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Research Article

New Modified Gamma and Beta Functions with an Application

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Abstract: This note presents a novel set of modified Gamma and Beta k functions. The authors introduce these new functions along with their first and second summation relations, functional relations, symmetry relations, Mellin transforms, and integral representations that incorporate algebraic powers and Mittag-Leffler functions, as well as those involving trigonometric terms, powers, and Mittag-Leffler functions. In contrast to other studies that generalize Gamma and Beta functions, this note highlights a statistical application, demonstrating the practical utility of the proposed functions across various fields.

***Keywords*:** Beta k function, Gamma k function, modified Mittag-Leffler k function

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1. Introduction

Mathematical special functions are intriguing and represent a significant field of study with numerous applications. While many of these functions have been utilized for centuries, several dozen have been developed in recent years. These functions are regarded as fundamental and provide the basis for more complex types of functions. Notably, the Beta and Gamma functions have experienced considerable advancements recently, owing to their advantageous properties and applications. The Beta function was first examined by Euler, while the Gamma function, a well-known improper integral, is analogous to the factorial function for natural numbers, also studied by Euler.

The classical Euler Gamma and Beta functions are examined by Chaudhry et al. [1] and their extended form in [2], who provides integral representations of these functions. The relationship between the Gamma and Beta functions is explored by Egan [3]. Additionally, the applications of both functions are addressed in [4, 5].

Recently, Diaz et al. [6–8] presented representations for the Beta and Gamma k functions, along with generalizations of these functions. They also examined Pochhammer's symbol and provided its representation. This work has garnered significant interest from researchers, including those cited in [9–12]. Mubeen et al. [13] discussed integral representations of the classical Beta and Gamma k functions. The generalizations introduced by [14] have proven beneficial for deriving various results. Further generalizations of the Beta and Gamma functions, including those related to q-calculus, can be found in [15–30].

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However, none of the existing papers have presented practical data applications of the Beta or Gamma functions. In this note, we propose modified versions of the Beta and Gamma functions that incorporate real data applications. These modifications utilize the Mittag-Leffler function, which has been widely applied in the field of special functions. Numerous researchers have reported various findings related to this function. For instance, Dorrego and Cerutti [31] introduced the *k* Mittag-Leffler function.

This note is organized into four primary sections: The first section presents the introduction and relevant literature. The second section covers the Mellin transform, along with symmetry and summation relations. The third section addresses integral representations. Finally, the fourth section includes a statistical application.

The classical Euler Gamma and Beta functions are [1]

$$\Gamma(\eta)=\int_{0}^{\infty} z^{\eta-1}e^{-z}dz, \text{ where } Re(\eta)>0$$

and

$$\beta(\eta,\ \zeta)=\int_{0}^{1} z^{\eta-1}(1-z)^{\zeta-1}dz, \text{ where } Re(\eta)>0,\ Re(\zeta)>0.$$

Similarly, the Gamma and Beta *k* functions are defined through [13]

$$\Gamma_k(\eta)=\int_{0}^{\infty} z^{\eta-1}e^{-\frac{z^k}{k}}dz, \text{ where } Re(\eta)>0,\ \ k>0 \tag{1}$$

and

$$\beta_k(\eta,\ \zeta)=\frac{1}{k}\int_{0}^{1} z^{\frac{\eta}{k}-1}(1-z)^{\frac{\zeta}{k}-1}dz, \text{ where } Re(\eta)>0,\ Re(\zeta)>0. \tag{2}$$

Next the relation between these functions and Whittaker functions are defined through [14]

$$\Gamma_{\alpha,\ k}(\eta)=\int_{0}^{\infty} z^{\eta-1}e^{-\frac{z^k}{k}}e^{-\frac{\alpha^k}{kz^k}}dz, \text{ where } Re(\eta)>0$$

and

$$\beta_k(\eta,\ \zeta;\ \alpha)=\frac{1}{k}\int_{0}^{1} z^{\frac{\eta}{k}-1}(1-z)^{\frac{\zeta}{k}-1}e^{-\frac{\alpha^k}{kz(1-z)}}dz, \text{ where } Re(\eta)>0,\ Re(\zeta)>0.$$

