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Research Article

On the Spatial Behaviour for Solutions to a Phase Transition Model Involving Two Temperatures

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Abstract: The aim, in this paper, is to study the spatial behaviour, in an unbounded domain, of solutions for a phase transition system involving two temperatures. When we want to determine the magnitude of the solution of an elliptic or parabolic partial differential align on a bounded domain, we generally use the maximum principle. This principle states that a solution function of such aligns has its maximum value on the boundary of the domain. Unfortunately, this property is no longer true when the domain of study of the function is not bounded. This leads us to apply a generalisation of the maximum principle known as the Phragmén-Lindelöf alternative. To apply it, we place ourselves in a domain comprising a bounded region and an unbounded region. If we can show that the solution does not explode in the bounded region, we can conclude that the solution does not explode in the whole domain.

Keywords: asymptotic behaviour, phase transition system involving two temperatures, unbounded domain, maximum principle, phragmén-Lindelöf alternative

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1. Introduction

The asymptotic behaviour of phase transition systems is a subject of interest to many researchers. While the study of asymptotic behaviour in time is fairly well mastered, this is not the case for spatial behaviour. Indeed, it is easier to find work done on bounded (see [1–10]) than on unbounded domains (see [11–14]). In this work, we will look at the spatial behaviour of a phase transition system defined on an unbounded domain.

Let us consider the initial boundary value problem below

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$$\frac{\partial u}{\partial t}-\Delta u+f(u)=\varphi-\Delta\varphi \text{ in } \Omega, \tag{1}$$

$$\frac{\partial \varphi}{\partial t}-\Delta\frac{\partial \varphi}{\partial t}-\Delta\varphi=-\frac{\partial u}{\partial t} \text{ in } \Omega, \tag{2}$$

$$u=\varphi=0 \text{ on } \partial\Omega, \tag{3}$$

$$u|_{t=0}=u_0,\ \varphi|_{t=0}=\varphi_0 \text{ in } \Omega. \tag{4}$$

The system (1)-(4) has been extensively studied in [15, 16]. The derivation aspects of the model, the existence and uniqueness of solutions as well as the analysis of the asymptotic behaviour of the solutions resulting from the bounded initial conditions, was treated in [16] whereas in [15], we showed that the global attractor obtained in [16] was of finite dimension. In this paper, we are interested in the spatial behaviour of the solutions. Actually, Our ambition is to extend the qualitative study of two-temperature phase transition models to $\mathbb{R}^3$, which is the most relevant field in physics. To reassure ourselves that such an extension is possible, we begin by proposing a study on a semi-infinite domain with the aim of establishing a Phragmén-Lindelöf-type alternative. To do this, we place ourselves in a semi-infinite cylinder and we prove that the energy expression for the solution to the problem must either increase exponentially or decrease exponentially with the axial distance from the end of a semi-infinite cylinder (See, for example [17–19]) and thus obtain the conditions not only for the explosion of solutions in the semi-infinite cylinder but also for the non-explosion which is of particular interest to us. Another consequence of the Phragmén-Lindelöf alternative is that the conditions thus obtained can be generalized to all unbounded domains and therefore to $\mathbb{R}^3$.

In section 1, we present the various notations and assumptions on which we will base our analysis. The last part of this article is devoted to describing the spatial behaviour of solutions by establishing a Phragmén-Lindelöf-type alternative.

2. Notations and assumptions

We adopt the assumptions of the article in [16]. They are an essential prerequisite for the rest of our work. These assumptions are as follows

$$-c_0\leq F(s)\leq f(s)s+c_1,\ c_0,\ c_1\geq 0,\ s\in\mathbb{R}, \text{ and } F(s)=\int_0^s f(\tau)\,d\tau, \tag{5}$$

$$|f'(s)|\leq c_2(|s|^{2p}+1),\ c_2,\ p\geq 1,\ s\in\mathbb{R}, \tag{6}$$

$$f'\geq -c_3,\ c_3\geq 0, \tag{7}$$

$$f(0)=0. \tag{8}$$

The constants may be designated by the same letter from one line to another, but they do not necessarily have the same value. The norms used in this work are identical to those used in the paper [16].

