}$$

$$\frac{d}{d\varphi}U_{2,\,n}(\varphi)+(\zeta+a_2(\varphi))U_{2,\,n}(\varphi)=0,\ n\geq 1 \tag{3}$$

$$\frac{d}{d\varphi}S_{i,\,n}(\varphi)+(\kappa+\zeta+\mu_i(\varphi))S_{i,\,n}(\varphi)=(\kappa+\zeta)S_{i,\,n-1}(\varphi),\ n\geq 1,\ i=1,\ 2 \tag{4}$$

$$\frac{d}{d\varphi}V_n(\varphi)+(\kappa+\zeta+\upsilon+\psi(\varphi))V_n(\varphi)=(\kappa+\zeta)V_{n-1}(\varphi),\ n\geq 1 \tag{5}$$

$$\frac{d}{d\varphi}G_n(\varphi)+(\kappa+\zeta+\xi(\varphi))G_n(\varphi)=(\kappa+\zeta)G_{n-1}(\varphi),\ n\geq 1 \tag{6}$$

The steady-state boundary conditions for $y=0$ and $\varphi=0$ are as follows:

$$\begin{aligned}U_{1,\,n}(0)=\beta\sigma\int_0^\infty S_{1,\,n}(\varphi)\mu_1(\varphi)d\varphi+(1-\beta)\int_0^\infty S_{1,\,n-1}(\varphi)\mu_1(\varphi)d\varphi+\beta\int_0^\infty V_n(\varphi)\psi(\varphi)d\varphi\\+(1-\beta)\int_0^\infty V_{n-1}(\varphi)\psi(\varphi)d\varphi\end{aligned} \tag{7}$$

$$U_{2,\,n}(0)=(1-\sigma)\int_0^\infty S_{2,\,n-1}(\varphi)\mu_2(\varphi)d\varphi \tag{8}$$

$$\begin{aligned}S_{1,\,n}(0)=\int_0^\infty U_{1,\,n+1}(\varphi)a_1(\varphi)d\varphi+\kappa\int_0^\infty U_{1,\,n}(\varphi)d\varphi+\int_0^\infty U_{2,\,n+1}(\varphi)a_2(\varphi)d\varphi\\+\zeta\int_0^\infty U_{2,\,n}(\varphi)d\varphi+\upsilon\int_0^\infty V_n(\varphi)d\varphi n\geq 1\end{aligned} \tag{9}$$

$$S_{2,\,n}(0)=p\int_0^\infty S_{1,\,n}(\varphi)\mu_1(\varphi)d\varphi \tag{10}$$

$$V_n(0)=\begin{cases}(\kappa+\zeta)U_0,\ n=0\\ 0,\ n\geq 1\end{cases} \tag{11}$$

$$G_n(0)=\gamma\int_0^\infty S_{1,\,n}(\varphi)\xi(\varphi)d\varphi \tag{12}$$

The state of normalization is

$$U_0+\sum_{n=1}^{\infty}\int_0^{\infty}U_{1,\ n}(\varphi)d\varphi+\sum_{n=1}^{\infty}\int_0^{\infty}U_{2,\ n}(\varphi)d\varphi+\sum_{n=0}^{\infty}\int_0^{\infty}\left(S_{i,\ n}(\varphi)d\varphi+V_n(\varphi)d\varphi+G_n(\varphi)d\varphi\right)=1 \tag{13}$$

3.2 *The solution in a steady state*

The steady-state solution of the RQ model is obtained using the generating function approach. The following are the GFs for $|\varpi|<1$ in order to solve the previously given equations:

$$U_1(\varphi,\ \varpi)=\sum_{n=0}^{\infty}U_n(\varphi)\varpi^n;\ U_1(0,\ \varpi)=\sum_{n=0}^{\infty}U_n(0)\varpi^n;$$

$$U_2(\varphi,\ \varpi)=\sum_{n=0}^{\infty}U_n(\varphi)\varpi^n;\ U_2(0,\ \varpi)=\sum_{n=0}^{\infty}U_n(0)\varpi^n;$$

$$S_i(\varphi,\ \varpi)=\sum_{n=0}^{\infty}S_{i,\ n}(\varphi)\varpi^n;\ S_i(0,\ \varpi)=\sum_{n=0}^{\infty}S_{i,\ n}(0)\varpi^n;\ i=1,\ 2$$

$$V(\varphi,\ \varpi)=\sum_{n=0}^{\infty}V_n(\varphi)\varpi^n;\ V(0,\ \varpi)=\sum_{n=0}^{\infty}V_n(0)\varpi^n;$$

$$G(\varphi,\ \varpi)=\sum_{n=0}^{\infty}G_n(\varphi)\varpi^n;\ G(0,\ \varpi)=\sum_{n=0}^{\infty}G_n(0)\varpi^n;$$

For each n, where $n=0,\ 1,\ 2,\ \ldots$, multiply the steady-state equation and boundary conditions by ϖ^n to move from (2) to (12). Then, add up all the factors we get.

$$\frac{\partial}{\partial\varphi}U_1(\varphi,\ \varpi)+(\kappa+a_1(\varphi))U_1(\varphi,\ \varpi)=0 \tag{14}$$

$$\frac{\partial}{\partial\varphi}U_2(\varphi,\ \varpi)+(\zeta+a_2(\varphi))U_2(\varphi,\ \varpi)=0 \tag{15}$$

$$\frac{\partial}{\partial\varphi}S_i(\varphi,\ \varpi)+((\kappa+\zeta)(1-\varpi)+\mu(\varphi))S_i(\varphi,\ \varpi)=0 i=1,\ 2 \tag{16}$$

$$\frac{\partial}{\partial\varphi}V(\varphi,\ \varpi)+(\upsilon+(1-\varpi)(\kappa+\zeta)+\psi(\varphi))V(\varphi,\ \varpi)=0 \tag{17}$$

$$\frac{\partial}{\partial\varphi}G(\varphi,\ \varpi)+((1-\varpi)(\kappa+\zeta)+\xi(\varphi))G(\varphi,\ \varpi)=0 \tag{18}$$

$$U_1(0,\ \varpi) = \sigma\beta \int_0^\infty S_1(\varphi,\ \varpi)\mu_1(\varphi)d\varphi + (1-\beta)\varpi \int_0^\infty S_1(\varphi,\ \varpi)\mu_1(\varphi)d\varphi + \beta \int_0^\infty V_n(\varphi,\ \varpi)\psi(\varphi)d\varphi$$

$$+ (1-\beta)\varpi \int_0^\infty V(\varphi,\ \varpi)\psi(\varphi)d\varphi - \kappa U_0 \tag{19}$$

$$U_2(0,\ \varpi) = (1-\sigma)\varpi \int_0^\infty S_2(\varphi,\ \varpi)\mu_2(\varphi)d\varphi \tag{20}$$

$$S_1(0,\ \varpi) = \frac{1}{\varpi}\int_0^\infty U_1(\varphi,\ \varpi)a_1(\varphi)d\varphi + \kappa \int_0^\infty U_1(\varphi,\ \varpi) + \frac{1}{\varpi}\int_0^\infty U_2(\varphi,\ \varpi)a_2(\varphi)d\varphi$$

$$+ \zeta \int_0^\infty U_2(\varphi,\ \varpi)d\varphi + \upsilon \int_0^\infty V(\varphi,\ \varpi)d\varphi \tag{21}$$

$$S_2(0,\ \varpi) = p\int_0^\infty S_1(\varphi,\ \varpi)\mu_1(\varphi)d\varphi \tag{22}$$

$$V(0,\ \varpi) = (\kappa+\zeta)U_0 \tag{23}$$

$$G(0,\ \varpi) = \gamma S_1(\varphi,\ \varpi) \tag{24}$$

After resolving partial differential equations (14) through (18), we get to

$$U_1(\varphi,\ \varpi) = U_1(0,\ \varpi)e^{-\kappa(\varphi)}[1-U_1(\varphi)] \tag{25}$$

$$U_2(\varphi,\ \varpi) = U_2(0,\ \varpi)e^{-\zeta(\varphi)}[1-U_2(\varphi)] \tag{26}$$

$$S_i(\varphi,\ \varpi) = S_i(0,\ \varpi)e^{-F_b(\varpi)\varphi}[1-S_i(\varphi)]i = 1,\ 2 \tag{27}$$

$$V(\varphi,\ \varpi) = V(0,\ \varpi)e^{-F_v(\varpi)\varphi}[1-V(\varphi)] \tag{28}$$

$$G(\varphi,\ \varpi) = G(0,\ \varpi)e^{-F_b(\varpi)\varphi}[1-G(\varphi)] \tag{29}$$

where $F_b(\varpi) = (\kappa+\zeta)(1-\varpi)$, $F_v(\varpi) = \upsilon + (\kappa+\zeta)(1-\varpi)$.

