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Research Article

On Qualitative Behaviour of Solutions of Third Order Matrix Delay Differential Equations

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Abstract: We analyze, using the Lyapunov-Krasovskii method, the conditions for the stability, boundedness and periodicity of solutions to a class of nonlinear matrix differential equation of third order with variable delay. Criteria under which the solutions to the equation considered possess solutions that are stable and bounded on the real line as well as existence of at least one periodic solution are given. Our results generalize and extend many existing results in the literature on scalar, vector and matrix differential equations with or without delay. The integrity of our results is demonstrated by two numerical examples included.

***Keywords*:** third order, stability, boundedness, lyapunov-Krasovskii functionali, matrix

MSC: 34K12, 34K20

1. Introduction

We shall be considering the matrix delay differential equation (MDDE),

$$\dddot{X} + F(X,\ \dot{X})\ddot{X} + B\dot{X} + H(X(t-r(t))) = P(t,\ X,\ \dot{X},\ \ddot{X}), \tag{1}$$

in which $X : \mathbb{R} \to \mathscr{M}$, $H : \mathscr{M} \to \mathscr{M}$, $B \in \mathscr{M}$ is a real $m \times m$-constant symmetric matrix, $F : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$, $P : \mathbb{R} \times \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ and $r(t)$ is continuous and differentiable, so that $0 \le r(t) \le \zeta_1$ and $r'(t) \le \zeta_2$, $(0 < \zeta_2 < 1)$, both ζ_1, ζ_2 are positive real constants and the dot in (1) denote differentiation with respect to t. All through this paper, $\mathscr{M}$ shall represents the space of all $m \times m$ matrices, $\mathbb{R} = (-\infty,\ \infty)$ and $\mathbb{R}^m$ is the m-dimensional Euclidean space. It is taken that $F,\ H$ and P are continuous and satisfied Lipschitz condition in their own arguments.

Generally, MDDEs are differential equations in which the time derivatives of the unknown function (i.e., $X \in \mathscr{M}$) at a current time is determined by the values of the function at the previous times. Numerous researchers in the last five decades and even more, have carried out various works on the qualitative behaviour of solutions to many scalar and vector

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differential equations by means of Lyapunov-Krasovskii (or Lyapunov direct) method (See, [1–27]). But, there is relatively little study on the qualitative behaviours of solutions to matrix differential equations (See, [14, 17, 28]). However, to the best of our knowledge, we are yet to see any work on matrix delay differential equations (MDDEs). Hence, there is need for the current work.

The conditions for boundedness and existence of at least one periodic solution to

$$\dddot{X} + F(X,\, Y)\ddot{X} + B\dot{X} + H(X) = P(t,\, X,\, \dot{X},\, \ddot{X}), \tag{2}$$

for which B is an $n \times n$-constant symmetric matrix and F, H and P are continuous vector functions was studied in [21]. Also, boundedness of solutions to (2) when $B\dot{X} = G(\dot{X})$ was considered in [1].

Similarly, Omeike [16] examined criteria for stable and bounded solutions to the delay differential equation (DDE)

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X(t - r(t))) = P(t),$$

where A and B are $n \times n$-constant symmetric matrices, $H(X)$ and $r(t)$ are continuous and differentiable functions. Later, Tunc and Mohammed [22] proved certain results on the stability of null solution and boundedness of solutions to the following DDE

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + B\dot{X}(t - \tau_1) + cX(t - \tau_1) = P(t).$$

Further more, Tunc [24] gave certain conditions for the stability and boundedness of solutions to

$$\dddot{X} + H(\dot{X})\ddot{X} + G(\dot{X}(t - r)) + cX(t - r) = P(t,\, X,\, \dot{X},\, \ddot{X}),$$

where $r > 0$ is a delay. Omeike [15] studied ultimate boundedness of solutions to the following DDE

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X(t - r)) = P(t,\, X,\, \dot{X},\, \ddot{X}),$$

where both A and B are $n \times n$-constant matrices, $r > 0$ is a delay and vector $H(X)$ is not required to be differentiable. Adeyanju and Tunc [7] gave some criteria for asymptotic stability and uniform ultimate boundedness of solutions to

$$\dddot{X} + F(X,\, \dot{X})\ddot{X} + B\dot{X} + H(X(t - r(t))) = P(t,\, X,\, \dot{X},\, \ddot{X}), \tag{3}$$

where $X : \mathbb{R} \to \mathbb{R}^n$ is the unknown, $H : \mathbb{R}^n \to \mathbb{R}^n$, B is a real $n \times n$-constant symmetric matrix and $r(t)$ is the delay.

Tejumola [28], proved some theorems on the stability, boundedness and existence of at least a periodic solution to the matrix differential equation (MDE)

$$\ddot{X} + A\dot{X} + H(X) = P(t,\, X,\, \dot{X}),$$

where $X : \mathscr{R} \to \mathscr{M}$ and A is an $n \times n$-constant symmetric matrix.

Later, Omeike [17] also studied conditions for boundedness and periodicity of solutions to MDE

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t,\, X,\, \dot{X},\, \ddot{X}), \tag{4}$$

where $X : \mathscr{R} \to \mathscr{M}$; $A,\, B$ are $n \times n$-constant matrices and $H(X)$ is a differentiable matrix function. In a recent paper by Olutimo and Omeike [14], the authors considered stability and ultimate boundedness of solutions to a rectangular MDE

$$\dddot{X} + A\ddot{X} + \Psi(\dot{X}) + H(X) = P(t,\, X,\, \dot{X},\, \ddot{X}), \tag{5}$$

where $X : \mathbb{R} \to \tilde{\mathscr{M}},\, \Psi(\dot{X}),\, H(X) : \tilde{\mathscr{M}} \to \tilde{\mathscr{M}}$, A is an $n \times n$ constant symmetric matrix and $\tilde{\mathscr{M}}$ is the space of $n \times m$ matrices.

Having derived motivation from papers [14, 15, 17, 28] and other referenced papers, we are encouraged to extend and generalize stability, boundedness and periodicity results of scalar, vector and matrix differential equation to MDDE using an appropriate Lyapunov-Krasovskii functional. This functional, has the property that, it is positive everywhere on the real line apart from the origin where it vanishes. On the other hand, the derivative of the functional along the solution path of the equation being examined is expected to be negative semi-definite. In the present paper, differentiability of $H(X) \in \mathscr{M}$ is not required but for any $X,\, Z \in \mathscr{M}$ (Similar to Afuwape [9], Meng [13] and Omeike [15]), there exists an $m \times m$ operator $C(X,\, Z)$ such that

$$H(X) = H(Z) + C(X,\, Z)(X - Z), \tag{6}$$

where $\lambda_i(C(X,\, Z))$ $(i = 1,\, 2,\, \ldots,\, m)$ are continuous eigenvalues of $C(X,\, Z)$ which satisfy

$$\Delta_h \geq \lambda_i(C(X,\, Z)) \geq \delta_h > 0,$$

for some real constants δ_h and Δ_h.

Remark 1

(i) Equation (1) reduces to (4) when $F(X,\, \dot{X}) = A$ and $r(t) = 0$. Thus, (1) is more general compare to (4).

(ii) The current research is an extension and generalization of the results in [7, 15, 17, 28] to MDDE.

Notation and definitions [17, 28].

We shall adopt some standard matrix notation as contained in [17, 28]. Let $\mathscr{M}$ represent the space of all real $m \times m$ matrices, $\mathbb{R}^m$ the real m-dimensional Euclidean space and $\mathbb{R} = (-\infty,\, \infty)$. For any $W,\, Z \in \mathscr{M}$, W^T and $w_{ij}(i,\, j = 1,\, 2,\, \ldots,\, m)$ denote the transpose and the elements of W respectively, while $(w_{ij})(z_{ij})$ will sometimes represent the product matrix WZ. $W_i = (w_{i1},\, w_{i2},\, \ldots,\, w_{im})$ and $W^j = (w_{1j},\, w_{2j},\, \ldots,\, w_{mj})$ stand for the ith-row and jth-column of W, respectively and $\underline{W} = (W_1,\, W_2,\, \ldots,\, W_m)$ is the m^2 column vector consisting of the m rows of W.