Generalizations of Beta and Gamma *k* functions involving the confluent hypergeometric function are defined by [15]

$$\Gamma_k^{(p_n,\, q_n)}(\eta,\ \alpha) = \int_0^\infty z^{\eta-1}\ {}_1F_{1,\, k}\left(p_n;\ q_n;\ -\frac{z^k}{k} - \frac{\alpha^k}{kz^k}\right) dz, \text{ where } p,\ q > 0$$

and

$$\beta_{\alpha,\, k}^{(p_n,\, q_n)}(\eta,\ \zeta) = \frac{1}{k}\int_0^1 z^{\frac{\eta}{k}-1}(1-z)^{\frac{\zeta}{k}-1}\ {}_1F_{1,\, k}\left(p_n,\ q_n;\ -\frac{\alpha^k}{kz(1-z)}\right) dz,$$

where

$${}_1F_{1,\, k}\left(p_n,\ q_n;\ -\frac{\alpha^k}{kz(1-z)}\right) = \sum_{j=0}^{\infty} \frac{(p)_{n,\, k}}{(q)_{n,\, k}\, j!}\left(-\frac{\alpha^k}{kz(1-z)}\right)^j.$$

Extended Gamma and Beta k functions defined using the Mittag-Leffler function are

$$\Gamma_k^p(s) = \int_0^\infty z^{s-1} E_{k,\, p}(-z) dz \tag{3}$$

and

$$\beta_{k,\, v}^p(s,\ t) = \frac{1}{k}\int_0^1 z^{\frac{s}{k}-1}(1-z)^{\frac{t}{k}-1} E_{k,\, p,\, q}^r\left(-vz^k(1-z)^k\right) dz. \tag{4}$$

The Mittag-Leffler function is defined by [31]

$$E_{k,\, p,\, q}^r(-m) = \sum_{j=0}^{\infty} \frac{(-1)^j (r)_{k,\, j}}{\Gamma(pj+q)} \frac{m^j}{j!}$$

and

$$E_{k,\, p}(-z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+p)} \frac{z^j}{j!}$$

for $Re(p) > 0$, $Re(q) > 0$ and $Re(r) > 0$, where $(r)_{k,\ j}$ is the Pochhammer's k symbol [6–8].

2. Main results

In this section, we study a new range of extended Beta and Gamma k functions and derive their properties such as functional relations and Mellin transforms.

Definition 1 Let $p,\ q,\ r \in \Re^{+}$ and $s \in \mathbb{C}$ be such that $Re(s) > 0$. Then, the extended Gamma k function is

$$\Gamma_{k,\ r}^{p,\ q}(s) = \int_{0}^{\infty} z^{s-1} E_{k,\ p,\ q}^{r}(-z) dz,$$

where

$$E_{k,\ p,\ q}^{r}(-m) = \sum_{j=0}^{\infty} \frac{(-1)^{j}(r)_{k,\ j}}{\Gamma_{k}(pj+q)} \frac{m^{j}}{j!}.$$

Remark 1 1. if $q = r = 1$ then $\Gamma_{k,\ r}^{p,\ q}(s) = \Gamma_{k}^{p}(s)$ given in (3). 2. if $p = q = r = 1$ then $\Gamma_{k,\ r}^{p,\ q}(s) = \Gamma_{k}(s)$ given in (1).

Lemma 1 Let $p,\ q,\ r,\ \in \Re^{+}$ and $s \in \mathbb{C}$. Then,

$$\Gamma_{k,\ r}^{p,\ q}(s) = \frac{\Gamma_{k}(s+1)\Gamma_{k}(1-(s+1))}{\Gamma_{k}(r-p(1+s))\Gamma_{k}(q-p(1+s))}.$$

Proof. Let $\sigma = s+1$. Then,

$$\Gamma_{k,\ r}^{p,\ q}(\sigma+1) = \Gamma_{k,\ r}^{p,\ q}(\sigma)$$

$$= \int_{0}^{\infty} z^{\sigma-1} E_{k,\ p,\ q}^{r}(-z) dz$$

$$= M\left[E_{k,\ p,\ q}^{r}(-m)\right](\sigma)$$

$$= \frac{\Gamma_{k}(\sigma)\Gamma_{k}(1-\sigma)}{\Gamma_{k}(r-p\sigma)\Gamma_{k}(q-p\sigma)},$$

where $M\left[E_{k,\ p,\ q}^{r}(-m)\right](\sigma)$ denotes Mellin transform. □

Definition 1 Let $v > 0$, $p,\ q,\ r \in \Re^{+}$ and $s,\ t \in \mathbb{C}$ be such that $Re(s),\ Re(t) > 0$. Then an extended Beta k function is