Inserting the eqns. (27) to (28) in (19) to (20) after doing some calculations, the result became,

$$U_1(0,\ \varpi) = S_1(0,\ \varpi)S_1^*(F_b(\varpi))[\beta\sigma + (1-\beta)\varpi] + V(0,\ \varpi)V^*(F_v(\varpi))[\beta + (1-\beta)\varpi] - \kappa U_0 \tag{30}$$

$$U_2(0,\ \varpi) = \varpi S_2(0,\ \varpi)(1-\sigma)S_2^*(F_b(\varpi)) - \zeta U_0 \tag{31}$$

combining the equation (25) to (29) in (21), we obtain

$$S_1(0,\ \varpi) = U_0\left[\frac{N(\varpi)}{D(\varpi)}\right] \tag{32}$$

$$N(\varpi) = [U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa\}-[U_2^*(\zeta)$$

$$+\varpi(1-U_2^*(\zeta))]\zeta+(\kappa+\zeta)W(\varpi)$$

$$D(\varpi) = \varpi-[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))][\beta\sigma+(1-\beta)\varpi]S_1^*(F_b(\varpi))-\varpi(1-\sigma)[U_2^*(\zeta)$$

$$+\varpi(1-U_2^*(\zeta))]S_2^*(F_b(\varpi))$$

Where

$$W(\varpi) = \frac{\upsilon[1-V^*(F_v(\varpi))]}{F_v(\varpi)}$$

Theorem 2 When the server is free and busy with both phases, reduced speed service, repair, and the problem that the server is free is supplied by $\rho < 1$ under the stability condition, the stationary distance of the number of patients in the ordinary and premium orbits

$$U_1(\varpi) = U_0\left[\frac{Ne(U_1(\varpi))}{D(\varpi)}\right] \tag{33}$$

$$Ne(U_1(\varpi)) = \frac{(1-U_1^*(\kappa))}{\kappa}\Bigg\{[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\{\kappa\varpi(1-\sigma)S_2^*(F_b(\varpi))-S_1^*(F_b(\varpi))[\beta\sigma$$

$$+(1-\beta)\varpi]\zeta-\varpi(1-\sigma)S_2^*(F_b(\varpi))(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]\}+S_1^*(F_b(\varpi))$$

$$[\beta\sigma+(1-\beta)\varpi](\beta+\zeta)W(\varpi)+\varpi(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa\varpi\Bigg\}$$

$$U_2(\varpi) = U_0\left[\frac{Ne(U_2(\varpi))}{D(\varpi)}\right] \tag{34}$$

$$Ne(U_2(\varpi)) = \frac{(1-U_2^*(\zeta))}{\zeta}\Bigg\{\varpi(1-\sigma)S_2^*(F_b(\varpi))PS_1^*(F_b(\varpi))\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]$$

$$(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]-\zeta[U_2^*(\zeta)$$

$$+\varpi(1-U_2^*(\zeta))]+(\kappa+\zeta)W(\varpi)\}-\zeta[\varpi-[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]$$

$$[\beta\sigma+(1-\beta)\varpi]S_1^*(F_b(\varpi))-[U_2^*(\zeta)+\varpi^2(1-U_2^*(\zeta))](1-\sigma)S_2^*(F_b(\varpi))]\Bigg\}$$

$$S_1(\varpi) = U_0\left[\frac{Ne(S_1(\varpi))}{D(\varpi)}\right] \tag{35}$$

$$Ne(S_1(\varpi)) = \frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)}\Bigg\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa\}$$

$$-[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\zeta+(\kappa+\zeta)W(\varpi)\Bigg\}$$

$$S_2(\varpi) = U_0\left[\frac{Ne(S_2(\varpi))}{D(\varpi)}\right] \tag{36}$$

$$Ne(S_2(\varpi)) = \frac{(1-S_2^*(F_b(\varpi))}{F_b(\varpi)}\Bigg(\varpi(1-\sigma)S_1^*(F_b(\varpi))\Bigg\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]$$

$$\{(\kappa+\zeta)V^*(F_v(\varpi))-\kappa\}-[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\zeta+(\kappa+\zeta)W(\varpi)\Bigg\}\Bigg)$$

$$V(\varpi) = U_0\left[\frac{(\kappa+\zeta)[1-V^*(F_v(\varpi))]}{F_v(\varpi)}\right] \tag{37}$$

$$G(\varpi) = \left[\frac{\gamma[1-G^*(F_b(\varpi))]}{F_b(\varpi)}\right]S_1(\varpi) \tag{38}$$

where

$$U_0 = \left[\frac{Ne(U_0)}{De(U_0)}\right] \tag{39}$$

$$Ne(U_0) = 1-[\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa)-E(S_1)(\kappa+\zeta)+1\}-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\}$$

$$De(U_0) = (1-[\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa)-E(S_1)(\kappa+\zeta)+1\}-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\})$$

$$(1+\left(\frac{(\kappa+\zeta)[1-V^*(\upsilon)]}{\upsilon}+(-\gamma E(G))\left(-E(S_1)\left[[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)((1-\beta))-E(V)\right.\right.\right.$$

$$\left.\left.\left.(\kappa+\zeta)^2(\beta+(1-\beta))-\kappa\}-[1-U_2^*(\zeta)]\zeta+(\kappa+\zeta)W'(1))\right]\right)\right)+\left(\frac{1-U_1^*(\kappa)}{\kappa}\right)\left[[1-U_2^*(\zeta)]\right.$$

$$\left\{\kappa(1-\sigma)-\kappa(1-\sigma)(\kappa+\zeta))E(S_2)+(\kappa+\zeta)E(S_1)[\beta\sigma+\zeta(1-\beta)]-\zeta(1-\beta)-(1-\sigma)V^*(\upsilon)\right.$$

$$(\kappa+\zeta)[\beta+(1-\beta)]+(1-\sigma)(\kappa+\zeta)^2E(S_2)V^*(\upsilon)[\beta+(1-\beta)]+(1-\sigma)(\kappa+\zeta)^2E(\nu)[(1-\beta)+\beta]$$

$$\left.-(1-\sigma)(1-\beta)V^*(\upsilon)(\kappa+\zeta)\right\}-E(S_1)(\kappa+\zeta)^2(\beta\sigma+(1-\beta))[1-V^*(\upsilon)]+(1-\beta)(\kappa+\zeta)$$

$$[1-V^*(\upsilon)]+[\beta\sigma+(1-\beta)](\kappa+\zeta)W^{'}(1)+V^*(\upsilon)(\kappa+\zeta)[(1-\beta)+\beta]-E(V)[\beta+(1-\beta)](\kappa+\zeta)^2$$

$$\left.+(1-\beta)V^*(\upsilon)(\kappa+\zeta)-\kappa\right]+\left(\frac{1-U_2^*(\zeta)}{\zeta}\right)\left[p(1-\sigma)\{1-E(S_2)(\kappa+\zeta)-E(S_1)(\kappa+\zeta)\}\right.$$

$$\left\{[1-U_1^*(\kappa)](\kappa+\zeta)V^*(\upsilon)((1-\beta)+\beta)-E(V)(\kappa+\zeta)^2((1-\beta)+\beta)+V^*(\upsilon)(\kappa+\zeta)(1-\beta)\right.$$

$$\left.-[1-U_1^*(\kappa)]\kappa-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W^{'}(1)\right\}-\left[1-[1-U_1^*(\kappa)]\zeta(\beta\sigma+(1-\beta))-(1-\beta)\right.$$

$$\left.\left.+E(S_1)(\kappa+\zeta)(\beta+(1-\beta))-(1-\sigma)[1-U_2^*(\zeta)]-(1-\sigma)+(\kappa+\zeta)(1-\sigma)E(S_2)\right]\right]$$

$$-E(S_1)\left[[1-U_1^*(\kappa)]\left\{(1-\beta)V^*(\upsilon)(\kappa+\zeta)(\kappa+\zeta)^2E(V)((1-\beta)+\beta)-\kappa\right\}\right.$$

$$\left.-[1-U_2^*(\zeta)]\zeta+W'(1)(\kappa+\zeta)\right]-E(S_2)\left[-p(\kappa+\zeta)E(S_1)\left\{[1-U_1^*(\kappa)]\left((\kappa+\zeta)V^*(\upsilon)\right.\right.\right.$$

$$\left.\left.\left.((1-\beta))-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\right)-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1)\right\}\right]$$

where

$$W'(1) = \frac{(\kappa+\zeta)[E(V)+[1-V^*(\upsilon)]]}{\upsilon}$$

Proof. Starting with equations (25)-(29), we integrate them concerning φ to determine the PGFs. $U_1(\varpi) = \int_0^\infty U_1(\varphi, \varpi)d\varphi$, $U_2(\varpi) = \int_0^\infty U_2(\varphi, \varpi)d\varphi$, $S_1(\varpi) = \int_0^\infty S_1(\varphi, \varpi)d\varphi$, $S_2(\varpi) = \int_0^\infty S_2(\varphi, \varpi)d\varphi$, $V(\varpi) = \int_0^\infty V(\varphi, \varpi)d\varphi$, $G(\varpi) = \int_0^\infty G(\varphi, \varpi)d\varphi$. Thus, we can determine the prob. that the server is free when there are no patients in the orbit by applying the normalizing condition (U_0). This is done by setting $\varpi = 1$ in equations (33)-(38) and using the rule l' hopital's whenever necessary. As a result, we obtain the equation $U_0 + U_1(1) + U_2(1) + S_1(1) + S_2(1) + V(1) + G(1) = 1$. □