Given any constant matrix $B \in \mathscr{M}$, there exists an associated $m^2 \times m^2$ matrix $\tilde{B}$ having m^2 diagonal and $m \times m$ matrix $(a_{ij}I_m)$ (where I_m is an $m \times m$ identity matrix) and such that $(a_{ij}I_m)$ belongs to the ith-row and jth-column of $\tilde{B}$. The inner product of $W,\, Z$ is $\langle W,\, Z\rangle = \text{trace } WZ^T$ and $\langle W,\, Z\rangle = \langle Z,\, W\rangle$. More so, $\| W - Z \|^2 = \langle W - Z,\, W - Z\rangle$ defines a norm on $\mathscr{M}$. Indeed, $\| W \| = |\underline{W}|_{m^2}$, where $|.|_{m^2}$ denotes the usual Euclidean norm in $\mathbb{R}^{m^2}$ and $\underline{W} \in \mathbb{R}^{m^2}$ is as defined earlier.

Lastly, let $x \in \mathbb{R}^m$, then, $|x|$ denotes the norm of x. For a given $r > 0$, $t_1 \in \mathbb{R}$, $C(t_1) = \{\phi : [t_1 - r,\, t_1] \to \mathbb{R}^m / \phi$ is continuous$\}$. Specifically, $C = C(0)$ stands for the space of continuous functions mapping the interval $[-r,\, 0]$ into $\mathbb{R}^m$ and for $\phi \in C$, $\|\phi\| = \sup_{-r \leq \theta \leq 0} |\phi(0)|$. $C_{\mathbf{H}}$ will denote the set of ϕ such that $\|\phi\| \leq \mathbf{H}$. For any continuous function $x(u)$ defined on $-h \leq u < A$, $A > 0$, and any fixed t, $0 \leq t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t-r,\ t]$, that is, x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$.

2. Preliminary results

The following preliminary results are necessary to prove our main results.

Lemma 1 [8]. Given that U is any real symmetric positive definite $m \times m$ matrix. Then for any Z in $\mathscr{M}$, we have

$$u_1 \|Z\|^2 \leq \langle UZ,\ Z\rangle \leq u_2 \|Z\|^2,$$

where $u_1,\ u_2$ are the least and the greatest eigenvalues of U, respectively.

Lemma 2 [9]. Suppose $U,\ V$ are any real $m \times m$ commuting symmetric matrices. Then

(i) the eigenvalues $\lambda_i(UV)$ $(i = 1,\ 2,\ \ldots,\ m)$ of the product matrix UV are all real and satisfy

$$\min_{1\leq j,\ k\leq m} \lambda_j(U)\lambda_k(V) \leq \lambda_i(UV) \leq \max_{1\leq j,\ k\leq m} \lambda_j(U)\lambda_k(V);$$

(ii) the eigenvalues $\lambda_i(U+V)$ $(i = 1,\ 2,\ \ldots,\ m)$ of the sum of matrices U and V are real and satisfy

$$\left\{\min_{1\leq j\leq m} \lambda_j(U) + \min_{1\leq k\leq m} \lambda_k(V)\right\} \leq \lambda_i(U+V) \leq \left\{\max_{1\leq j\leq m} \lambda_j(U) + \max_{1\leq k\leq m} \lambda_k(V)\right\}.$$

Lemma 3 [15]. Let $H \in \mathscr{C}(\mathscr{M})$ be a continuous matrix function and that $H(0) = 0$. Then,

$$H(U) = C(U,\ 0)X(t) - C(U,\ 0)\int_{t-r(t)}^{t} X_1(s)ds,$$

where $U = X(t-r(t))$ and $X,\ X_1 \in \mathscr{M}$.

Proof. By setting $X = X(t-r(t))$ and $Z = X_1(t-r(t))$ in (6), we obtain

$$H(X(t-r(t))) = H(X_1(t-r(t))) + C\big(X(t-r(t)),\ X_1(t-r(t))\big)\big(X(t-r(t)) - X_1(t-r(t))\big). \tag{7}$$

Again, we set $X_1(t-r(t)) = 0$ in (7) and note that

$$X(t-r(t)) = X(t) - \int_{t-r(t)}^{t} X_1(s)ds,$$

where

$$\dot{X}(t) = \frac{dX(t)}{dt} = X_1(t).$$

Then, we have

$$H(X(t-r(t))) = C(X(t-r(t)),\ 0)X(t) - C(X(t-r(t)),\ 0)\int_{t-r(t)}^{t} X_1(s)ds. \tag{8}$$

On letting $U = X(t-r(t))$ in (8), we have

$$H(U) = C(U,\ 0)X(t) - C(U,\ 0)\int_{t-r(t)}^{t} X_1(s)ds. \tag{9}$$

□

Consider the following equation

$$\dot{x} = F(t,\ x_t),\ x_t(\theta) = x(t+\theta),\ -r \leq \theta \leq 0, \tag{10}$$

where $F : \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}^m$ is a continuous mapping which takes bounded set into bounded sets. Hence, by Burton [10] we have the followings.

Lemma 4 [10] Let $V(t,\ \phi) : \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. $V(t,\ 0) = 0$, and such that:

(i) $W_1(|x(t)|) \leq V(t,\ x_t) \leq W_2(|x(t)|) + W_3\left(\int_{t-r(t)}^{t} W_4(|x(s)|)ds\right)$,

(ii) $\dot{V}_{(2.4)}(t,\ x(t)) \leq -W_4(|x(0)|)$,

where, $W_i(i = 1,\ 2,\ 3,\ 4)$ are wedges. Then the null solution of (10) is uniformly asymptotically stable.

Lemma 5 [10] Let $V(t,\ \phi) : \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. $V(t,\ 0) = 0$, and such that:

(i) $W_1(|x(t)|) \leq V(t,\ x_t) \leq W_2(|x(t)|) + W_3\left(\int_{t-r(t)}^{t} W_4(|x(s)|)ds\right)$,

(ii) $\dot{V}_{(3.1)} \leq -W_3(|x(s)|) + M$,

for some positive constant M, where $W_i(i = 1,\ 2,\ 3,\ 4)$ are wedges. Then the solutions of (10) are uniformly bounded and uniformly ultimately bounded for bound $\mathbf{M}$, $\mathbf{M} > 0$ is a constant.

3. Statement of main results

For convenience, we set $\dot{X} = X_1$, $\ddot{X} = X_2$ and $\dddot{X} = \dot{X}_2$ in (1) to obtain

$$\dot{X} = X_1$$

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = -F(X,\ X_1)X_2 - BX_1 - H(X(t-r(t))) + P(t,\ X,\ X_1,\ X_2). \tag{11}$$

Also, from now on, we shall simply write $F(X,\ X_1)$ as F and $P(t,\ X,\ X_1,\ X_2)$ as P.