3. Spatial behaviour of solutions

The purpose of this part is to understand how solutions to problem (1)-(4) behave from a spatial point of view. In this respect, we introduce the following notations. We consider the region $R = (0,\ +\infty) \times D$ of space $\mathbb{R}^n,\ n = 2$ or 3, with D a limited area of $\mathbb{R}^{n-1}$. The region R is supposed to be sufficiently smooth to apply the divergence theorem. For $n = 2$, one has, in that case, a rectangular strip along the direction x_1. On the other side, for $n = 3$, R is a semi-infinite cylinder following the direction x_1. It is precisely this last case which will be the subject of our analysis. Furthermore, we hypothesize that such solutions exist. Let the system given by (1)-(2) be taken in the region R, when the dimension $n = 3$, then the following boundary and initial conditions are associated to it, respectively

$$u = \varphi = 0 \text{ on } (0,\ +\infty) \times \partial D \times (0,\ T), \tag{9}$$

$$u(0,\ x_2,\ x_3,\ t) = p(x_2,\ x_3,\ t),\ \varphi(0,\ x_2,\ x_3,\ t) = q(x_2,\ x_3,\ t) \text{ on } \{0\} \times \partial D \times (0,\ T), \tag{10}$$

where $T > 0$ is a given final time and

$$u|_{t=0} = \varphi|_{t=0} = 0 \text{ on } R. \tag{11}$$

The functions F and f satisfy, respectively

$$F(s) + ds^2 \geq 0 \text{ and } f(s)s + ds^2,\ \ d > 0. \tag{12}$$

Remark 1 In our case, the function F stands for an antiderivative of f and it can be seen that the function $f(s) = s^3 - s$ satifies conditions listed above. Actually, any function of the form $f(s) = a|s|^k s - bs,\ a,\ b > 0$, satisfies these hypotheses.

First of all, consider the function

$$F_w(z,\ t) = \int_0^t \int_{D(z)} exp(-ws)(u_s u_{,\ 1} + \varphi(\varphi_{,\ 1} + \varphi_{,\ 1s}) + \varphi_s \varphi_{,\ 1})\ dads, \tag{13}$$

where $D(z) = \{x \in R,\ x_1 = z\}$ and w is an arbitrary positive constant to be fixed later, here $u_s = \dfrac{\partial u}{\partial s}$ and $u_{,\ 1} = \dfrac{\partial u}{\partial x_1}$. We have, after using the divergence theorem and taking into account (9)-(11),

$$\begin{aligned} F_w(z+h,\ t) - F_w(z,\ t) = {} & \frac{e^{-wt}}{2} \int_{R(z,\ z+h)} (|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla \varphi|^2 + |\Delta \varphi|^2)\ dx \\ & + \int_0^t \int_{R(z,\ z+h)} e^{-ws}(|u_s|^2 + |\nabla \varphi|^2 + |\Delta \varphi|^2)\ dxds \\ & + \frac{w}{2} \int_0^t \int_{R(z,\ z+h)} e^{-ws}(|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla \varphi|^2 + |\Delta \varphi|^2)\ dxds, \end{aligned} \tag{14}$$

where $R(z,\ z+h)=\{x_1\in R,\ z<x_1<z+h\}$.

Divide by h, and allow h to tend towards 0, one gets

$$\frac{\partial F_w}{\partial z}(z,\ t)=\frac{e^{-wt}}{2}\int_{D(z)}(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2)\ da$$
$$+\int_0^t\int_{D(z)}e^{-ws}(|u_s|^2+|\nabla\varphi|^2+|\Delta\varphi|^2)\ dads \tag{15}$$
$$+\frac{w}{2}\int_0^t\int_{D(z)}e^{-ws}(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2)\ dads.$$

Now, let us look at another function

$$G_w(z,\ t)=\int_0^t\int_{D(z)}e^{-ws}(uu_{,\,1}+\varphi(\theta_{,\,1}+\varphi_{,\,1}))\ dads, \tag{16}$$

where $\theta=\int_0^t\varphi(s)\ ds$.