Theorem 3 The PGF of the total number of patients in the system and the orbit size distribution at a stationary point in time are computed under the stability constraint $\rho < 1$.

$$K_s(\varpi) = U_0 \frac{Ne_s(\varpi)}{(1-\varpi)D(\varpi)} \tag{40}$$

$$Ne_s(\varpi) = (1-\varpi)\Bigg\{ \frac{(1-U_1^*(\kappa))}{\kappa} \Bigg([U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\{\kappa\varpi(1-\sigma)S_2^*(F_b(\varpi)) - S_1^*(F_b(\varpi))$$

$$[\beta\sigma+(1-\beta)\varpi]\zeta - \varpi(1-\sigma)S_2^*(F_b(\varpi))(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]\} + S_1^*(F_b(\varpi))$$

$$[\beta\sigma+(1-\beta)\varpi](\beta+\zeta)W(\varpi) + \varpi(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi] - \kappa\varpi \Bigg) + \frac{(1-U_2^*(\zeta))}{\zeta}$$

$$\Bigg(\varpi(1-\sigma)S_2^*(F_b(\varpi))PS_1^*(F_b(\varpi))\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))](\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]$$

$$-\kappa[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))] - \zeta[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))] + (\kappa+\zeta)W(\varpi)\} - \zeta[\varpi - [U_1^*(\kappa)$$

$$+\varpi(1-U_1^*(\kappa))][\beta\sigma+(1-\beta)\varpi]S_1^*(F_b(\varpi)) - [U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))](1-\sigma)\varpi S_2^*(F_b(\varpi))] \Bigg) \Bigg\}$$

$$+\varpi(1-\varpi)D(\varpi)\Bigg[\frac{(\kappa+\zeta)(1-V^*(F_v(\varpi)))}{F_v(\varpi)} + \frac{\gamma[1-G^*(F_b(\varpi))]}{F_b(\varpi)} \Bigg(\frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)}$$

$$\Bigg\{ [U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi] - \kappa\} - [U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]$$

$$\zeta + (\kappa+\zeta)W(\varpi) \Bigg\} \Bigg) \Bigg] + \varpi \Bigg\{ \frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)} ([U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))$$

$$[\beta+(1-\beta)\varpi]-\kappa\}-[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\zeta+(\kappa+\zeta)W(\varpi))+\frac{(1-S_2^*(F_b(\varpi))}{F_b(\varpi)}$$

$$(\varpi(1-\sigma)S_1^*(F_b(\varpi))\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))-\kappa\}$$

$$-[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\zeta+(\kappa+\zeta)W(\varpi)\})\Big\}$$

$$H_o(\varpi)=U_0\frac{Ne_o(\varpi)}{(1-\varpi)D(\varpi)} \tag{41}$$

$$Ne_s(\varpi)=(1-\varpi)\Big\{\frac{(1-U_1^*(\kappa))}{\kappa}\Big([U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\{\kappa\varpi(1-\sigma)S_2^*(F_b(\varpi))-S_1^*(F_b(\varpi))$$

$$[\beta\sigma+(1-\beta)\varpi]\zeta-\varpi(1-\sigma)S_2^*(F_b(\varpi))(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]\}+S_1^*(F_b(\varpi))$$

$$[\beta\sigma+(1-\beta)\varpi](\beta+\zeta)W(\varpi)+\varpi(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa\varpi\Big)+\frac{(1-U_2^*(\zeta))}{\zeta}$$

$$\Big(\varpi(1-\sigma)S_2^*(F_b(\varpi))PS_1^*(F_b(\varpi))\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))](\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]$$

$$-\kappa[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]-\zeta[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]+(\kappa+\zeta)W(\varpi)\}-\zeta[\varpi-[U_1^*(\kappa)$$

$$+\varpi(1-U_1^*(\kappa))][\beta\sigma+(1-\beta)\varpi]S_1^*(F_b(\varpi))-[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))](1-\sigma)\varpi S_2^*(F_b(\varpi))]\Big)\Big\}$$

$$+(1-\varpi)D(\varpi)\Big[\frac{(\kappa+\zeta)(1-V^*(F_v(\varpi)))}{F_v(\varpi)}+\frac{\gamma[1-G^*(F_b(\varpi))]}{F_b(\varpi)}(\frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)}\Big\{[U_1^*(\kappa)$$

$$+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa\}-[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\zeta$$

$$+(\kappa+\zeta)W(\varpi)\Big\})\Big]+\Big\{\frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)}([U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]$$

$$-\kappa\}-[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\zeta+(\kappa+\zeta)W(\varpi))+\frac{(1-S_2^*(F_b(\varpi))}{F_b(\varpi)}(\varpi(1-\sigma)S_1^*(F_b(\varpi))\{[U_1^*(\kappa)$$

$$+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))-\kappa\}-[U_2^*(\zeta)+\varpi(1-U_2^*(\zeta))]\zeta+(\kappa+\zeta)W(\varpi)\})\Big\}$$

where eqn. (39) denotes U_0.

Proof. Lastly, using equations (33)-(38) and a little mathematical arithmetic, the PGF of the number of patients in the system, $K_s(\varpi)$, is derived as $K_s(\varpi) = U_0 + U_1(\varpi) + U_2(\varpi) + \varpi S_1(\varpi) + \varpi S_2(\varpi) + \varpi W(\varpi) + \varpi G(\varpi)$, which corresponds to equation (40). To find the PGF for the number of available patients in the orbit, use the formula as $H_0(\varpi) = U_0 + U_1(\varpi) + U_2(\varpi) + S_1(\varpi) + S_2(\varpi) + W(\varpi) + G(\varpi)$, which allows us to compute equation (41). □

4. System performance measures

We found the mean busy cycle and mean busy time of the model, several pertinent system probabilities, and system efficiency measures even if the system is in various states.

4.1 *System state probabilities*

We arrive at the following results by applying equations (33)-(38), setting $\varpi \to 1$, and applying l'Hospital's rule till it is practicable.

(i) During the process of the ordinary retrial, the steady-state probability of the server being available is denoted by U_1,

$$U_1 = U_1(1) = U_0 \left\{ \frac{Ne_1(1)}{De(1)} \right\}$$

$$\begin{aligned} Ne_1(1) = & \frac{(1-U_1^*(\kappa))}{\kappa} \Bigg\{ [1-U_2^*(\zeta)] \Bigg[\kappa(1-\sigma) - \kappa(1-\sigma)(\kappa+\zeta))E(S_2) + (\kappa+\zeta)E(S_1)\zeta[\beta\sigma+(1-\beta)] \\ & - \zeta(1-\beta) - (1-\sigma)V^*(\upsilon)(\kappa+\zeta)[\beta+(1-\beta)] + (\kappa+\zeta)^2(1-\sigma)E(S_2)V^*(\upsilon)[\beta+(1-\beta)] \\ & + (1-\sigma)(\kappa+\zeta)^2 E(v)[\beta+(1-\beta)] - (\kappa+\zeta)(1-\beta)(1-\sigma)V^*(\upsilon) \Bigg] - (\kappa+\zeta)^2 E(S_1) \\ & (\beta\sigma+(1-\beta))[1-V^*(\upsilon)] + (1-\beta)(\kappa+\zeta)[1-V^*(\upsilon)] + [\beta\sigma+(1-\beta)](\kappa+\zeta)W'(1) \\ & + [\beta+(1-\beta)](\kappa+\zeta)V^*(\upsilon) - (\kappa+\zeta)^2 E(V)[\beta+(1-\beta)] + (1-\beta)V^*(\upsilon)(\kappa+\zeta) - \kappa \Bigg\} \end{aligned}$$

$$De(1) = 1 - [\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa) - E(S_1)(\kappa+\zeta)+1\} - (1-\sigma)\{1-U_2^*(\zeta) - (\kappa+\zeta)E(S_2)+1\}$$