Theorem 1 Further to the earlier assumptions on F, $H(X)$, B and $r(t)$ contained in (11),

(i) there is an $m \times m$ real continuous operator $C(X,\ X_1)$, $X,\ X_1 \in \mathscr{M}$ so that:

$$H(X) = H(X_1) + C(X,\, X_1)(X - X_1),\ (H(0) = 0);$$

(ii) $\tilde{F}$ and $\tilde{B}$ commute with each other and also with $\tilde{C}(X,\, X_1)$. The eigenvalues $\lambda_i(\tilde{F}),\ \lambda_i(\tilde{B}),\ \lambda_i(\tilde{F} - \tilde{I}),\ \lambda_i(\tilde{F} - \delta_a\tilde{I})$ and $\lambda_i(\tilde{C}(X,\, X_1))$ of symmetric matrices $\tilde{F},\ \tilde{B},\ (\tilde{F} - \tilde{I}),\ (\tilde{F} - \delta_a\tilde{I})$ and $(\tilde{C}(X,\, X_1))$ $(i = 1,\, 2,\, \ldots,\, m^2)$ satisfy

$$\delta_a \le \lambda_i(\tilde{F}) \le \Delta_a,\ \delta_b \le \lambda_i(\tilde{B}) \le \Delta_b,$$

and

$$0 \le \min\{\lambda_i(\tilde{F} - \tilde{I}),\ \lambda_i(\tilde{F} - \delta_a\tilde{I})\} \le \max\{\lambda_i(\tilde{F} - \tilde{I}),\ \lambda_i(\tilde{F} - \delta_a\tilde{I})\} \le \varepsilon,$$

$$0 < \delta_h \le \lambda_i(\tilde{C}(X,\, X_1)) \le \Delta_h;$$

where $\tilde{I}$ is an $m^2 \times m^2$-identity matrix, $\delta_a,\ \delta_b,\ \Delta_a,\ \Delta_b$, and ε are positive constants with ε satisfying

$$\varepsilon = \min\left\{\frac{3\eta\delta_b^2}{\delta_a^2 + \eta^2},\ \frac{\eta\delta_a}{2(2\delta_b + 1)},\ \frac{\eta(1-\rho)\delta_b^2 - 4\Delta_h(\eta + \delta_a)^2}{\eta\delta_b\Delta_b^2(1-\rho)\delta_h^{-1}}\right\},$$

and

$$\Delta_h \le k\delta_a\delta_b\ (k < 1),$$

where $k > 0$ is a constant such that

$$k = \frac{\eta(1-\rho)}{4}\min\left\{\frac{\delta_b}{\delta_a(\eta + \delta_a)^2},\ \frac{1}{(1 + 2\eta)^2}\right\}.$$

If

$$\zeta_1 < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h},\ \frac{2\rho\delta_a\delta_b(1-\zeta_2)}{\Delta_h\left[(1-\rho)\Delta_b + (1+2\eta) + (2-\zeta_2)(\eta + \delta_a)\right]},\ \frac{\eta\delta_a}{2\Delta_h(1+2\eta)}\right\}.$$

Then the null solution of (11) is stable, asymptotically stable and uniformly asymptotically stable when $P(t,\, X,\, X_1,\, X_2) \equiv 0$.

Proof. We make use of the Lyapunov-Krasovskii functional (LKF) $V : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to \mathbb{R}$ defined for any $X,\, X_1,\, X_2 \in \mathscr{M}$ so that $V = V(X,\, X_1,\, X_2)$ is given by

$$2V = \rho(1-\rho)\langle BX,\, BX\rangle + 2\eta\langle BX_1,\, X_1\rangle + \rho\langle BX_1,\, X_1\rangle + \eta\langle X_2,\, X_2\rangle + \eta\langle X_1+X_2,\, X_1+X_2\rangle$$

$$+\langle X_2+\delta_a X_1+(1-\rho)BX,\, X_2+\delta_a X_1+(1-\rho)BX\rangle + \gamma\int_{-r(t)}^{0}\int_{t+s}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta ds, \tag{12}$$

where $\rho,\ \gamma$ and η are constants satisfying $0<\rho<1,\ \eta>0,\ \gamma>0$ and δ_a is as defined above.

The LKF defined in (12) is positive definite, since the coefficient of each term appearing in it is positive and it vanishes at $X=0,\ X_1=0$ and $X_2=0$.

In view of Lemmas 1, 2 and the definition of our norm (under "Notation and definitions"), we obtain for the first term in (12), the following.

$$\begin{aligned}\rho(1-\rho)\delta_b^2 \parallel X \parallel^2 &\le \rho(1-\rho)\langle BX,\, BX\rangle \\ &= \rho(1-\rho)\sum_{i=1}^{n}|BX^i|_n^2 \\ &\le \rho(1-\rho)\Delta_b^2 \parallel X \parallel^2 .\end{aligned}$$

Similar estimates, can easily be obtained for other terms of (12) save the last term. The last term appearing in Eq. (12) satisfies

$$0<\gamma\int_{-r(t)}^{0}\int_{t+s}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta ds.$$

Therefore, we have

$$\delta_1(\parallel X\parallel^2+\parallel X_1\parallel^2+\parallel X_2\parallel^2)\le V\le \delta_2(\parallel X\parallel^2+\parallel X_1\parallel^2+\parallel X_2\parallel^2)+\gamma\int_{-r(t)}^{0}\int_{t+s}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta ds, \tag{13}$$

where $\delta_1=\frac{1}{2}\min\{\rho(1-\rho)\delta_b^2,\ (2\eta+\rho)\delta_b,\ \eta\}$ and

$$\delta_2=\frac{1}{2}\max\left\{\Delta_b(1-\rho)(\Delta_b+\delta_a+1),\ \delta_a\big(\delta_a+\Delta_b(1-\rho)+1\big)+\Delta_b(2\eta+\rho)+2\eta,\ \delta_a+\Delta_b(1-\rho)+3\eta+1\right\}.$$

Inequality defined by (13) implies that $V\to\infty$ as $\parallel X\parallel^2+\parallel X_1\parallel^2+\parallel X_2\parallel^2\to\infty$.

Suppose $(X,\ X_1,\ X_2)$ is any given solution of (11) when $P\equiv 0$. Then, derivative of (12) with respect to t along (11) is obtained as

$$
\begin{aligned}
\frac{dV}{dt} =& -\langle (1-\rho)BX,\, H(X(t-r(t)))\rangle - \langle \eta BX_1,\, X_1\rangle - \langle \rho\delta_a X_1,\, BX_1\rangle - \langle (1+2\eta)X_2,\, H(X(t-r(t)))\rangle \\
& -\langle (\eta+\delta_a)X_1,\, H(X(t-r(t)))\rangle - \langle \eta F X_2,\, X_2\rangle + \gamma\frac{d}{dt}\int_{-r(t)}^{0}\int_{t+s}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta ds \\
& -\langle (F-\delta_a I)X_2 + \eta(F-I)X_2,\, X_2\rangle - \langle (1-\rho)(F-\delta_a I)X_2,\, BX\rangle - \langle (F-\delta_a I)X_2,\, \delta_a X_1\rangle - \langle \eta(F-I)X_2,\, X_1\rangle.
\end{aligned}
$$

But,

$$
\begin{aligned}
\gamma\frac{d}{dt}\int_{-r(t)}^{0}\int_{t+s}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta ds =& \gamma\int_{-r(t)}^{0}\Big(\frac{d}{dt}\int_{t+s}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta\Big)ds + \gamma\int_{t+s}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta\,\frac{d}{dt}\int_{-r(t)}^{0} ds \\
=& \gamma\int_{-r(t)}^{0}\Big(\langle X_1(t),\, X_1(t)\rangle - \langle X_1(t+s),\, X_1(t+s)\rangle\Big)ds \\
& + \gamma r'(t)\int_{t-r(t)}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta \\
=& \gamma\langle X_1(t),\, X_1(t)\rangle\int_{-r(t)}^{0} ds - \gamma\int_{-r(t)}^{0}\langle X_1(t+s),\, X_1(t+s)\rangle ds \\
& + \gamma r'(t)\int_{t-r(t)}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta \\
=& \gamma r(t)\langle X_1(t),\, X_1(t)\rangle - \gamma\int_{-r(t)}^{0}\langle X_1(t+s),\, X_1(t+s)\rangle ds \\
& + \gamma r'(t)\int_{t-r(t)}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta.
\end{aligned}
$$

On substituting $\theta = t+s$ and $d\theta = ds$ in the above, we have

$$
\gamma\frac{d}{dt}\int_{-r(t)}^{0}\int_{t+s}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta ds = \gamma r(t)\langle X_1(t),\, X_1(t)\rangle - \gamma(1-r'(t))\int_{t-r(t)}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta.
$$