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t) = \frac{1}{k} \int_{0}^{1} z^{\frac{s}{k}-1}(1-z)^{\frac{t}{k}-1} E_{k,\ p,\ q}^{r}\left(-vz^{k}(1-z)^{k}\right) dz. \tag{5}$$

Remark 2 1. If $q = r = 1$ then $\beta_{k,\ v,\ r}^{p,\ q}(s,\ t) = \beta_{v,\ k}^{p}(s,\ t)$ given in (4). 2. If $p = q = r = 1$ and $v = 0$ then $\beta_{k,\ v,\ r}^{p,\ q}(s,\ t) = \beta_k(s,\ t)$ given in (2).

Theorem 1 (Functional relation) Let $v > 0,\ p,\ q,\ r,\ \in \Re^+$ and $s,\ t \in \mathbb{C}$ be such that $Re(s+1),\ Re(t+1) > 0.$ Then,

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t+1) + \beta_{k,\ v,\ r}^{p,\ q}(s+1,\ t) = \beta_{k,\ v,\ r}^{p,\ q}(s,\ t).$$

Proof. Starting from the left hand side,

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t+1) + \beta_{k,\ v,\ r}^{p,\ q}(s+1,\ t)$$

$$= \frac{1}{k}\int_0^1 z^{\frac{s}{k}-1}(1-z)^{\frac{t}{k}} E_{k,\ p,\ q}^{r}\left(-vz^k(1-z)^k\right) dz + \frac{1}{k}\int_0^1 z^{\frac{s}{k}}(1-z)^{\frac{t}{k}-1} E_{k,\ p,\ q}^{r}\left(-vz^k(1-z)^k\right) dz$$

$$= \frac{1}{k}\int_0^1 \left[z^{-1}(1-z)^{-1}\right] z^{\frac{s}{k}}(1-z)^{\frac{t}{k}} E_{k,\ p,\ q}^{r}\left(-vz^k(1-z)^k\right) dz$$

$$= \frac{1}{k}\int_0^1 z^{\frac{s}{k}-1}(1-z)^{\frac{t}{k}-1} E_{k,\ p,\ q}^{r}\left(-vz^k(1-z)^k\right) dz$$

$$= \beta_{k,\ v,\ r}^{p,\ q}(s,\ t).$$

The proof is complete. □

Theorem 2 (Symmetry relation) Let $v > 0$ and $Re(s),\ Re(t) > 0.$ Then,

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t) = \beta_{k,\ v,\ r}^{p,\ q}(t,\ s).$$

Proof. Using (5) and setting $m = 1 - u$, we obtain the stated result. □

Theorem 3 (Mellin transform) Let $v > 0,\ p,\ q,\ r,\ \in \Re^+$ and $s,\ g \in \mathbb{C}$ be such that $Re(s-g),\ Re(t-g),\ Re(g) > 0.$ Then,

$$M\left[\beta_{k,\ v,\ r}^{p,\ q}(s,\ t);\ g\right] = \beta_k\left(s - k^2 g,\ t - k^2 g\right)\Gamma_{k,\ r}^{p,\ q}(s).$$

Proof. Note that

$$M\left[\beta_{k,\,v,\,r}^{p,\,q}(s,\,t);\,g\right]=\int_0^{\infty}v^{g-1}\left(\frac{1}{k}\int_0^1 z^{\frac{s}{k}-1}(1-z)^{\frac{t}{k}-1}E_{k,\,p,\,q}^{r}\left(-vz^k(1-z)^k\right)dz\right)dv$$

$$=\frac{1}{k}\int_0^1 z^{\frac{s}{k}-1}(1-z)^{\frac{t}{k}-1}\int_0^{\infty}v^{g-1}E_{k,\,p,\,q}^{r}\left(-vz^k(1-z)^k\right)dzdv$$

$$=\beta_k\left(s-k^2g,\,t-k^2g\right)\Gamma_{k,\,r}^{p,\,q}(s),$$

where the order of integration was changed using uniform convergence, $v=uz^{-1}(1-z)^{-1}$ and $z=w$. □

3. Integral representations

In this section, we give integral representations of the extended Beta function.