(ii) Taking U_2 as the steady-state probability of the server's availability for the Premium retry,

$$U_2 = U_2(1) = U_0 \left\{ \frac{Ne_2(1)}{De(1)} \right\}$$

$$Ne_2(1) = \left(\frac{1-U_2^*(\zeta)}{\zeta}\right)\Bigg\{p(1-\sigma)-(\kappa+\zeta)E(S_2)p(1-\sigma)-p(1-\sigma)(\kappa+\zeta)E(S_1)\Bigg\{[1-U_1^*(\kappa)](\kappa+\zeta)$$

$$V^*(\upsilon)(\beta+(1-\beta))-E(V)(\kappa+\zeta)^2(\beta+(1-\beta))+V^*(\upsilon)(\kappa+\zeta)(1-\beta)-\kappa[1-U_1^*(\kappa)]$$

$$-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1)\Bigg\}-\zeta\left[1-(\beta\sigma+(1-\beta))[1-U_1^*(\kappa)]-(1-\beta)\right.$$

$$\left.+(\kappa+\zeta)E(S_1)(\beta+(1-\beta))-[1-U_2^*(\zeta)](1-\sigma)-(1-\sigma)+(\kappa+\zeta)E(S_2)(1-\sigma)\right]\Bigg\}$$

(iii) The steady-state probability that the server is delivering first-phase service is denoted by S_1.

$$S_1 = S_1(1) = U_0\left\{\frac{Nes_1(1)}{De(1)}\right\}$$

$$Nes_1(1) = -E(S_1)\Bigg\{[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)(1-\beta)-E(V)(\kappa+\zeta)^2(\beta+(1-\beta))-\kappa\}$$

$$-\zeta[1-U_2^*(\zeta)]+W'(1)(\kappa+\zeta)\Bigg\}$$

(iv) S_2 represents the steady-state probability that the server is utilizing the second phase service.

$$S_2 = S_2(1) = U_0\left\{\frac{Nes_2(1)}{De(1)}\right\}$$

$$Nes_2(1) = -E(S_2)\Bigg\{-p(\kappa+\zeta)E(S_1)\Bigg([1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)(1-\beta)-E(V)(\kappa+\zeta)^2$$

$$(\beta+(1-\beta)-\kappa\}-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1)\Bigg)\Bigg\}$$

(v) The steady-state probability that the server is taking a working vacation is denoted by V.

$$V = V(1) = U_0\left[\frac{(\kappa+\zeta)[1-V^*(\upsilon)]}{\upsilon}\right]$$

(vi) The steady-state probability of the server being repaired is G.

$$G=G(1)=-\gamma E(G)\Bigg(-E(S_1)\Bigg\{[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)(1-\beta)-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\}$$

$$-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1)\Bigg\}\Bigg)$$

4.2 The average size of a system and the size of an orbit

After bringing the system to a steady state,

(i) The expected amount of patients in the orbit (L_q) is found given ϖ, (41), and $\varpi=1$.

$$L_q=H_0^{'}(1)=\lim_{\varpi\to 1}\frac{d}{d\varpi}H_0(\varpi)=U_0\left[\frac{Ne_q^{'''}(1)De_q^{''}(1)-De_q^{'''}(1)Ne_q^{''}(1)}{3(De_q^{''}(1))^2}\right] \tag{42}$$

$$Ne_q^{''}(1)=-2\Bigg\{\left(\frac{1-U_1^*(\kappa)}{\kappa}\right)\Bigg([1-U_2^*(\zeta)]\Big[(1-\sigma)\kappa-\kappa E(S_2)(1-\sigma)(\kappa+\zeta))+(\kappa+\zeta)E(S_1)\zeta[\beta\sigma$$

$$+(1-\beta)]-\zeta(1-\beta)-(1-\sigma)V^*(\upsilon)[\beta+(1-\beta)](\kappa+\zeta)+(\kappa+\zeta)^2(1-\sigma)E(S_2)V^*(\upsilon)[\beta+(1-\beta)]$$

$$+(\kappa+\zeta)^2E(v)(1-\sigma)[\beta+(1-\beta)]-(1-\beta)(1-\sigma)(\kappa+\zeta)V^*(\upsilon)\Big]-(\kappa+\zeta)^2E(S_1)(\beta\sigma+(1-\beta))$$

$$[1-V^*(\upsilon)]+(1-\beta)(\kappa+\zeta)[1-V^*(\upsilon)]+[\beta\sigma+(\kappa+\zeta)W^{'}(1)(1-\beta)]+V^*(\upsilon)(\kappa+\zeta)[\beta+(1-\beta)]$$

$$-(\kappa+\zeta)^2E(V)[\beta+(1-\beta)]+(\kappa+\zeta)(1-\beta)V^*(\upsilon)-\kappa\Bigg)+\left(\frac{1-U_2^*(\zeta)}{\zeta}\right)\Bigg(p(1-\sigma)\{1-E(S_2)$$

$$(\kappa+\zeta)-E(S_1)(\kappa+\zeta)\}\Bigg([1-U_1^*(\kappa)](\kappa+\zeta)V^*(\upsilon)(\beta+(1-\beta))-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))$$

$$+V^*(\upsilon)(1-\beta)(\kappa+\zeta)-\kappa[1-U_1^*(\kappa)]-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W^{'}(1)\Bigg)-\zeta\Big[1-[1-U_1^*(\kappa)]$$

$$(\beta\sigma+(1-\beta))-(1-\beta)+(\kappa+\zeta)E(S_1)(\beta+(1-\beta))-[1-U_2^*(\zeta)](1-\sigma)$$

$$-(1-\sigma)+(1-\sigma)E(S_2)(\kappa+\zeta)\Big]\Bigg)\Bigg\}+2(\kappa+\zeta)E(S_1)\Bigg\{[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)(1-\beta)$$

$$-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\}-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1)\Bigg\}+2(\kappa+\zeta)E(S_2)$$

$$\Bigg\{-pE(S_1)(\kappa+\zeta)\{[1-U_1^*(\kappa)]\{(\kappa+\zeta)V^*(\upsilon)((1-\beta))-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\Big)$$

$$-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1)\}\Bigg\}-2(1-(\beta\sigma+(1-\beta))\{1-U_1^*(\kappa)-E(S_1)(\kappa+\zeta)+1\}$$

$$-(1-\sigma)\{1-U_2^*(\zeta)-E(S_2)(\kappa+\zeta)+1\})\Bigg(\Bigg[\frac{(\kappa+\zeta)[1-V^*(\upsilon)]}{\upsilon}\Bigg]+(-\gamma E(G))\Bigg(-E(S_1)$$

$$\Bigg[[1-U_1^*(\kappa)]\{(\kappa+\zeta)V^*(\upsilon)((1-\beta))-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\}-\zeta[1-U_2^*(\zeta)]$$

$$+(\kappa+\zeta)W'(1))\Bigg]\Bigg)\Bigg)$$

$$De_q''(1)=-2(1-[\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa)-E(S_1)(\kappa+\zeta)+1\}-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\})$$

$$Ne_q'''(1)=-3\Bigg\{\Bigg(\frac{1-U_1^*(\kappa)}{\kappa}\Bigg)\Bigg([1-U_2^*(\zeta)]\Bigg[\kappa(1-\zeta)E(S_2)^2(\kappa+\zeta)^2-2\kappa(1-\sigma)(\kappa+\zeta)E(S_2)$$

$$-\zeta(\kappa+\zeta)^2E(S_1)^2[\beta\sigma+(1-\beta)]+2(\kappa+\zeta)E(S_1)(1-\beta)\zeta+2(1-\sigma)(\kappa+\zeta)^2E(S_2)V^*(\upsilon)(\beta$$

$$+(1-\beta))+2(1-\sigma)(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-(\kappa+\zeta)V^*(\upsilon)(1-\beta)(1-\sigma)-(1-\sigma)$$

$$(\kappa+\zeta)^3E(S_2)^2V^*(\upsilon)(\beta+(1-\beta))-2(1-\sigma)(\kappa+\zeta)^3E(S_2)E(V)(\beta+(1-\beta))$$

$$+2(1-\beta)(1-\sigma)(\kappa+\zeta)^2E(S_2)V^*(\upsilon)-(1-\sigma)(\kappa+\zeta)^3E(V)^2(\beta+(1-\beta))+2(1-\beta)$$

$$(1-\sigma)(\kappa+\zeta)^2E(V)-(1-\beta)(1-\sigma)(\kappa+\zeta)V^*(\upsilon)\Bigg]+2(1-\beta)(\kappa+\zeta)W'(1)+[\beta\sigma$$

$$+(1-\beta)]W''(1)(\kappa+\zeta)-2E(V)(\kappa+\zeta)^2(\beta+(1-\beta))+E(V)^2(\kappa+\zeta)^3(\beta+(1-\beta))-2(1-\beta)$$

$$(\kappa+\zeta)^2E(V)+2(1-\beta)(\kappa+\zeta)V^*(\upsilon)\Bigg)+\Bigg(\frac{1-U_2^*(\zeta)}{\zeta}\Bigg)\Bigg(-2p(\kappa+\zeta)(1-\sigma)[E(S_2)+E(S_1)]$$

$$+2p(1-\sigma)(\kappa+\zeta)^2E(S_1)E(S_2)+p(\kappa+\zeta)^2(1-\sigma)[E(S_2)^2+E(S_1)^2]\{-2[1-U_1^*(\kappa)]E(V)(\kappa+\zeta)^2$$