Thus,

$$\frac{dV}{dt} = -\langle (1-\rho)BX,\ H(X(t-r(t)))\rangle - \langle \eta BX_1,\ X_1\rangle - \langle \rho\delta_a X_1,\ BX_1\rangle - \langle (1+2\eta)X_2,\ H(X(t-r(t)))\rangle$$

$$-\langle (\eta+\delta_a)X_1,\ H(X(t-r(t)))\rangle - \langle \eta FX_2,\ X_2\rangle + \gamma r(t)\langle X_1,\ X_1\rangle - \gamma(1-r'(t))\int_{t-r(t)}^{t}\langle X_1(\theta),\ X_1(\theta)\rangle d\theta$$

$$-\langle (F-\delta_a I)X_2 + \eta(F-I)X_2,\ X_2\rangle - \langle (1-\rho)(F-\delta_a I)X_2,\ BX\rangle - \langle (F-\delta_a I)X_2,\ \delta_a X_1\rangle - \langle \eta(F-I)X_2,\ X_1\rangle.$$

Using (9) in the above, we get

$$\frac{dV}{dt} = -\langle (1-\rho)BX,\ C(U,\ 0)X\rangle - \langle \eta BX_1,\ X_1\rangle - \langle \rho\delta_a X_1,\ BX_1\rangle - \langle (1+2\eta)X_2,\ C(U,\ 0)X\rangle$$

$$-\langle (\eta+\delta_a)X_1,\ C(U,\ 0)X\rangle - \langle \eta FX_2,\ X_2\rangle + \gamma r(t)\langle X_1,\ X_1\rangle - \gamma(1-r'(t))\int_{t-r(t)}^{t}\langle X_1(\theta),\ X_1(\theta)\rangle d\theta$$

$$-\langle (F-\delta_a I)X_2 + \eta(F-I)X_2,\ X_2\rangle - \langle (1-\rho)(F-\delta_a I)X_2,\ BX\rangle - \langle (F-\delta_a I)X_2,\ \delta_a X_1\rangle - \langle \eta(F-I)X_2,\ X_1\rangle$$

$$+\int_{t-r(t)}^{t}\langle (1-\rho)BX(s) + (\eta+\delta_a)X_1(s) + (1+2\eta)X_2(s),\ C(U,\ 0)X_1(s)\rangle ds.$$

For ease of computation, we shall write $\frac{dV}{dt}$ as

$$\frac{dV}{dt} = -U_1 - U_2 - U_3 + U_4, \tag{14}$$

where,

$$U_1 = \frac{1-\rho}{2}\langle BX,\ C(U,\ 0)X\rangle + \langle \rho\delta_a X_1,\ BX_1\rangle + \frac{\eta}{4}\langle FX_2,\ X_2\rangle,$$

$$U_2 = \frac{1-\rho}{4}\langle BX,\ C(U,\ 0)X\rangle + \eta\langle BX_1,\ X_1\rangle + \langle (\eta+\delta_a)X_1,\ C(U,\ 0)X\rangle + \frac{\eta}{2}\langle FX_2,\ X_2\rangle + \langle (F-\delta_a I)X_2,\ \delta_a X_1\rangle$$

$$+\langle (F-\delta_a I)X_2,\ X_2\rangle + \eta\langle (F-I)X_2,\ X_2\rangle + (1-\rho)\langle (F-\delta_a I)X_2,\ BX\rangle + \eta\langle (F-I)X_2,\ X_1\rangle,$$

$$U_3 = \frac{1-\rho}{4}\langle BX,\ C(U,\ 0)X\rangle + \frac{\eta}{4}\langle FX_2,\ X_2\rangle + (1+2\eta)\langle X_2,\ C(U,\ 0)X\rangle,$$

$$U_4=\int_{t-r(t)}^{t}\langle(1-\rho)BX(s)+(\eta+\delta_a)X_1(s)+(1+2\eta)X_2(s),\ C(U,\ 0)X_1(s)\rangle ds$$

$$+\gamma r(t)\langle X_1,\ X_1\rangle-\gamma(1-r'(t))\int_{t-r(t)}^{t}\langle X_1(\theta),\ X_1(\theta)\rangle d\theta.$$

We now find an estimate for each of $U_i,\ (i=1,\ 2,\ 3,\ 4)$. Starting with $U_1,$ we have from the conditions of Theorem 1 and Lemmas 1, 2.

$$\begin{aligned}U_1&=\frac{1-\rho}{2}\langle BX,\ C(U,\ 0)X\rangle+\langle\rho\delta_aX_1,\ BX_1\rangle+\frac{\eta}{4}\langle FX_2,\ X_2\rangle\\&=\underline{X}^T\left[\frac{1-\rho}{2}\tilde{B}\tilde{C}(U,\ 0)\right]\underline{X}+\underline{X}_1^T\left[\rho\delta_a\tilde{B}\right]\underline{X}_1+\underline{X}_2^T\left[\frac{\eta}{4}\tilde{F}\right]\underline{X}_2\\&\geq\frac{1-\rho}{2}\delta_b\delta_h\langle X,\ X\rangle+\rho\delta_a\delta_b\langle X_1,\ X_1\rangle+\frac{\eta}{4}\delta_a\langle X_2,\ X_2\rangle\\&\geq\delta_3(\|X\|^2+\|X_1\|^2+\|X_2\|^2),\end{aligned}$$

where $\delta_3=\min\left\{\frac{1-\rho}{2}\delta_b\delta_h,\ \rho\delta_a\delta_b,\ \frac{\eta}{4}\delta_a\right\}.$

Given that $k_i>0(i=1,\ 2,\ ..,\ 5)$ are some constants whose values are to be estimated later. Then, from the conditions of Theorem 1, Lemmas 1 and 2, we have the following estimates

$$\begin{aligned}\langle(\eta+\delta_a)X_1,\ C(U,\ 0)X\rangle&=\|k_1(\eta+\delta_a)^{\frac{1}{2}}X_1+\frac{1}{2}k_1^{-1}(\eta+\delta_a)^{\frac{1}{2}}C(U,\ 0)X\|^2\\&\quad-\langle k_1^2(\eta+\delta_a)X_1,\ X_1\rangle-\frac{1}{4}k_1^{-2}\langle(\eta+\delta_a)C(U,\ 0)X,\ C(U,\ 0)X\rangle\\&\geq-k_1^2\langle(\eta+\delta_a)X_1,\ X_1\rangle-\frac{1}{4}k_1^{-2}\langle(\eta+\delta_a)C(U,\ 0)X,\ C(U,\ 0)X\rangle;\end{aligned}\tag{15}$$

$$\begin{aligned}\langle(F-\delta_aI)X_2,\delta_aX_1\rangle&=\|k_2(F-\delta_aI)^{\frac{1}{2}}\delta_a^{\frac{1}{2}}X_2+\frac{1}{2}k_2^{-1}(F-\delta_aI)^{\frac{1}{2}}\delta_a^{\frac{1}{2}}X_1\|^2-k_2^2\langle(F-\delta_aI)\delta_aX_2,\ X_2\rangle\\&\quad-\frac{1}{4}k_2^{-2}\langle(F-\delta_aI)X_1,\ \delta_aX_1\rangle\\&\geq-\underline{X}_2^T[k_2^2(\tilde{F}-\delta_a\tilde{I})\delta_a]\underline{X}_2-\underline{X}_1^T[\frac{1}{4}k_2^{-2}(\tilde{F}-\delta_a\tilde{I})\delta_a]\underline{X}_1\end{aligned}$$

$$\geq -k_2^2\varepsilon\delta_a \parallel X_2 \parallel^2 - \frac{1}{4}k_2^{-2}\varepsilon\delta_a \parallel X_1 \parallel^2; \tag{16}$$

$$(1-\rho)\langle (F-\delta_a I)X_2,\, BX\rangle = (1-\rho)\|k_3(F-\delta_a I)^{\frac{1}{2}}B^{\frac{1}{2}}X_2 + \frac{1}{2}k_3^{-1}(F-\delta_a I)^{\frac{1}{2}}B^{\frac{1}{2}}X\|^2$$

$$- k_3^2(1-\rho)\langle (F-\delta_a I)BX_2,\, X_2\rangle - \frac{1}{4}k_3^{-2}(1-\rho)\langle (F-\delta_a I)X,\, BX\rangle$$