Theorem 4 The following integral transforms hold for $p,\,q,\,r,\,\in\Re^+$, $k>0$ and $s,\,t\in\mathbb{C}$, $\min\{Re(p),\,Re(q),\,Re(r),\,Re(v)\}>0$

$$\beta_{k,\,v,\,r}^{p,\,q}(s,\,t)=2\frac{1}{k}\int_0^{\frac{\pi}{2}}\cos^{\frac{2s}{k}-1}\sin^{\frac{2t}{k}-1}E_{k,\,p,\,q}^{r}\left(-v\cos^{2k}y\sin^{2k}y\right)dy \tag{6}$$

$$=n\frac{1}{k}\int_0^1(u^n)^{\frac{s}{k}-1}(1-u^n)^{\frac{t}{k}-1}E_{k,\,p,\,q}^{r}\left(-v\left(u^n\left(1-u^n\right)\right)^k\right)du \tag{7}$$

$$=\frac{1}{k}\frac{1}{\eta^{\frac{s}{k}+\frac{t}{k}-1}}\int_0^{\eta}u^{\frac{s}{k}-1}(\eta-u)^{\frac{t}{k}-1}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{u(\eta-u)}{\eta^2}\right)^k\right)du \tag{8}$$

$$=\frac{1}{k}(1+\eta)^{\frac{s}{k}-1}\eta^{\frac{t}{k}-1}\int_0^1\frac{u^{\frac{s}{k}-1}(1-u)^{\frac{t}{k}-1}}{(t+\eta)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{\eta(1+\eta)u(1-u)}{(u+\eta^2)}\right)^k\right)du. \tag{9}$$

Proof. In (5), taking $z=\cos^2 y$ with $dz=-2\cos y\sin y dy$, we obtain

$$\beta_{k,\,v,\,r}^{p,\,q}(s,\,t)=\frac{1}{k}\int_{\frac{\pi}{2}}^{0}\cos^{\frac{2s}{k}-2}\sin^{\frac{2t}{k}-2}E_{k,\,p,\,q}^{r}\left(-v\cos^{2k}y\sin^{2k}y\right)(-2\cos y\sin y dy)$$

and hence (6). In (5), taking $z=u^n$ with $dz=nu^{n-1}du$, we obtain

$$\beta_{k,\,v,\,r}^{p,\,q}(s,\,t)=\frac{1}{k}\int\limits_{0}^{1}(u^{n})^{\frac{s}{k}-1}(1-u^{n})^{\frac{t}{k}-1}E_{k,\,p,\,q}^{r}\left(-v\left(u^{n}\left(1-u^{n}\right)\right)^{k}\right)nu^{n-1}du$$

and hence (7). In (5), taking $z=\frac{u}{\eta}$ with $dz=\frac{du}{\eta}$, we obtain

$$\beta_{k,\,v,\,r}^{p,\,q}(s,\,t)=\frac{1}{k}\int\limits_{0}^{\eta}\left(\frac{u}{\eta}\right)^{\frac{s}{k}-1}\left(\frac{\eta-u}{\eta}\right)^{\frac{t}{k}-1}E_{k,\,p,\,q}^{r}\left(-v\left(\left(\frac{u}{\eta}\right)\left(\frac{\eta-u}{\eta}\right)\right)^{k}\right)\frac{du}{\eta}$$

and hence (8). In (5), taking $z=\frac{(1+\eta)u}{(u+\eta)}$ with $dz=\frac{\eta(1+\eta)}{(u+\eta)^{2}}du$, we obtain

$$\beta_{k,\,v,\,r}^{p,\,q}(s,\,t)=\frac{1}{k}\int\limits_{0}^{1}\left(\frac{(1+\eta)u}{u+\eta}\right)^{\frac{s}{k}-1}\left(\frac{(1-u)\eta}{u+\eta}\right)^{\frac{t}{k}-1}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{\eta(1+\eta)u(1-u)}{(u+\eta^{2})}\right)^{k}\right)\frac{\eta(1+\eta)}{(u+\eta)^{2}}du$$

and hence (9). □

Theorem 5 The following integral transforms hold for $p,\ q,\ r,\in\Re^{+},\ k>0$ and $s,\ t\in\mathbb{C}$, $\min\{Re(p),\ Re(q),\ Re(r),\ Re(v)\}>0$

$$\beta_{k,\,v,\,r}^{p,\,q}(s,\,t)=\frac{1}{k}\int\limits_{0}^{\infty}\frac{u^{\frac{s}{k}-1}}{(1+u)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{u}{(1+u)^{2}}\right)^{k}\right)du \tag{10}$$