$$(\beta+(1-\beta))+2(1-\beta)[1-U_1^*(\kappa)]V^*(\upsilon)(\kappa+\zeta)+(\kappa+\zeta)^3E(V)^2(\beta+(1-\beta))-2(1-\beta)$$

$$(\kappa+\zeta)^2E(V)+(\kappa+\zeta)W''(1)\}-\zeta\{-2[1-U_1^*(\kappa)](1-\beta)+2[1-U_1^*(\kappa)](\beta\sigma+(1-\beta))$$

$$(\kappa+\zeta)E(S_1)+2(\kappa+\zeta)E(S_1)(1-\beta)-(\beta\sigma+(1-\beta))(\kappa+\zeta)^2E(S_1)^2-2(1-\sigma)[1-U_2^*(\zeta)]$$

$$+2(1-\sigma)[1-U_2^*(\zeta)](\kappa+\zeta)E(S_2)+2(1-\sigma)(\kappa+\zeta)E(S_2)-(1-\sigma)(\kappa+\zeta)^2E(S_2)^2\}\Big)\Big\}$$

$$+3\Big\{-(\kappa+\zeta)^2E(S_1)^2\Big([1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)(1-\beta)-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\}$$

$$-[1-U_2^*(\zeta)]\zeta+(\kappa+\zeta)W'(1)\Big)\Big\}+3\Big\{E(S_1)(\kappa+\zeta)\Big([1-U_1^*(\kappa)]\Big\{(\kappa+\zeta)^3E(V)^2(\beta+(1-\beta))$$

$$-2(1-\beta)(\kappa+\zeta)^2E(V)\Big\}+(\kappa+\zeta)W''(1)\Big)\Big\}+3\Big\{-(\kappa+\zeta)^2E(S_2)^2\Big(-p(\kappa+\zeta)E(S_1)$$

$$\Big[[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)((1-\beta))-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\}-\zeta[1-U_2^*(\zeta)]+W'(1)$$

$$(\kappa+\zeta)\Big]\Big)\Big\}+3\Big\{(\kappa+\zeta)E(S_2)\Big(p(\kappa+\zeta)^2E(S_1)^2\Big[[1-U_1^*(\kappa)]\{(\kappa+\zeta)^3E(V)^2(\beta+(1-\beta))$$

$$-2(1-\beta)(\kappa+\zeta)^2E(V)\}+(\kappa+\zeta)W''(1)\Big]\Big)\Big\}-6(1-(\beta\sigma+(1-\beta))\{1-U_1^*(\kappa)-(\kappa+\zeta)E(S_1)+1\}$$

$$-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\})\Big[W'(1)+-\gamma E(G)\Big(-E(S_1)\Big\{[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)$$

$$(1-\beta)-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\}-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1))\Big\}\Big)\Big]-3([1-U_1^*(\kappa)]\beta\sigma$$

$$-(\beta\sigma+(1-\beta))(\kappa+\zeta)E(S_1)-(1-\beta)(\kappa+\zeta)E(S_1)+(\beta\sigma+(1-\beta))(\kappa+\zeta)^2E(S_1)^2-(1-\sigma)\{2$$

$$[1-U_2^*(\zeta)]-2(\kappa+\zeta)E(S_2)-2[1-U_2^*(\zeta)](\kappa+\zeta)E(S_2)+(\kappa+\zeta)^2E(S_2)^2\})$$

$$\Big(\Big[\frac{(\kappa+\zeta)[1-V^*(\upsilon)]}{\upsilon}\Big]+(-\gamma E(G))\Big(-E(S_1)\Big[[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)(1-\beta)-(\kappa+\zeta)^2E(V)$$

$$(\beta+(1-\beta))-\kappa\}-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1))\Big]\Big)\Big)$$

$$De_q^{'''}(1) = -3\big([1-U_1^*(\kappa)]\beta\sigma - (\beta\sigma + (1-\beta))E(S_1)(\kappa+\zeta) - (1-\beta)E(S_1)(\kappa+\zeta)$$

$$+(\beta\sigma + (1-\beta))(\kappa+\zeta)^2 E(S_1)^2 - (1-\sigma)\{2[1-U_2^*(\zeta)] - 2(\kappa+\zeta)E(S_2)$$

$$-2[1-U_2^*(\zeta)](\kappa+\zeta)E(S_2) + (\kappa+\zeta)^2 E(S_2)^2\}\big)$$

where

$$W''(1) = \frac{\upsilon(\kappa+\zeta)^2 E(V)[3\upsilon - \upsilon E(V) - 1] + 2[1-V^*(\upsilon)](\kappa+\zeta)^2}{\upsilon^3}$$

(ii) When ϖ is taken into account, (40) and $\varpi = 1$ yield the number of patients (L_s) that are expected in the system.

$$L_s = K_s'(1) = \lim_{\varpi \to 1} \frac{d}{d\varpi} K_s(\varpi) = U_0 \left[\frac{Ne_s^{'''}(1) De_q^{''}(1) - De_q^{'''}(1) Ne_q^{''}(1)}{3(De_q^{''}(1))^2} \right] \tag{43}$$

$$Ne_s^{'''}(1) = Nr_q^{'''}(1) + 6(\kappa+\zeta)E(S_1)\Bigg((-E(S_1))\Bigg\{[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)(1-\beta) - (\kappa+\zeta)^2 E(V)$$

$$(\beta + (1-\beta)) - \kappa\} - [1-U_2^*(\zeta)]\zeta + W'(1)(\kappa+\zeta))\Bigg\}\Bigg) + 6(\kappa+\zeta)E(S_2)\Bigg((-E(S_2))$$

$$\Bigg\{-p(\kappa+\zeta)E(S_1)\big[[1-U_1^*(\kappa)]\{(\kappa+\zeta)V^*(\upsilon)((1-\beta)) - (\kappa+\zeta)^2 E(V)(\beta + (1-\beta))$$

$$-\kappa\} - [1-U_2^*(\zeta)]\zeta + (\kappa+\zeta)W'(1)\big]\Bigg\}\Bigg) - 6(1 - (\beta\sigma + (1-\beta))\{1 - U_1^*(\kappa) - E(S_1)(\kappa+\zeta) + 1\}$$

$$-(1-\sigma)\{1 - U_2^*(\zeta) - E(S_2)(\kappa+\zeta) + 1\})\Bigg(\frac{(\kappa+\zeta)[1-V^*(\upsilon)]}{\upsilon} + -\gamma E(G)(-E(S_1)$$

$$\Bigg\{[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)(1-\beta) - (\kappa+\zeta)^2 E(V)(\beta + (1-\beta)) - \kappa\} - \zeta[1-U_2^*(\zeta)]$$

$$+W'(1)(\kappa+\zeta)\Bigg\})\Bigg)$$

(iii) The estimated time the patient will spend in the queue (W_q) and the system (W_s) is calculated using Little's technique. Specifically, $W_s = \dfrac{L_s}{\kappa}$ and $W_q = \dfrac{L_q}{\kappa}$.

4.3 *The busy cycle and the average busy period*

$$U_0 = \frac{A(T_0)}{A(T_y)+A(T_0)};\ A(T_y) = \frac{1}{\kappa}\left(\frac{1}{U_0}-1\right);\ A(T_{\varpi}) = \frac{1}{\kappa U_0} = A(T_0)+A(T_y). \tag{44}$$

where T_0 represents the duration that the system was empty. due to the exponential delay in time between two patients' arrivals. $A(T_0) = (1/\kappa)$ is the result when κ is used as the parameter. We can get (44) by putting it into (39) and using the previously found data.

$$A(T_y) = \frac{1}{\kappa}\times\left\{\frac{Ne(\varpi)}{De(\varpi)}-1\right\} \tag{45}$$

$$Ne(\varpi) = (1-[\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa)-E(S_1)(\kappa+\zeta)+1\}-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\})$$

$$(1+\left(\frac{(\kappa+\zeta)[1-V^*(\upsilon)]}{\upsilon}+(-\gamma E(G))\left(-E(S_1)\left[[1-U_1^*(\kappa)]\{V^*(\upsilon)(\kappa+\zeta)((1-\beta))\right.\right.\right.$$

$$\left.\left.\left.-E(V)(\kappa+\zeta)^2(\beta+(1-\beta))-\kappa\}-[1-U_2^*(\zeta)]\zeta+(\kappa+\zeta)W'(1))\right]\right)\right)+\left(\frac{1-U_1^*(\kappa)}{\kappa}\right)$$

$$\left[[1-U_2^*(\zeta)]\left\{\kappa(1-\sigma)-\kappa(1-\sigma)(\kappa+\zeta))E(S_2)+(\kappa+\zeta)E(S_1)[\beta\sigma+\zeta(1-\beta)]-\zeta(1-\beta)\right.\right.$$

$$-(1-\sigma)V^*(\upsilon)(\kappa+\zeta)[\beta+(1-\beta)]+(1-\sigma)(\kappa+\zeta)^2E(S_2)V^*(\upsilon)[\beta+(1-\beta)]$$