$$\geq -\underline{X}_2^T[k_3^2(1-\rho)(\tilde{F}-\delta_a\tilde{I})\tilde{B}]\underline{X}_2 - \underline{X}^T[\frac{1}{4}k_3^{-2}(1-\rho)(\tilde{F}-\delta_a\tilde{I})\tilde{B}]\underline{X}$$

$$\geq -k_3^2\varepsilon\Delta_b(1-\rho) \parallel X_2 \parallel^2 - \frac{1}{4}k_3^{-2}\varepsilon\Delta_b(1-\rho) \parallel X \parallel^2; \tag{17}$$

$$\eta\langle (F-I)X_2,\, X_1\rangle = \eta\|k_4(F-I)^{\frac{1}{2}}X_2 + \frac{1}{2}k_4^{-1}(F-I)^{\frac{1}{2}}X_1\|^2 - k_4^2\eta\langle (F-I)X_2,\, X_2\rangle - \frac{1}{4}k_4^{-2}\eta\langle (F-I)X_1,\, X_1\rangle$$

$$\geq -\underline{X}_2^T[k_4^2\eta(\tilde{F}-\tilde{I})]\underline{X}_2 - \underline{X}_1^T[\frac{1}{4}k_4^{-2}\eta(\tilde{F}-\tilde{I})]\underline{X}_1$$

$$\geq -k_4^2\eta\varepsilon \parallel X_2 \parallel^2 - \frac{1}{4}k_4^{-2}\eta\varepsilon \parallel X_1 \parallel^2; \tag{18}$$

and finally,

$$(1+2\eta)\langle X_2,\, C(U,\, 0)X\rangle = (1+2\eta)\|k_5X_2 + \frac{1}{2}k_5^{-1}C(U,\, 0)X\|^2 - k_5^2(1+2\eta)\langle X_2,\, X_2\rangle$$

$$- \frac{1}{4}k_5^{-2}(1+2\eta)\langle C(U,\, 0)X,\, C(U,\, 0)X\rangle. \tag{19}$$

Thus, by Lemmas 1, 2 and estimates (15)-(18), we have

$$U_2 \geq \underline{X}^T\frac{1}{4}[(1-\rho)\delta_b - k_1^{-2}\tilde{C}(U,\, 0)(\eta+\delta_a)]\tilde{C}(U,\, 0)\underline{X} + \eta\delta_b \parallel X_1 \parallel^2 - k_1^2(\eta+\delta_a) \parallel X_1 \parallel^2 + \frac{\eta}{2}\delta_a \parallel X_2 \parallel^2$$

$$- k_2^2\varepsilon\delta_a \parallel X_2 \parallel^2 - \frac{1}{4}k_2^{-2}\varepsilon\delta_a \parallel X_1 \parallel^2 + \underline{X}_2^T[(\tilde{F}-\delta_a\tilde{I})]\underline{X}_2 + \underline{X}_2^T[\eta(\tilde{F}-\tilde{I})]\underline{X}_2 - k_3^2\varepsilon\Delta_b(1-\rho) \parallel X_2 \parallel^2$$

$$- \frac{1}{4}k_3^{-2}\varepsilon\Delta_b(1-\rho) \parallel X \parallel^2 - k_4^2\eta\varepsilon \parallel X_2 \parallel^2 - \frac{1}{4}k_4^{-2}\eta\varepsilon \parallel X_1 \parallel^2$$

$$\geq \frac{1}{4}\left[[(1-\rho)\delta_b - k_1^{-2}\Delta_h(\eta+\delta_a)]\delta_h - k_3^{-2}\varepsilon\Delta_b(1-\rho)\right] \parallel X \parallel^2$$

$$+\left[\eta\delta_b - k_1^2(\eta+\delta_a) - \frac{1}{4}k_2^{-2}\varepsilon\delta_a - \frac{1}{4}k_4^{-2}\eta\varepsilon\right] \parallel X_1 \parallel^2$$

$$+\left[\frac{\eta}{2}\delta_a - \varepsilon(k_2^2\delta_a + k_3^2\Delta_b + k_4^2\eta)\right] \parallel X_2 \parallel^2 .$$

Similarly, using Lemmas 1, 2 and estimate (19) in U_3, we obtain

$$U_3 \geq \underline{X}^T \frac{1}{4}\left[(1-\rho)\delta_b - k_5^{-2}\tilde{C}(U,\ 0)(1+2\eta)\right]\tilde{C}(U,\ 0)\underline{X} + \underline{X}_2^T\left[\frac{\eta}{4}\tilde{F}\right]\underline{X}_2 - k_5^2(1+2\eta) \parallel X_2 \parallel^2$$

$$\geq \frac{1}{4}\left[(1-\rho)\delta_b - k_5^{-2}\Delta_h(1+2\eta)\right]\delta_h \parallel X \parallel^2 + \frac{1}{4}\left[\eta\delta_a - 4k_5^2(1+2\eta)\right] \parallel X_2 \parallel^2 .$$

If we choose $k_1^2 = \dfrac{\eta\delta_b}{4(\eta+\delta_a)}$, $k_2^2 = \dfrac{\delta_b}{\delta_a}$, $k_3^2 = \dfrac{1}{\Delta_b}$, $k_4^2 = \dfrac{\delta_b}{\eta}$ and $k_5^2 = \dfrac{\eta\delta_a}{4(1+2\eta)}$, then we have

$$U_2 \geq 0$$

with

$$\Delta_h \leq \frac{k_1^2(1-\rho)\delta_b}{(\eta+\delta_a)} = \frac{\eta(1-\rho)\delta_b^2}{4(\eta+\delta_a)^2}, \tag{20}$$

and

$$U_3 \geq 0,$$

with

$$\Delta_h \leq \frac{k_5^2(1-\rho)\delta_b}{(1+2\eta)} = \frac{\eta(1-\rho)\delta_a\delta_b}{4(1+2\eta)^2}. \tag{21}$$

It then follows from inequalities (20) and (21) that for all $X,\ X_1,\ X_2 \in \mathscr{M}$, $U_2 \geq 0$ and $U_3 \geq 0$, whenever

$$\Delta_h \leq k\delta_a\delta_b$$

such that

$$k=\frac{\eta(1-\rho)}{4}\min\left\{\frac{\delta_b}{\delta_a(\eta+\delta_a)^2},\ \frac{1}{(1+2\eta)^2}\right\}.$$

By the fact that $2|\langle e_1,\ e_2\rangle|\le\|e_1\|^2+\|e_2\|^2,\ U_4$ becomes

$$|U_4|=\int_{t-r(t)}^{t}\left[(1-\rho)\tilde{B}\underline{X}^T(s)+(\eta+\delta_a)\underline{X}_1^T(s)+(1+2\eta)\underline{X}_2^T(s)\right]\tilde{C}(U,\ 0)\underline{X}_1(s)ds$$

$$+\gamma r(t)\langle X_1,\ X_1\rangle-\gamma(1-r'(t))\int_{t-r(t)}^{t}\langle X_1(\theta),\ X_1(\theta)\rangle d\theta$$

$$\le\frac{1}{2}(1-\rho)\Delta_b\Delta_h r(t)\|X\|^2+\frac{1}{2}(\eta+\delta_a)\Delta_h r(t)\|X_1\|^2+\frac{1}{2}(1+2\eta)\Delta_h r(t)\|X_2\|^2$$

$$+\left\{\frac{1}{2}(1-\rho)\Delta_b\Delta_h+\frac{1}{2}(\eta+\delta_a)\Delta_h+\frac{1}{2}(1+2\eta)\Delta_h\right\}\int_{t-r(t)}^{t}\langle X_1(s),\ X_1(s)\rangle ds$$

$$+\gamma r(t)\langle X_1,\ X_1\rangle-\gamma(1-r'(t))\int_{t-r(t)}^{t}\langle X_1(\theta),\ X_1(\theta)\rangle d\theta$$

$$\le\frac{1}{2}(1-\rho)\Delta_b\Delta_h\zeta_1\|X\|^2+\frac{1}{2}(\eta+\delta_a)\Delta_h\zeta_1\|X_1\|^2+\frac{1}{2}(1+2\eta)\Delta_h\zeta_1\|X_2\|^2$$

$$+\left\{\frac{1}{2}(1-\rho)\Delta_b\Delta_h+\frac{1}{2}(\eta+\delta_a)\Delta_h+\frac{1}{2}(1+2\eta)\Delta_h\right\}\int_{t-r(t)}^{t}\langle X_1(s),\ X_1(s)\rangle ds$$

$$+\gamma\zeta_1\langle X_1,\ X_1\rangle-\gamma(1-\zeta_2)\int_{t-r(t)}^{t}\langle X_1(\theta),\ X_1(\theta)\rangle d\theta. \tag{22}$$