$$=\frac{1}{2k}\int\limits_{0}^{\infty}\frac{u^{\frac{s}{k}-1}+u^{\frac{t}{k}-1}}{(1+u)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{u}{(1+u)^{2}}\right)^{k}\right)du \tag{11}$$

$$=\frac{1}{k}\eta^{\frac{s}{k}}\zeta^{\frac{t}{k}}\int\limits_{0}^{\infty}\frac{u^{\frac{s}{k}-1}}{(\zeta+\eta u)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{\eta\zeta u}{(\zeta+\eta u)^{2}}\right)^{k}\right)du \tag{12}$$

$$=\frac{1}{k}2\eta^{\frac{s}{k}}\zeta^{\frac{t}{k}}\int\limits_{0}^{\frac{\pi}{2}}\frac{\sin^{2\frac{s}{k}-1}y\cos^{\frac{2t}{k}-1}y}{\left(\cos^{2}y+\eta\sin^{2}y\right)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{\eta\zeta\tan^{2}y}{(\zeta+\eta u\tan^{2}y)^{2}}\right)^{k}\right)dy. \tag{13}$$

Proof. In (5), taking $z=\frac{u}{(1+u)}$ with $dz=\frac{du}{(1+u)^{2}}$, we obtain

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t)=\frac{1}{k}\int_{0}^{\infty}\frac{u^{\frac{s}{k}-1}}{(1+u)^{\frac{s}{k}+\frac{t}{k}}}\frac{1}{(1+u)^{\frac{t}{k}-1}}E_{k,\ p,\ q}^{r}\left(-v\left(\frac{u}{(1+u)^{2}}\right)^{k}\right)\frac{du}{(1+u)^{2}}$$

$$=\frac{1}{k}\int_{0}^{\infty}\frac{u^{\frac{s}{k}-1}}{(1+u)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\ p,\ q}^{r}\left(-v\left(\frac{u}{(1+u)^{2}}\right)^{k}\right)du \tag{14}$$

as required in (10). By symmetry,

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t)=\frac{1}{k}\int_{0}^{\infty}\frac{u^{\frac{t}{k}-1}}{(1+u)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\ p,\ q}^{r}\left(-v\left(\frac{u}{(1+u)^{2}}\right)^{k}\right)du. \tag{15}$$

Adding (14) and (15) gives (11). In (5), taking $z=u\frac{\eta}{\zeta}$ with $dz=\frac{\eta}{\zeta}du$, we obtain

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t)=\frac{1}{k}\int_{0}^{\infty}\frac{\left(\frac{\eta}{\zeta}u\right)^{\frac{s}{k}-1}}{\left(1+\frac{\eta}{\eta}u\right)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\ p,\ q}^{r}\left(-v\left(\frac{\frac{\eta}{\zeta}u}{\left(1+\frac{\eta}{\zeta}u\right)^{2}}\right)^{k}\right)\frac{\eta}{\zeta}du$$

and hence (12). In (5), taking $z=\tan^{2}y$ with $dz=2\tan y\sec^{2}y dy$, we obtain

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t)=\frac{1}{k}\eta^{\frac{s}{k}}\zeta^{\frac{t}{k}}\int_{0}^{\frac{\pi}{2}}\frac{\left(\tan^{2}y\right)^{\frac{s}{k}-1}}{(1+\tan^{2}y)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\ p,\ q}^{r}\left(-v\left(\frac{\eta\zeta\tan^{2}y}{(\eta+\zeta\tan^{2}y)^{2}}\right)^{k}\right)2\tan y\sec y dy$$

and hence (13). □

Theorem 6 The following integral representations hold for $p,\ q,\ r,\ \in\mathfrak{R}^{+},\ k>0$ and $s,\ t\in\mathbb{C},\ \min\{Re(p),\ Re(q),\ Re(r),\ Re(v)\}>0$

$$\beta_{k,\ v,\ r}^{p,\ q}(s,\ t)=\frac{1}{k}\zeta^{\frac{s}{k}}\eta^{\frac{t}{k}}\int_{0}^{1}\frac{u^{\frac{s}{k}-1}(1-u)^{\frac{t}{k}-1}}{(\zeta+(\eta-\zeta)u)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\ p,\ q}^{r}\left(-v\left(\frac{\eta\zeta u(1-u)}{(\zeta+(\eta-\zeta)u)^{2}}\right)^{k}\right)du \tag{16}$$