$$\left.+(1-\sigma)(\kappa+\zeta)^2E(V)[(1-\beta)+\beta]-(1-\sigma)(1-\beta)V^*(\upsilon)(\kappa+\zeta)\right\}-E(S_1)(\kappa+\zeta)^2$$

$$(\beta\sigma+(1-\beta))[1-V^*(\upsilon)]+(1-\beta)(\kappa+\zeta)[1-V^*(\upsilon)]+[\beta\sigma+(1-\beta)](\kappa+\zeta)W^{'}(1)$$

$$\left.+V^*(\upsilon)(\kappa+\zeta)[(1-\beta)+\beta]-E(V)[\beta+(1-\beta)](\kappa+\zeta)^2+(1-\beta)V^*(\upsilon)(\kappa+\zeta)-\kappa\right]$$

$$+\left(\frac{1-U_2^*(\zeta)}{\zeta}\right)\left[p(1-\sigma)\{1-E(S_2)(\kappa+\zeta)-E(S_1)(\kappa+\zeta)\}\left\{[1-U_1^*(\kappa)](\kappa+\zeta)V^*(\upsilon)\right.\right.$$

$$((1-\beta)+\beta)-E(V)(\kappa+\zeta)^2((1-\beta)+\beta)+V^*(\upsilon)(\kappa+\zeta)(1-\beta)-[1-U_1^*(\kappa)]\kappa$$

$$-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W^{'}(1)\Big\}-\big[1-[1-U_1^*(\kappa)]\zeta(\beta\sigma+(1-\beta))-(1-\beta)$$

$$+E(S_1)(\kappa+\zeta)(\beta+(1-\beta))-(1-\sigma)[1-U_2^*(\zeta)]$$

$$-(1-\sigma)+(\kappa+\zeta)(1-\sigma)E(S_2)]\Big]-E(S_1)\Big[[1-U_1^*(\kappa)]\Big\{(1-\beta)V^*(\upsilon)(\kappa+\zeta)-(\kappa+\zeta)^2E(V)$$

$$((1-\beta)+\beta)-\kappa\Big\}-[1-U_2^*(\zeta)]\zeta+W'(1)(\kappa+\zeta)\Big]-E(S_2)\Big[-p(\kappa+\zeta)E(S_1)\Big\{[1-U_1^*(\kappa)]$$

$$\Big((\kappa+\zeta)V^*(\upsilon)((1-\beta))-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\Big)-\zeta[1-U_2^*(\zeta)]+(\kappa+\zeta)W'(1)\Big\}\Big]$$

$$De(\varpi)=1-[\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa)-E(S_1)(\kappa+\zeta)+1\}-(1-\sigma)\{1-U_2^*(\zeta)-(\kappa+\zeta)E(S_2)+1\}$$

$$A(T_\varpi)=\frac{1}{\kappa}\times\left\{\frac{Ne(\varpi)}{De(\varpi)}\right\} \tag{46}$$

5. Special cases

This segment examines a few real-world uses of our technique that align with the current research.

Case (i): No premium orbit and No ordinary orbit

Let $U_2^*(\zeta)=U_1^*(\kappa)=1$ and our approach to an $M/G/1$ RQ with WVs. These results are consistent with Gao et al [25].

$$K_s(\varpi)=U_0\frac{Ne_s(\varpi)}{De_s(\varpi)}$$

$$Ne_s(\varpi)=\varpi(1-\varpi)(\varpi-[\beta\sigma+(1-\beta)\varpi]S_1^*(F_b(\varpi))-\varpi(1-\sigma)S_2^*(F_b(\varpi)))\Bigg(\frac{(\kappa+\zeta)(1-V^*(F_v(\varpi)))}{F_v(\varpi)}$$

$$+\frac{\gamma[1-G^*(F_b(\varpi))]}{F_b(\varpi)}\Big(\frac{(1-S_1^*(F_b(\varpi)))}{F_b(\varpi)}\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa\}-\zeta+(\kappa+\zeta)W(\varpi)\Big)\Bigg)$$

$$+\frac{\varpi(1-S_1^*(F_b(\varpi)))}{F_b(\varpi)}\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa\}-\zeta+(\kappa+\zeta)W(\varpi)+\frac{\varpi(1-S_2^*(F_b(\varpi)))}{F_b(\varpi)}$$

$$(\varpi(1-\sigma)S_1^*(F_b(\varpi))\{(\kappa+\zeta)V^*(F_v(\varpi))-\kappa\}-\zeta)+(\kappa+\zeta)W(\varpi)$$

$$De_s(\varpi) = (1-\varpi)(\varpi - [\beta\sigma + (1-\beta)\varpi]S_1^*(F_b(\varpi)) - \varpi(1-\sigma)S_2^*(F_b(\varpi)))$$

where,

$$U_0 = \left[\frac{Ne(U_0)}{De(U_0)}\right]$$

$$Ne(U_0) = 1 - [\beta\sigma + (1-\beta)]\{-(\kappa+\zeta)E(S_1)+1\} - (1-\sigma)\{-(\kappa+\zeta)E(S_2)+1\}$$

$$De(U_0) = (1 - [\beta\sigma + (1-\beta)]\{-E(S_1)(\kappa+\zeta)+1\} - (1-\sigma)\{-(\kappa+\zeta)E(S_2)+1\})$$

$$(1+\left(\frac{(\kappa+\zeta)[1-V^*(\upsilon)}{\upsilon} + \gamma E(G)E(S_1)[(\kappa+\zeta)W^{'}(1)]\right)) - (\kappa+\zeta)E(S_1)W^{'}(1)$$

$$+ p(\kappa+\zeta)^2 E(S_1)E(S_2)W^{'}(1)$$

Case (ii): No working vacation and No premium orbit

Let $\upsilon = 0$; and $U_2^*(\zeta) = 1$. Our approach to an $M/G/1$ RQ with general retrial times. These results are consistent with Gomez-Corral's [26].

$$K_s(\varpi) = \frac{Ne_s(\varpi)}{De_s(\varpi)}$$

$$Ne_s(\varpi) = (1-\varpi)\left\{\frac{(1-U_1^*(\kappa))}{\kappa}\left(\{\kappa\varpi(1-\sigma)S_2^*(F_b(\varpi)) - S_1^*(F_b(\varpi))[\beta\sigma + (1-\beta)\varpi]\zeta\right.\right.$$

$$-\varpi(1-\sigma)S_2^*(F_b(\varpi))(\kappa+\zeta)V^*((\kappa+\zeta)(1-\varpi))[\beta+(1-\beta)\varpi]\} + \varpi(\kappa+\zeta)V^*(\kappa+\zeta)(1-\varpi)$$

$$\left.\left.[\beta+(1-\beta)\varpi] - \kappa\varpi\right)\right\} + \varpi(1-\varpi)De_s(\varpi)\left[\frac{(\kappa+\zeta)(1-V^*(\kappa+\zeta)(1-\varpi))}{(\kappa+\zeta)(1-\varpi)} + \frac{\gamma[1-G^*(F_b(\varpi))]}{F_b(\varpi)}\right.$$

$$\left.\left(\frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)}\left([U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(\kappa+\zeta)(1-\varpi)[\beta+(1-\beta)\varpi]-\kappa\}-\zeta\right)\right)\right]$$

$$+\varpi\left\{\frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)}\left([U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(\kappa+\zeta)(1-\varpi)[\beta+(1-\beta)\varpi]-\kappa\}-\zeta\right)\right.$$

$$\left.+\frac{(1-S_2^*(F_b(\varpi))}{F_b(\varpi)}\left(\varpi(1-\sigma)S_1^*(F_b(\varpi))\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(\kappa+\zeta)(1-\varpi)-\kappa\}-\zeta\right)\right\}$$

$$De_s(\varpi)=(1-\varpi)[\varpi-[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))][\beta\sigma+(1-\beta)\varpi]S_1^*(F_b(\varpi))-\varpi(1-\sigma)S_2^*(F_b(\varpi))]$$

where,

$$U_0=\left[\frac{Ne(U_0)}{De(U_0)}\right]$$

$$Ne(U_0)=1-[\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa)-E(S_1)(\kappa+\zeta)+1\}-(1-\sigma)\{-(\kappa+\zeta)E(S_2)+1\}$$

$$De(U_0)=(1-[\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa)-E(S_1)(\kappa+\zeta)+1\}-(1-\sigma)\{-(\kappa+\zeta)E(S_2)+1\})$$

$$(1+\gamma E(G)E(S_1)[1-U_1^*(\kappa)]\{(\kappa+\zeta)(1-\beta)-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\})$$

$$+\left(\frac{1-U_1^*(\kappa)}{\kappa}\right)\left[(\kappa+\zeta)[\beta+(1-\beta)]-(\kappa+\zeta)^2E(V)[\beta+(1-\beta)]+(1-\beta)(\kappa+\zeta)-\kappa\right]$$

$$-E(S_1)\left[[1-U_1^*(\kappa)]\{(1-\beta)(\kappa+\zeta)-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\}\right]$$

$$-E(S_2)\left[-p(\kappa+\zeta)E(S_1)\{[1-U_1^*(\kappa)]((1-\beta)(\kappa+\zeta)-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa)\}\right]$$