If we set

$$\gamma=\frac{\Delta_h}{2(1-\zeta_2)}\left[(1-\rho)\Delta_b+(\eta+\delta_a)+(1+2\eta)\right]$$

in (22), then we obtain

$$|U_4|\le\frac{1}{2}(1-\rho)\Delta_b\Delta_h\zeta_1\|X\|^2+\frac{1}{2}(1+2\eta)\Delta_h\zeta_1\|X_2\|^2$$

$$+\frac{\Delta_h\zeta_1}{2(1-\zeta_2)}\left[(1-\rho)\Delta_b+(1+2\eta)+(2-\zeta_2)(\eta+\delta_a)\right]\|X_1\|^2.$$

On plugging back the values for $U_i (i = 1,\ 2,\ 3,\ 4)$ into (14), we obtain

$$\frac{dV}{dt} \le -\frac{1}{2}(1-\rho)[\delta_b\delta_h - \Delta_b\Delta_h\zeta_1]\|X\|^2 - \frac{1}{4}\left[\eta\delta_a - 2\Delta_h\zeta_1(1+2\eta)\right]\|X_2\|^2$$

$$-\left[\rho\delta_a\delta_b - \frac{\Delta_h\zeta_1}{2(1-\zeta_2)}\left[(1-\rho)\Delta_b + (1+2\eta) + (2-\zeta_2)(\eta+\delta_a)\right]\right]\|X_1\|^2. \tag{23}$$

Let

$$\zeta_1 < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h},\ \frac{2\rho\delta_a\delta_b(1-\zeta_2)}{\Delta_h\left[(1-\rho)\Delta_b + (1+2\eta) + (2-\zeta_2)(\eta+\delta_a)\right]},\ \frac{\eta\delta_a}{2\Delta_h(1+2\eta)}\right\},$$

then we have for some positive constants $D_2,\ D_3$ and D_4

$$\frac{dV}{dt} \le -D_2 \parallel X \parallel^2 - D_3 \parallel X_1 \parallel^2 - D_4 \parallel X_2 \parallel^2$$

$$\le -D_5\{\parallel X \parallel^2 + \parallel X_1 \parallel^2 + \parallel X_2 \parallel^2\}, \tag{24}$$

where $D_5 = \min\{D_2,\ D_3,\ D_4\}$. Thus, by inequalities (13) and (24), we established uniform stability of null solution to (11).

To conclude the proof, we define for any $X,\ X_1,\ X_2 \in \mathscr{M}$, a set Q,

$$Q \equiv \{(X,\ X_1,\ X_2) : \dot{V}(X,\ X_1,\ X_2) = 0\}.$$

Applying LaSalle's invariance principle to Q, it is obvious that $(X,\ X_1,\ X_2) \in Q$ shows that $X = X_1 = X_2 = 0$, i.e, $(X,\ X_1,\ X_2) = (0,\ 0,\ 0)$. This in turn implies that the largest invariant set found in Q is $(0,\ 0,\ 0) \in Q$. Hence, conditions of Lemma 4 hold. Therefore, the null solution of (1) or (11) is uniformly asymptotically stable. Thus, the result is established. □

Theorem 2 Suppose all the assumptions of Theorem 1 hold and $P \neq 0$. Furthermore, we assume

(iii) there are some constants $D_0 \ge 0$ and $D_1 \ge 0$ so that

$$\|P(t,\ X,\ X_1,\ X_2)\| \le D_0 + D_1(\|X\| + \|X_1\| + \|X_2\|), \tag{25}$$

uniformly in t, for all $X,\ X_1,\ X_2 \in \mathscr{M}$. Then, if D_1 is adequately small, the solutions of (11) are uniformly ultimately bounded if

$$\zeta_1 < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h},\ \frac{2\rho\delta_a\delta_b(1-\zeta_2)}{\Delta_h\left[(1-\rho)\Delta_b + (1+2\eta) + (2-\zeta_2)(\eta+\delta_a)\right]},\ \frac{\eta\delta_a}{2\Delta_h(1+2\eta)}\right\}.$$

Proof. We still depend on the LKF defined in (12) for the proof of this theorem. Thus, inequality (13) earlier obtained is still valid for $P(t,\, X,\, X_1,\, X_2) \neq 0$. Under the conditions of Theorem 2, derivative of V is given by

$$\frac{dV}{dt} = -\langle(1-\rho)BX,\, H(X(t-r(t)))\rangle - \langle \eta BX_1,\, X_1\rangle - \langle \rho\delta_a X_1,\, BX_1\rangle - \langle(1+2\eta)X_2,\, H(X(t-r(t)))\rangle$$

$$-\langle(\eta+\delta_a)X_1,\, H(X(t-r(t)))\rangle - \langle \eta F X_2,\, X_2\rangle + \gamma r(t)\langle X_1,\, X_1\rangle - \gamma(1-r'(t))\int_{t-r(t)}^{t}\langle X_1(\theta),\, X_1(\theta)\rangle d\theta$$

$$-\langle(F-\delta_a I)X_2 + \eta(F-I)X_2,\, X_2\rangle - \langle(1-\rho)(F-\delta_a I)X_2,\, BX\rangle - \langle(F-\delta_a I)X_2,\, \delta_a X_1\rangle - \langle\eta(F-I)X_2,\, X_1\rangle$$

$$+\langle(1-\rho)BX + (\eta+\delta_a)X_1 + (1+2\eta)X_2,\, P\rangle.$$

By Schwarz's inequality and (25), we have

$$\begin{aligned} &|\langle(1-\rho)BX + (\eta+\delta_a)X_1 + (1+2\eta)X_2,\, P\rangle| \\ &\leq [(1-\rho)\Delta_b \parallel X \parallel + (\eta+\delta_a) \parallel X_1 \parallel + (1+2\eta) \parallel X_2 \parallel] \parallel P \parallel \\ &\leq \delta_4(\|X\| + \|X_1\| + \|X_2\|)\,[D_0 + D_1(\|X\| + \|X_1\| + \|X_2\|)], \end{aligned} \tag{26}$$

where $\delta_4 = \max\{(1-\rho)\Delta_b,\, (\eta+\delta_a),\, (1+2\eta)\}$.

Hence, if we carefully follow the same pattern used to obtain (23) of Theorem 1 or simply combine (23) and (26), we obtain

$$\frac{dV}{dt} \leq -\frac{1}{2}(1-\rho)[\delta_b\delta_h - \Delta_b\Delta_h\zeta_1]\|X\|^2 - \left[\rho\delta_a\delta_b - \frac{\Delta_h\zeta_1}{2(1-\zeta_2)}[(1-\rho)\Delta_b + (1+2\eta) + (2-\zeta_2)(\eta+\delta_a)]\right]\|X_1\|^2$$

$$-\frac{1}{4}[\eta\delta_a - 2\Delta_h\zeta_1(1+2\eta)]\,\|X_2\|^2 + \delta_4(\|X\| + \|X_1\| + \|X_2\|)\,[D_0 + D_1(\|X\| + \|X_1\| + \|X_2\|)].$$