$$=(\zeta+\xi)^{\frac{s}{k}}\zeta^{\frac{t}{k}}\int_{0}^{1}\frac{u^{\frac{s}{k}-1}(1-u)^{\frac{t}{k}-1}}{(\zeta+\xi u)^{\frac{s}{k}+\frac{t}{k}}}E_{k,\ p,\ q}^{r}\left(-v\left(\frac{\eta\zeta u(1-u)}{(\eta-\xi u)^{2}}\right)^{k}\right)du. \tag{17}$$

Proof. In (5), taking $\frac{\eta}{u}-\frac{\zeta}{z}=\eta-\zeta$ with $dz=\frac{\eta\zeta}{[\eta+(\zeta-\eta)u]^{2}}du$, we obtain

$$\beta_{k,\,v,\,r}^{p,\,q}(s,\,t)=\frac{1}{k}\eta^{\frac{s}{k}-1}\zeta^{\frac{t}{k}-1}\int\limits_{0}^{1}\frac{u^{\frac{s}{k}-1}(1-u)^{\frac{t}{k}-1}}{[\eta+(\zeta-\eta)u]^{\frac{s}{k}+\frac{t}{k}}}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{\eta\zeta u(1-u)}{[\eta+(\zeta-\eta)u]^{2}}\right)^{k}\right)\frac{\eta\zeta}{(\eta+(\zeta-\eta)u)^{2}}du$$

and hence (16). Changing η and ζ and setting $\eta-\zeta=\xi$ gives (17). □

Theorem 7 The following integral representations hold for $p,\ q,\ r,\ \in\Re^{+},\ k>0$ and $s,\ t\in\mathbb{C},\ \min\{Re(p),\ Re(q),\ Re(r),\ Re(v)\}>0$ with $\zeta\neq\eta$

$$\beta_{k,\,v,\,r}^{p,\,q}(s,\,t)=(\zeta-\eta)^{1-\frac{s}{k}-\frac{t}{k}}\int\limits_{\eta}^{\zeta}(u-\eta)^{\frac{s}{k}-1}(\eta-u)^{\frac{t}{k}-1}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{(u-\eta)(\zeta-u)}{(\zeta-\eta)^{2}}\right)^{k}\right)du \tag{18}$$

$$=2^{1-\frac{s}{k}-\frac{t}{k}}\int\limits_{-1}^{1}(u+1)^{\frac{s}{k}-1}(1-u)^{\frac{t}{k}-1}E_{k,\,p,\,q}^{r}\left(-v\left(\frac{(u+1)(1-u)}{4}\right)^{k}\right)du. \tag{19}$$

Proof. In (5), taking $z=\dfrac{u-\eta}{\zeta-\eta}$ with $dz=\dfrac{du}{\zeta-\eta}$ gives (18). Set $\eta=-1$ and $\zeta=1$ to obtain (19). □

4. A generalized Beta distribution

Let X denote a random variable with the following probability density function for $p,\ q,\ r,\ \in\Re^{+},\ k>0$ and $s,\ t\in\mathbb{C},\ \min\{Re(p),\ Re(q),\ Re(r),\ Re(v)\}>0$

$$f(x)=\begin{cases}\dfrac{1}{k\beta_{k,\,v,\,l}^{p,\,q}(s,\,t)}x^{\frac{s}{k}-1}(1-x)^{\frac{t}{k}-1}E_{k,\,p,\,q}^{l}\left(-vx^{k}(1-x)^{k}\right),\ 0<m<1\\ \\ 0,\ \ \text{otherwise}\end{cases} \tag{20}$$

where $s,\ t\in\Re$ and $v,\ p,\ q,\ l\in\Re^{+}$. We shall write $X\sim\beta_{k,\,v,\,l}^{p,\,q}(s,\,t)$.