Case (iii): No repair and No premium orbit

Let $\gamma=0$; and $U_2^*(\zeta)=1$. Our approach to an $M/G/1$ RQ with general retrial times. These findings align with those of Zhang and Hou [27].

$$K_s(\varpi)=U_0\frac{Ne_s(\varpi)}{De_s(\varpi)}$$

$$Ne_s(\varpi) = (1-\varpi)\left\{\frac{(1-U_1^*(\kappa))}{\kappa}\left(\{\kappa\varpi(1-\sigma)S_2^*(F_b(\varpi)) - S_1^*(F_b(\varpi))[\beta\sigma+(1-\beta)\varpi]\zeta - \varpi(1-\sigma)S_2^*(F_b(\varpi))\right.\right.$$

$$(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]\} + S_1^*(F_b(\varpi))[\beta\sigma+(1-\beta)\varpi](\beta+\zeta)W(\varpi) + \varpi(\kappa+\zeta)V^*(F_v(\varpi))$$

$$\left.\left.[\beta+(1-\beta)\varpi] - \kappa\varpi\right)\right\} + \varpi(1-\varpi)D(\varpi)\left[\frac{(\kappa+\zeta)(1-V^*(F_v(\varpi)))}{F_v(\varpi)} + \frac{\gamma[1-G^*(F_b(\varpi))]}{F_b(\varpi)}\right.$$

$$\left(\frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)}\left\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]-\kappa\} - \zeta\right.\right.$$

$$\left.\left.\left.+(\kappa+\zeta)W(\varpi)\right\}\right)\right] + \varpi\left\{\frac{(1-S_1^*(F_b(\varpi))}{F_b(\varpi)}([U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]\{(\kappa+\zeta)V^*(F_v(\varpi))[\beta+(1-\beta)\varpi]\right.$$

$$-\kappa\} - \zeta + (\kappa+\zeta)W(\varpi)) + \frac{(1-S_2^*(F_b(\varpi))}{F_b(\varpi)}(\varpi(1-\sigma)S_1^*(F_b(\varpi))\{[U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))]$$

$$\left.\{(\kappa+\zeta)V^*(F_v(\varpi))-\kappa\} - \zeta + (\kappa+\zeta)W(\varpi)\})\right\}$$

$$De_s(\varpi) = (1-\varpi)[\varpi - [U_1^*(\kappa)+\varpi(1-U_1^*(\kappa))][\beta\sigma+(1-\beta)\varpi]S_1^*(F_b(\varpi)) - \varpi(1-\sigma)S_2^*(F_b(\varpi))]$$

$$U_0 = \left[\frac{Ne(U_0)}{De(U_0)}\right]$$

$$Ne(U_0) = 1 - [\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa) - E(S_1)(\kappa+\zeta)+1\} - (1-\sigma)\{-E(S_2)(\kappa+\zeta)+1\}$$

$$De(U_0) = (1 - [\beta\sigma+(1-\beta)]\{1-U_1^*(\kappa) - E(S_1)(\kappa+\zeta)+1\} - (1-\sigma)\{-E(S_2)(\kappa+\zeta)+1\})$$

where,

$$(1+\left(\frac{(\kappa+\zeta)[1-V^*(\upsilon)]}{\upsilon}\right)) + \left(\frac{1-U_1^*(\kappa)}{\kappa}\right)\left[-(\kappa+\zeta)^2E(S_1)(\beta\sigma+(1-\beta))[1-V^*(\upsilon)]\right.$$

$$+(1-\beta)(\kappa+\zeta)[1-V^*(\upsilon)] + [\beta\sigma+(1-\beta)]W'(1)(\kappa+\zeta)$$

$$\left.+(\kappa+\zeta)V^*(\upsilon)[\beta+(1-\beta)] - (\kappa+\zeta)^2E(V)[\beta+(1-\beta)] + (1-\beta)(\kappa+\zeta)V^*(\upsilon) - \kappa\right]$$

$$-E(S_1)\left[[1-U_1^*(\kappa)]\left\{(\kappa+\zeta)V^*(\upsilon)(1-\beta)-E(V)(\kappa+\zeta)^2(\beta+(1-\beta))-\kappa\right\}+(\kappa+\zeta)W'(1)\right]$$

$$-E(S_2)\Bigg[-p(\kappa+\zeta)E(S_1)\Bigg\{[1-U_1^*(\kappa)]\Big((\kappa+\zeta)V^*(\upsilon)(1-\beta)-(\kappa+\zeta)^2E(V)(\beta+(1-\beta))-\kappa\Big)$$

$$+(\kappa+\zeta)W'(1)\Bigg\}\Bigg]$$

6. Numerical results

The many settings that impact system behavior measurements are illustrated in this section using MATLAB. The exponential distributions of service times, premium retrial times, working vacation periods, and ordinary retrial times are all examined. Numerical measurements that satisfy the stability criteria are chosen at random. The computed values for the following parameters in our queueing model are presented in Tables 1, 2, and 3: the average system size (L_s), average queue size (L_q), average waiting time in the queue (W_q), working vacation time $\psi_\upsilon(1)$, busy time $\mu(1)$, server idle time (U_0), and the server's idle time during ordinary and premium retrial times $U_1(1)$ and $U_2(1)$.

Table 1 displays that ordinary retrial rate a_1 hikes, U_0, L_q, $U_1(1)$, and W_q are diminished.

Table 2 displays that premium retrial rate a_2 hikes, U_0, L_q, $U_2(1)$, and W_q are diminished.

Table 3 displays that feedback probability β escalates, U_0, L_q, and W_q are diminished, and $U_1(1)$ escalates.

Table 1. U_0 and L_q for distinct Ordinary retrial rate (a_1) for the values of $\sigma=0.5$, $\kappa=0.3$, $\upsilon=1.7$, $\zeta=0.5$, $\beta=0.4$, $\gamma=2$

Retrial rate a_1	U_0	$U_1(1)$	L_q	W_q
1	2.3324	3.1291	14.7507	49.1689
1.1	2.2996	3.0605	14.3881	47.9604
1.2	2.2669	2.9909	14.0254	46.7516
1.3	2.2305	2.9558	13.8414	46.1383
1.4	2.1951	2.8132	13.1080	43.6934
1.5	2.0997	2.7770	12.9344	43.1147
1.6	2.0097	2.6673	12.3956	41.3189

Table 2. U_0 and L_q for distinct Premium retrial rate (a_2) for the values of $\sigma = 0.5$, $\kappa = 1.5$, $\upsilon = 1.7$, $\zeta = 0.4$, $\beta = 0.5$, $\gamma = 3$

Retrial rate a_2	U_0	$U_2(1)$	L_q	W_q
1	11.8694	14.7480	38.8878	25.9252
1.1	11.6198	12.4166	37.6648	25.1099
1.2	11.3701	10.2119	36.7037	24.4691
1.3	11.1205	8.3034	35.7424	23.8283
1.4	10.8709	6.5016	34.5730	23.0487
1.5	10.6212	4.8575	33.4338	22.2892
1.6	10.3716	2.3629	32.3256	21.5504

Table 3. U_0 and L_q for distinct feedback probabilities (β) for the values of $\sigma = 0.7$, $\kappa = 0.6$, $\upsilon = 1.1$, $\zeta = 0.2$, $\gamma = 4$

Feedback β	U_0	$U_1(1)$	L_q	W_q
0.2	3.3586	0.1125	6.5492	10.9153
0.3	3.3057	0.1565	6.2701	10.4502
0.4	3.2527	0.1965	5.5089	9.1815
0.5	3.1998	0.2327	4.7975	7.9957
0.6	3.1468	0.2650	4.1357	6.8928
0.7	2.9939	0.2933	3.5237	5.8727
0.8	2.9409	0.3178	2.9613	4.9355

The system's performance data are displayed on a three-dimensional graph. With the impact of the parameters κ, υ, ζ, β, σ, and γ, see Figures 2-4. As Figure 2 shows, the surface shows the escalation in the ordinary retry rate (a_1), whereas (L_q) and (W_q) diminish. In Figure 3, we found that (L_q) and (W_q) diminish while increasing the premium retrial rate a_2. In Figure 4, we found that (L_q) and (W_q) diminish while increasing the feedback probability β. And now we obtain the effect of parameters κ, υ, ζ, β, σ, and γ in Figures 5-8. Present the two-dimensional graph displaying the system's performance metrics. The escalation of the ordinary retrial rates (a_1), (L_q), and (W_q) diminishes in Figure 5. In Figure 6, we found that (L_q) and (W_q) diminish while increasing the premium retrial rate a_2. As the feedback probability (β) escalates, we find that (L_q) and (W_q) diminish, as shown in Figure 7. In Figure 8, we found that premium waiting time is reduced compared to the ordinary waiting time.