By letting

$$\zeta_1 < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h},\, \frac{2\rho\delta_a\delta_b(1-\zeta_2)}{\Delta_h[(1-\rho)\Delta_b + (1+2\eta) + (2-\zeta_2)(\eta+\delta_a)]},\, \frac{\eta\delta_a}{2\Delta_h(1+2\eta)}\right\},$$

we get

$$\frac{dV}{dt} \leq -\delta_5(\|X\|^2 + \|X_1\|^2 + \|X_2\|^2) + 3\delta_4 D_1(\|X\|^2 + \|X_1\|^2 + \|X_2\|^2) + \delta_4 D_0(\|X\| + \|X_1\| + \|X_2\|)$$

$$= -(\delta_5 - 3\delta_4 D_1)(\|X\|^2 + \|X_1\|^2 + \|X_2\|^2) + \delta_4 D_0(\|X\| + \|X_1\| + \|X_2\|),$$

where

$$0 < \delta_5 < \min\{(1-\rho)[\delta_b\delta_h - \Delta_b\Delta_h\zeta_1],\ 2\rho\delta_a\delta_b - \frac{\Delta_h\zeta_1}{(1-\zeta_2)}[(1-\rho)\Delta_b + (1+2\eta) + (2-\zeta_2)(\eta+\delta_a)],$$

$$\frac{1}{2}[\eta\delta_a - 2\Delta_h\zeta_1(1+2\eta)]\}.$$

If we choose $D_1 < \frac{\delta_5}{3\delta_4}$, then, we can find some constants $\theta_1 > 0$, such that

$$\frac{dV}{dt} \leq -\theta_1(\|X\|^2 + \|X_1\|^2 + \|X_2\|^2) + n\theta_1(\|X\| + \|X_1\| + \|X_2\|)$$

$$= -\frac{\theta_1}{2}(\|X\|^2 + \|X_1\|^2 + \|X_2\|^2) - \frac{\theta_1}{2}\{(\|X\| - n)^2 + (\|X_1\| - n)^2 + (\|X_2\| - n)^2\} + \frac{3\theta_1}{2}n^2$$

$$\leq -\frac{\theta_1}{2}(\|X\|^2 + \|X_1\|^2 + \|X_2\|^2) + \frac{3\theta_1}{2}n^2,$$

for some n and θ_1. Clearly, conditions of Lemma 5 hold with $M = \frac{3\theta_1}{2}n^2$. Therefore, all solutions of (1) or (11) are uniformly ultimately bounded. □

Theorem 3 Further to the basic conditions of Theorem 2, let $P(t, X, X_1, X_2) = P(t+\omega, X, X_1, X_2)$ uniformly for all $X, X_1, X_2 \in \mathscr{M}$. Then (11) has at least one ω-periodic solution.

Proof. The proof follows the same pattern as in the proof of [Theorem 3, Meng [13]]. □

4. Example

To show the validity of our results, we give the following numerical examples as special cases of (1) or (11) for $n = 2$.

In (11), let

$$P(t, X, X_1, X_2) \equiv 0,\ X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix},\ X_1 = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix},\ X_2 = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$$

$$F(X,\, X_1) = \begin{bmatrix} 1.0005 + \dfrac{1}{x_1^2 y_1^2 + x_2^2 y_2^2 + 1{,}000} & 0 \\ 0 & 1.0005 + \dfrac{1}{x_3^2 y_3^2 + x_4^2 y_4^2 + 1{,}000} \end{bmatrix},$$

$$H(X(t-r(t))) = \begin{bmatrix} \dfrac{x_1(t-r(t))}{20+\cos x_1(t-r(t))} & \dfrac{x_3(t-r(t))}{20+\cos x_2(t-r(t))} \\ \dfrac{x_2(t-r(t))}{20+\cos x_1(t-r(t))} & \dfrac{x_4(t-r(t))}{20+\cos x_2(t-r(t))} \end{bmatrix},\ r(t) = \frac{1+\cos t}{13+\cos t},\ B = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$$

From this example, we show that all the conditions of Theorem (1) are realized.

(i) The function $r(t)$ certainly satisfies $0 \le r(t) \le \dfrac{1}{7} = \zeta_1$ while its derivative $r'(t) = \dfrac{-12\sin t}{(13+\cos t)^2} \le \dfrac{1}{12} = \zeta_2$.

(ii) $H(X(t-\mu(t))$ can be written in the form of (6) (with $H(0) = 0$) as

$$H(X(t-r(t))) = \begin{bmatrix} \dfrac{1}{20+\cos x_1(t-r(t))} & 0 \\ 0 & \dfrac{1}{20+\cos x_2(t-r(t))} \end{bmatrix} \begin{bmatrix} x_1(t-r(t)) & x_2(t-r(t)) \\ x_3(t-r(t)) & x_4(t-r(t)) \end{bmatrix},$$

so that,

$$C_H((X(t-r(t))),\, 0) = \begin{bmatrix} \dfrac{1}{20+\cos x_1(t-r(t))} & 0 \\ 0 & \dfrac{1}{20+\cos x_2(t-r(t))} \end{bmatrix}.$$

The associated matrix to $C_H((X(t-r(t))),\, 0)$ based on our notation is

$$\tilde{C}_H((X(t-r(t))),\, 0) = \begin{bmatrix} \dfrac{1}{20+\cos x_1(t-r(t))} & 0 & 0 & 0 \\ 0 & \dfrac{1}{20+\cos x_1(t-r(t))} & 0 & 0 \\ 0 & 0 & \dfrac{1}{20+\cos x_2(t-r(t))} & 0 \\ 0 & 0 & 0 & \dfrac{1}{20+\cos x_2(t-r(t))} \end{bmatrix}.$$

This matrix has the following eigenvalues:

$$\lambda_{1,\,2}(\tilde{C}_H) = \frac{1}{20+\cos x_1(t-r(t))},\ \lambda_{3,\,4}(\tilde{C}_H) = \frac{1}{20+\cos x_2(t-r(t))},$$

such that

$$\delta_h = \frac{1}{21} \leq \lambda_i(\tilde{C}_H) \leq \frac{1}{19} = \Delta_h,\ (i = 1,\ 2,\ 3,\ 4).$$

(iii) The matrix $\tilde{B}$ associated with the matrix B as defined under notation and definition is

$$\tilde{B} = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \end{bmatrix}.$$

This matrix is clearly symmetric and has eigenvalues

$$\lambda_{1,\ 2}(\tilde{B}) = 4 - \sqrt{2},\ \lambda_{3,\ 4}(\tilde{B}) = 4 + \sqrt{2},$$

which implies that

$$\delta_b = 4 - \sqrt{2} \leq \lambda_i(\tilde{B}) \leq 4 + \sqrt{2} = \Delta_b,\ (i = 1,\ 2,\ 3,\ 4).$$

(iv) Further more,

$$\tilde{F} = \begin{bmatrix} 1.0005 + \chi_1 & 0 & 0 & 0 \\ 0 & 1.0005 + \chi_1 & 0 & 0 \\ 0 & 0 & 1.0005 + \chi_2 & 0 \\ 0 & 0 & 0 & 1.0005 + \chi_2 \end{bmatrix},$$

where $\chi_1 = \frac{1}{x_1^2 y_1^2 + x_2^2 y_2^2 + 1000}$, $\chi_2 = \frac{1}{x_3^2 y_3^2 + x_4^2 y_4^2 + 1000}$ and the eigenvalues of $\tilde{F}$ are:

$$\lambda_{1,\ 2}(\tilde{F}) = 1.0005 + \chi_1,\ \lambda_{3,\ 4}(\tilde{F}) = 1.0005 + \chi_2,$$

such that

$$\delta_a = 1.0005 \leq \lambda_i(\tilde{F}) \leq 1.0015 = \Delta_a,\ (i = 1,\ 2,\ 3,\ 4).$$

Similarly, it can be shown that

$$(\tilde{F}-\tilde{I})=\begin{bmatrix} 0.0005+\chi_1 & 0 & 0 & 0 \\ 0 & 0.0005+\chi_1 & 0 & 0 \\ 0 & 0 & 0.0005+\chi_2 & 0 \\ 0 & 0 & 0 & 0.0005+\chi_2 \end{bmatrix},$$

with eigenvalues,

$$\lambda_{1,\,2}(\tilde{F}-\tilde{I})=0.0005+\chi_1,\ \lambda_{3,\,4}(\tilde{F}-\tilde{I})=0.0005+\chi_2.$$