We find, proceeding as above

$$G_w(z+h,\ t)-G_w(z,\ t)=\frac{e^{-wt}}{2}\int_{R(z,\ z+h)}(|u|^2+|\nabla\theta|^2)\ dx$$
$$+\int_0^t\int_{R(z,\ z+h)}e^{-ws}(|\nabla u|^2+f(u)u+u\Delta\varphi+|\varphi|^2+|\nabla\varphi|^2)\ dxds \tag{17}$$
$$+\frac{w}{2}\int_0^t\int_{R(z,\ z+h)}e^{-ws}(|u|^2+|\nabla\theta|^2)\ dxds.$$

Calculating the differential of G_w, yields

$$\frac{\partial G_w}{\partial z}(z,\ t)=\frac{e^{-wt}}{2}\int_{D(z)}(|u|^2+|\nabla\theta|^2)\ da$$
$$+\int_0^t\int_{D(z)}e^{-ws}(|\nabla u|^2+f(u)u+u\Delta\varphi+|\varphi|^2+|\nabla\varphi|^2)\ dads \tag{18}$$
$$+\frac{w}{2}\int_0^t\int_{D(z)}e^{-ws}(|u|^2+|\nabla\theta|^2)\ dads.$$

We choose a sufficiently large w so that

$$f(u)u+\frac{w}{2}u^2 \geq c_1,\ c_1 \geq 0,\ (w>2). \tag{19}$$

We set $H_w = F_w + \tau G_w$, where τ is large enough. We have

$$\begin{aligned}&(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2)+\tau(|u|^2+|\nabla\theta|^2)\\ &\geq c_2(|u|^2+|\nabla u|^2+|\varphi|^2+|\nabla\varphi|^2+|\Delta\varphi|^2+|\nabla\theta|^2)\end{aligned} \tag{20}$$

and

$$\begin{aligned}&(|u_s|^2+|\Delta\varphi|^2)+\tau(|\nabla u|^2+f(u)u+u\Delta\varphi+|\varphi|^2+|\nabla\varphi|^2)\\ &+\frac{w}{2}(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2)+\tau(|u|^2+|\nabla\theta|^2)\\ &\geq c_2(|u|^2+|\nabla u|^2+|u_s|^2+|\varphi|^2+|\nabla\varphi|^2+|\Delta\varphi|^2+|\nabla\theta|^2),\end{aligned} \tag{21}$$

where c_2 is a strictly positive constant.

Noting that

$$\frac{\partial H_w}{\partial z} = \frac{\partial F_w}{\partial z} + \tau\frac{\partial G_w}{\partial z} \tag{22}$$

and assuming w and τ large enough, we get

$$\begin{aligned}\frac{\partial H_w}{\partial z}(z,\, t) \geq &\int_0^t\int_{D(z)} e^{-ws}(|u|^2+|\nabla u|^2+|u_s|^2+|\varphi|^2+|\nabla\varphi|^2)\ dads\\ &+c_2\int_0^t\int_{D(z)} e^{-ws}(|\Delta\varphi|^2+|\nabla\theta|^2)\ dads.\end{aligned} \tag{23}$$