The provided numerical results accurately measure the influence of characteristics on the system's assessment standards, ensuring that they represent actual conditions.

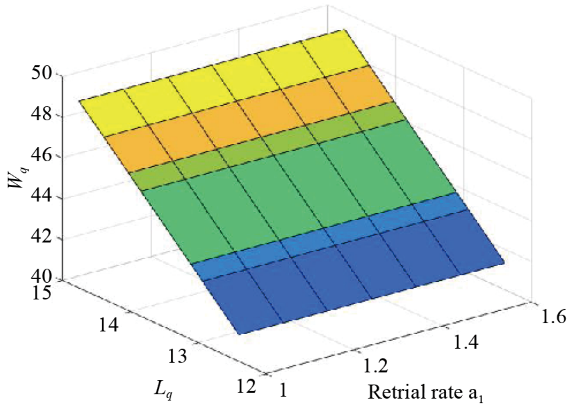

Figure 2. U_0 verses ordinary retrial rate a_1

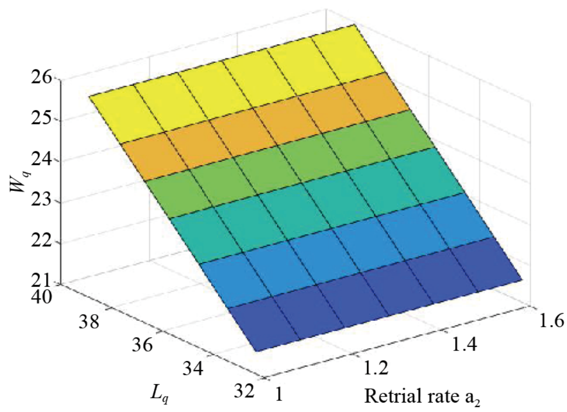

Figure 3. U_0 verses premium retrial rate a_2

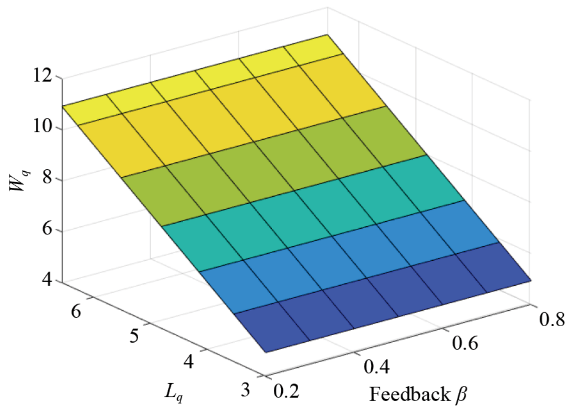

Figure 4. U_0 verses feedback probability β

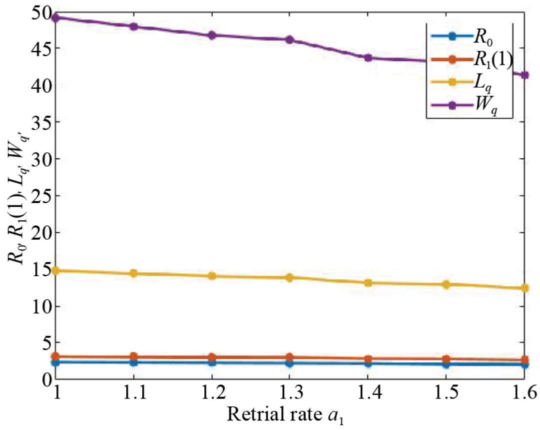

Figure 5. U_0 verses ordinary retrial rate a_1

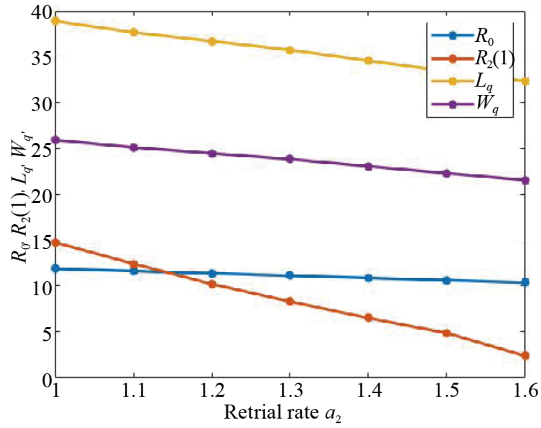

Figure 6. U_0 verses premium retrial rate a_2

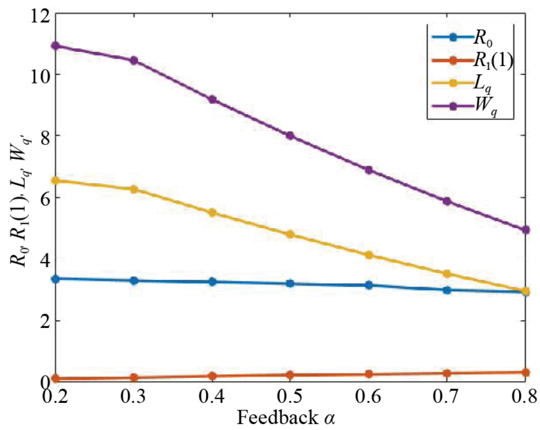

Figure 7. U_0 verses feedback probability β

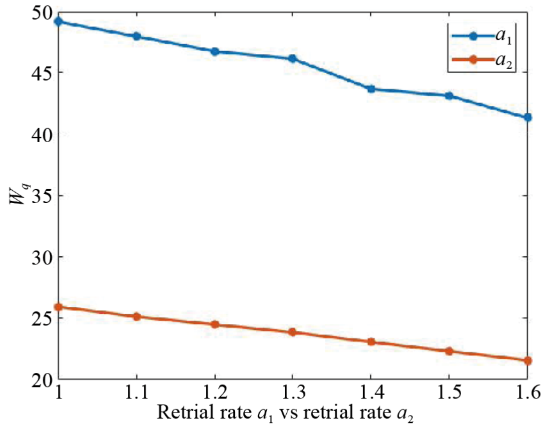

Figure 8. U_0 verses ordinary and premium retrial rates

7. Conclusion

In this paper, a thorough analysis of a sophisticated double-orbit retrial queue model is presented, along with the introduction of two different patient types into the model. The idea of a double orbit, which is essential for precisely describing queue dynamics and reliability in systems with an unreliable single server, is incorporated into this model, marking a substantial expansion of the conventional retrial queue concept. The mathematical model is developed in the paper using a non-Markovian approach, which allows for a thorough examination of queue behavior and system reliability. The model is further enhanced by the inclusion of a two-phase optional service and repair mechanism. In the first phase, upon the server's availability, patients may choose to undergo an initial basic service. Once this service is finished, patients have the option to proceed to a second, more comprehensive service phase if required. This two-phase service structure introduces an additional layer of complexity to the model, reflecting more realistic scenarios where different levels of service may be necessary depending on patient needs. Additionally, the model accounts for the possibility of server failure and subsequent repair. If the server becomes unreliable and fails during the service, it undergoes a repair process before resuming operation. This repair mechanism is crucial in maintaining system reliability and ensuring continuous service, albeit with possible delays due to repair times. The introduction of server repair dynamics allows for a more robust and realistic representation of real-world queuing systems, particularly in environments where server reliability is a critical concern. Moreover, the study incorporates a feedback loop mechanism, where patients who have completed service may rejoin the queue if they require additional assistance or if the initial service was insufficient. This feedback loop adds another layer of realism to the model, as it captures scenarios where patients may need further attention after their initial service, thus affecting the overall queue dynamics and system performance. Analyzes system parameters and queue length using probability generating functions (PGF), among other important performance indicators. The number of patients in the system and the length of the queue determine these parameters, as well as the complexities introduced by optional service phases, repair mechanisms, and feedback loops. Numerical simulations are performed to analyze the effects of these system characteristics on overall performance. The research specifically explores the application of a double-orbit retrial queuing model with two-phase optional service, feedback, and repair within hospital management systems. In this model, premium orbit patients experience shorter queue times compared to ordinary patients, a reduction that is critical in lowering patient risk. The inclusion of optional services, feedback, and repair mechanisms enhances the model's suitability, especially in scenarios involving pandemics and healthcare, where queuing systems' dependability and effectiveness are critical. Steady-state results validate the analytical methodologies and show how resilient the model is. In the end, the research offers a thorough examination of the double-orbit retrial queue model, taking into account feedback mechanisms, working vacations, and two-phase optional service and repair. This paradigm is very applicable to the optimization of queuing systems with unreliable servers, especially in healthcare settings where prompt medical attention and efficient patient flow management are important.

Confilict of interest

The authors declare no competing financial interest.

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