Hence,

$$0.0005\leq\lambda_i(\tilde{F}-\tilde{I})\leq 0.0015,\ (i=1,\ 2,\ 3,\ 4).$$

Also,

$$(\tilde{F}-\delta_a\tilde{I})=(\tilde{F}-1.0005\tilde{I})=\begin{bmatrix} \chi_1 & 0 & 0 & 0 \\ 0 & \chi_1 & 0 & 0 \\ 0 & 0 & \chi_2 & 0 \\ 0 & 0 & 0 & \chi_2 \end{bmatrix},$$

has the following eigenvalues,

$$\lambda_{1,\,2}(\tilde{F}-1.0005\tilde{I})=\chi_1,\ \lambda_{3.4}(\tilde{F}-1.0005\tilde{I})=\chi_2.$$

This implies that,

$$0\leq\lambda_i(\tilde{F}-1.0005\tilde{I})\leq\Delta_a=0.001.$$

(v) By choosing $\rho=\dfrac{1}{5}$ and $\eta=\dfrac{1}{4}$, we estimate the value of ε.

$$\varepsilon = \min\left\{\frac{3\eta\delta_b^2}{\delta_a^2+\eta^2},\ \frac{\eta\delta_a}{2(2\delta_b+1)},\ \frac{\eta(1-\rho)\delta_b^2-4\Delta_h(\eta+\delta_a)^2}{\eta\delta_b\Delta_b^2(1-\rho)\delta_h^{-1}}\right\}$$

$$= \min\left\{\frac{3\times\frac{1}{4}(4-\sqrt{2})^2}{1.0005^2+(0.25)^2},\ \frac{0.25\times 1.0005}{2(2(4-\sqrt{2})+1)},\ \frac{0.25(1-\frac{1}{5})(4-\sqrt{2})^2-4\times\frac{1}{19}(0.25+1.0005)^2}{0.25(4-\sqrt{2})(4+\sqrt{2})^2(1-\frac{1}{5})\times 21}\right\}$$

$$= \min\{4.7153,\ 0.01651,\ 0.0032\}$$

$$= 0.0032.$$

Therefore, from our calculations above, we have

$$0 \leq \min\{\lambda_i(\tilde{F}-\tilde{I}),\ \lambda_i(\tilde{F}-\delta_a\tilde{I})\} \leq \max\{\lambda_i(\tilde{F}-\tilde{I}),\ \lambda_i(\tilde{F}-\delta_a\tilde{I})\} = 0.001 \leq \varepsilon = 0.0032.$$

(vi) Going further, we have

$$k = \frac{\eta(1-\rho)}{4}\min\left\{\frac{\delta_b}{\delta_a(\eta+\delta_a)^2},\ \frac{1}{(1+2\eta)^2}\right\}$$

$$= \frac{0.25(1-\frac{1}{5})}{4}\min\left\{\frac{4-\sqrt{2}}{1.0005(0.25+1.0005)^2},\ \frac{1}{(1+\frac{1}{2})^2}\right\}$$

$$= \frac{4}{5}\min\{0.1034,\ 0.0278\}$$

$$= 0.02224 < 1.$$

And

$$\Delta_h = \frac{1}{19} \leq k\delta_a\delta_b = 0.02224\times 1.0005(4-2\sqrt{2})$$

$$0.05263 < 0.05754.$$

(vii) Lastly,

$$\frac{1}{7} = \zeta_1 < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h},\ \frac{2\rho\delta_a\delta_b(1-\zeta_2)}{\Delta_h\left[(1-\rho)\Delta_b+(1+2\eta)+(2-\zeta_2)(\eta+\delta_a)\right]},\ \frac{\eta\delta_a}{2\Delta_h(1+2\eta)}\right\}$$

$$= \min\left\{\frac{(4-\sqrt{2})\frac{1}{21}}{(4+\sqrt{2})\frac{1}{19}},\ \frac{2\times\frac{1}{5}\times 1.0005\times(4-\sqrt{2})(1-\frac{1}{12})}{\frac{1}{19}\left[(1-\frac{1}{5})(4+\sqrt{2})+(1+\frac{1}{2})+(2-\frac{1}{12})(\frac{1}{4}+1.0005)\right]},\ \frac{\frac{1}{4}\times 1.0005}{\frac{2}{19}(1+\frac{1}{2})}\right\}$$

$$= \min\{0.4320,\ 2.1902,\ 1.5841\}$$

$$\zeta_1 = 0.1429 < 0.4320.$$

This example satisfies all the conditions of Theorem (1). Hence the null solution is uniformly asymptotically stable. The next example is when $P(t,\ X,\ X_1,\ X_2) \neq 0$.

Example 2 In addition to Example (1), let

$$P(t,\ X,\ X_1,\ X_2) = \frac{1}{(25+\cos t)^4}\begin{bmatrix} 2+x_1+y_1+z_1 \\ 2+x_2+y_2+z_2. \end{bmatrix}$$

On taking the norm of P, we have,

$$\parallel P(t,\ X,\ X_1,\ X_2)\parallel \leq \frac{\sqrt{5}}{(25+\cos t)^2}\Big(2+\parallel X\parallel+\parallel X_1\parallel+\parallel X_2\parallel\Big)$$

$$\leq \frac{\sqrt{5}}{288}+\frac{\sqrt{5}}{576}\Big(\parallel X\parallel+\parallel X_1\parallel+\parallel X_2\parallel\Big).$$

Thus, $D_0 = \frac{\sqrt{5}}{288}$ and $D_1 = \frac{\sqrt{5}}{576}$.

Finally. we show that $D_1 < \frac{\delta_5}{3\delta_4}$. Using the estimates obtained for various constants in Example (1), we have

$$\delta_4 = \max\{(1-\rho)\Delta_b,\ (\eta+\delta_a),\ (1+2\eta)\}$$

$$= \max\Big\{(1-0.2)(4+\sqrt{2}),\ (0.25+1.0005),\ (1+0.5)\Big\}$$

$$= \max\{4.33,\ 1.25,\ 1.5\}$$

$$= 4.33.$$

Also,

$$\delta_5 < \min\{(1-\rho)[\delta_b\delta_h - \Delta_b\Delta_h\zeta_1],\ 2\rho\delta_a\delta_b - \frac{\Delta_h\zeta_1}{(1-\zeta_2)}[(1-\rho)\Delta_b + (1+2\eta) + (2-\zeta_2)(\eta+\delta_a)],$$

$$\frac{1}{2}[\eta\delta_a - 2\Delta_h\zeta_1(1+2\eta)]\}$$

$$= \min\{\frac{4}{5}[(4-\sqrt{2})\frac{1}{21} - (4+\sqrt{2})\frac{1}{133}],\ 0.4002(4-\sqrt{2}) - \frac{12}{1463}\left[\frac{4}{5}(4+\sqrt{2}) + 1.5 + \frac{23}{12}(1.2505)\right],$$

$$\frac{1}{2}\left[\frac{1.0005}{4} - \frac{3}{133}\right]\}$$

$$= \min\{0.0659,\ 0.9673,\ 0.1138\}$$

$$= 0.0659.$$

Therefore, $D_1 = \frac{\sqrt{5}}{576} = 0.0039 < \frac{\delta_5}{3\delta_4} = \frac{0.0659}{12.99} = 0.0051$.
So, all the conditions of Theorem (2) hold for this example. Consequently, Theorem (2) is verified.

5. Conclusion

In this paper, we have proved by LKF approach, some theorems on asymptotic stability of null solution when the matrix function $P(t, X, X_1, X_2) \equiv 0$ and uniform ultimate boundedness of all solutions when $P(t, X, X_1, X_2) \neq 0$ to a class of nonlinear third order matrix differential equations with variable delay. The results of this paper include and improve some existing results in literature. In our future research, we hope to generalize (1) by replacing matrix B with a matrix function and also introduce delay term in $F(X, \dot{X})$.

Conflict of interest

The authors declare no competing financial interest.

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