For any real r, the rth moment of X is

$$E\left(X^{r}\right)=\frac{\beta_{k,\,v,\,l}^{p,\,q}(s+r,\,t)}{\beta_{k,\,v,\,l}^{p,\,q}(s,\,t)}.$$

The mean and variance of X are

$$E(X)=\frac{\beta_{k,\,v,\,l}^{p,\,q}(s+1,\,t)}{\beta_{k,\,v,\,l}^{p,\,q}(s,\,t)}$$

and

$$Var(X) = \frac{\beta^{p,\, q}_{k,\, v,\, l}(s,\, t)\beta^{p,\, q}_{k,\, v,\, l}(s+2,\, t) - \left[\beta^{p,\, q}_{k,\, v,\, l}(s+1,\, t)\right]^2}{\left[\beta^{p,\, q}_{k,\, v,\, l}(s,\, t)\right]^2},$$

respectively. The moment generating function of X is

$$M(y) = \frac{1}{\beta^{p,\, q}_{k,\, v,\, l}}(s,\, t) \sum_{f=0}^{\infty} \beta^{p,\, q}_{k,\, v,\, l}(s+f,\, t)\frac{y^f}{r!}.$$

The cumulative distribution function of X is

$$F(y) = \frac{\beta^{p,\, q}_{k,\, v,\, l,\, y}(s,\, t)}{\beta^{p,\, q}_{k,\, v,\, l}(s,\, t)},$$

where

$$\beta^{p,\, q}_{k,\, v,\, l,\, y}(s,\, t) = \frac{1}{k}\int_0^1 y^{\frac{s}{k}-1}(1-y)^{\frac{t}{k}-1}E^{l}_{k,\, p,\, q}\left(-vy^k(1-y)^k\right)dy$$

is an extended modified incomplete Beta k function.

The particular case of (20) for $v = 0$ and $k = 1$ is the standard Beta distribution. To illustrate the better fit of (20) over the standard Beta distribution, we fitted both to a data comprises the proportions of individuals voting “Remain” in the Brexit (EU referendum) polls, encompassing 126 polls conducted from January 2016 until the referendum date in June 2016. The specific values of the data were 0.52, 0.55, 0.49, 0.44, 0.54, 0.48, 0.41, 0.45, 0.42, 0.53, 0.45, 0.44, 0.44, 0.42, 0.42 0.37, 0.46, 0.43, 0.39, 0.45, 0.44, 0.46, 0.40, 0.48, 0.42, 0.44, 0.45, 0.43, 0.43, 0.48 0.41, 0.43, 0.40, 0.41, 0.42, 0.44, 0.51, 0.44, 0.44, 0.41, 0.41, 0.45, 0.55, 0.44, 0.44 0.52, 0.55, 0.47, 0.43, 0.55, 0.38, 0.36, 0.38, 0.44, 0.42, 0.44, 0.43, 0.42, 0.49, 0.39 0.41, 0.45, 0.43, 0.44, 0.51, 0.51, 0.49, 0.48, 0.43, 0.53, 0.38, 0.40, 0.39, 0.35, 0.45 0.42, 0.40, 0.39, 0.44, 0.51, 0.39, 0.35, 0.41, 0.51, 0.45, 0.49, 0.40, 0.48, 0.41, 0.46 0.47, 0.43, 0.45, 0.48, 0.49, 0.40, 0.40, 0.40, 0.39, 0.41, 0.39, 0.48, 0.48, 0.37, 0.38 0.42, 0.51, 0.45, 0.40, 0.54, 0.36, 0.43, 0.49, 0.41, 0.36, 0.42, 0.38, 0.55, 0.44, 0.54 0.41, 0.52, 0.42, 0.38, 0.42, 0.44.

The fit of the standard Beta distribution gave the log-likelihood $= 202.1$ while the fit of (20) with $k = 1$ gave the log-likelihood $= 230.2$. The likelihood ratio test shows that (20) provides a significantly better fit.

5. Conclusions

In this note, we have defined modified Gamma k and Beta k functions by using the Mittag-Leffler function. We have investigated some special cases and integral representations of these functions. Further, we have introduced a generalized Beta distribution, derived its mean, variance, moment generating function, and cumulative distribution function and illustrated its better fit over the standard Beta distribution for real data. This new distribution can be useful in application areas of the standard Beta distribution, including (but not limited to) Bayesian statistics, project management for modeling completion times and quality control for assessing the probability of defects, modeling asset returns, machine learning for probabilistic modeling, and environmental science for modeling proportions of species in ecological studies. We are not aware of other modified Gamma and Beta functions shown to have practical appeal. Future work is to extend the proposed Gamma k and Beta k functions to bivariate, multivariate, complex variate, and matrix variate cases.

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Conflict of interest

The authors declare no competing financial interest.

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