The next stage is to establish an estimate of $|H_w|$ based on $\dfrac{\partial H_w}{\partial z}$. One has

$$
\begin{aligned}
|F_w(z,\, t)| \leq & \left(\int_0^t \int_{D(z)} e^{-ws} u_s^2 \; dads\right)^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} e^{-ws} u_{,1}^2 \; dads\right)^{\frac{1}{2}} \\
& + \left(\int_0^t \int_{D(z)} e^{-ws} \varphi^2 \; dads\right)^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} e^{-ws} \varphi_{,1}^2 \; dads\right)^{\frac{1}{2}} \\
& + \left(\int_0^t \int_{D(z)} e^{-ws} \varphi^2 \; dads\right)^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} e^{-ws} \varphi_{,1s}^2 \; dads\right)^{\frac{1}{2}} \\
& + \left(\int_0^t \int_{D(z)} e^{-ws} \varphi_s^2 \; dads\right)^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} e^{-ws} \varphi_{,1}^2 \; dads\right)^{\frac{1}{2}} \\
\leq & \; c_3 \int_0^t \int_{D(z)} e^{-ws} (|\nabla u|^2 + |u_s|^2 + |\varphi|^2 + |\nabla \varphi|^2 + |\varphi_s|^2 + |\nabla \varphi_s|^2) \; dads,
\end{aligned} \tag{24}
$$

where c_3 is a positive constant that can be calculated explicitly. Likewise,

$$
\begin{aligned}
|G_w(z,\, t)| \leq & \left(\int_0^t \int_{D(z)} e^{-ws} u^2 \; dads\right)^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} e^{-ws} u_{,1}^2 \; dads\right)^{\frac{1}{2}} \\
& + \left(\int_0^t \int_{D(z)} e^{-ws} \varphi^2 \; dads\right)^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} e^{-ws} \theta_{,1}^2 \; dads\right)^{\frac{1}{2}} \\
& + \left(\int_0^t \int_{D(z)} e^{-ws} \varphi^2 \; dads\right)^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} e^{-ws} \varphi_{,1s}^2 \; dads\right)^{\frac{1}{2}} \\
& + \left(\int_0^t \int_{D(z)} e^{-ws} \varphi_s^2 \; dads\right)^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} e^{-ws} \varphi_{,1}^2 \; dads\right)^{\frac{1}{2}} \\
\leq & \; c_4 \int_0^t \int_{D(z)} e^{-ws} (|u|^2 + |\nabla u|^2 + |\varphi|^2 + |\nabla \varphi|^2 + |\nabla \theta|^2) \; dads,
\end{aligned} \tag{25}
$$

c_4 may be determined explicitly in terms of parameters and cross-section geometry. We deduce from (24)-(25) that

$$
|H_w(z,\, t)| \leq c_5 \frac{\partial H_w}{\partial z}(z,\, t), \tag{26}
$$

where $c_5 = \dfrac{c_3 + c_4}{c_2}$.

Remark 2 The inequality (26) is well known in the study of spatial estimates and leads to the so-called Phragmén-Lindelöf alternative (see [20, 21]). In particular, if there exists $z_0 \geq 0$ such that $H_w(z_0,\ t) > 0$, then the solution satisfies

$$H_w(z,\ t) \geq H_w(z_0,\ t)e^{c_5^{-1}(z-z_0)},\ z \geq z_0. \tag{27}$$

This last estimate provides information in terms of measure defined in the cylinder. Indeed, from (27), it follows that

$$\begin{aligned}
&\frac{e^{-wt}}{2}\int_{R(0,\ z)}(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2)\ dx+\tau\frac{e^{-wt}}{2}\int_{R(0,\ z)}(|u|^2+|\nabla\theta|^2)\ dx\\
&+\frac{e^{-wt}}{2}\int_{R(0,\ z)}(|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2)\ dx+\int_0^t\int_{R(0,\ z)}e^{-ws}(|u_s|^2+|\nabla\varphi|^2+|\Delta\varphi|^2)\ dxds\\
&+\tau\frac{w}{2}\int_0^t\int_{R(0,\ z)}e^{-ws}(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2)\ dxds \qquad (28)\\
&+\tau\int_0^t\int_{R(0,\ z)}e^{-ws}(|\nabla u|^2+f(u)u+u\Delta\varphi+|\varphi|^2+2|\nabla\varphi|^2\ dxds\\
&+\tau\frac{w}{2}\int_0^t\int_{R(0,\ z)}e^{-ws}(|u|^2+|\nabla\theta|^2)\ dxds
\end{aligned}$$

tends exponentially towards infinity. On the other hand, if $H_w(z,\ t) \leq 0$, for every $z \geq 0$, we deduce that the solution decays and we get an inequality of the form

$$-H_w(z,\ t) \leq -H_w(0,\ t)e^{c_5^{-1}(z-z_0)},\ z \geq 0. \tag{29}$$

This estimate implies that $H_w(z,\ t)$ tends to zero as z goes to infinity.

Besides, from (29), it follows that

$$E_w(z,\ t) \leq E_w(0,\ t)e^{-c_5^{-1}z},\ z \geq 0, \tag{30}$$

where

$$
\begin{aligned}
E_w(z,\ t) = \frac{e^{-wt}}{2} & \int_{R(z)} (|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2)\ dx \\
& + \tau \frac{e^{-wt}}{2} \int_{R(z)} (|u|^2 + |\nabla\theta|^2)\ dx + \frac{e^{-wt}}{2} \int_{R(z)} (|\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2)\ dx \\
& + \int_0^t \int_{R(z)} e^{-ws} (|u_s|^2 + |\nabla\varphi|^2 + |\Delta\varphi|^2)\ dxds \\
& + \tau \frac{w}{2} \int_0^t \int_{R(z)} e^{-ws} (|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2)\ dxds \\
& + \tau \int_0^t \int_{R(z)} e^{-ws} (|\nabla u|^2 + f(u)u + u\Delta\varphi + |\varphi|^2 + 2|\nabla\varphi|^2\ dxds \\
& + \tau \frac{w}{2} \int_0^t \int_{R(z)} e^{-ws} (|u|^2 + |\nabla\theta|^2)\ dxds
\end{aligned} \tag{31}
$$

and $R(z) = \{x \in R,\ x_1 > z\}$. We proved the following result.

Theorem 1 Let $(u,\ \varphi)$ be a regular solution to the problem defined by (1)-(2), boundary conditions (9)-(10) and initial data (11). Therefore, either this solutions satifies the growth (27) or it satisfies the decay estimate.

$$
\mathscr{E}_w(z,\ t) \leq E_w(0,\ t) e^{wt - c_5^{-1} z},\ \ z \geq 0, \tag{32}
$$

where

$$
\begin{aligned}
\mathscr{E}_w(z,\ t) = \frac{1}{2} & \int_{R(z)} (|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2)\ dx \\
& + \frac{\tau}{2} \int_{R(z)} (|u|^2 + |\nabla\theta|^2)\ dx + \frac{1}{2} \int_{R(z)} (|\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2)\ dx \\
& + \int_0^t \int_{R(z)} (|u_s|^2 + |\nabla\varphi|^2 + |\Delta\varphi|^2)\ dxds \\
& + \tau \frac{w}{2} \int_0^t \int_{R(z)} (|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2)\ dxds \\
& + \tau \int_0^t \int_{R(z)} (|\nabla u|^2 + f(u)u + u\Delta\varphi + |\varphi|^2 + 2|\nabla\varphi|^2\ dxds \\
& + \tau \frac{w}{2} \int_0^t \int_{R(z)} (|u|^2 + |\nabla\theta|^2)\ dxds.
\end{aligned} \tag{33}
$$

Remark 3 The theorem stated above corresponds to the Phragmén-Lindelöf alternative (see [22]).

4. Conclusion

In this work, we studied the spatial behaviour of solutions of a phase transition system. More precisely, we obtained explosion/non-explosion conditions for solutions on a semi-infinite cylinder by proving a Phragmén-Lindelöf type alternative for solutions. As a follow-up to this work, and for a better description of the decay, we could obtain an upper bound for the amplitude term using boundary conditions. A consequence of the Phragmén-Lindelöf alternative is that we can state that these conditions can be apply to any non-bounded subset of $\mathbb{R}^3$.

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Authors contributions

These authors contributed equally to this work.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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