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Research Article

A Recursive Formula for Sums of Values of Degenerate Falling Factorials

Dae San Kim[1], Taekyun Kim[2*], Jongkyum Kwon[3], Hyunseok Lee[2]

[1]Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
[3]Department of Mathematics Education, Gyeongsang National University, Jinju, 52828, Republic of Korea
E-mail: tkkim@kw.ac.kr

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Abstract: The classical Faulhaber's formula expresses the sum of a fixed positive integer powers of the first n positive integers in terms of Bernoulli polynomial. As a degenerate version of this, we may consider sums of values of degenerate falling factorials, which reduce to aforementioned sum as λ tends to 0. The aim of this note is to derive a recursive formula for sums of values of degenerate falling factorials by using probabilistic method. In this manner, we obtain a new recursive formula for such sums, which involves the (signed) Stirling numbers of the first kind.

***Keywords*:** sums of values of degenerate falling factorials, uniform distribution

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1. Introduction

For $m, n \in \mathbb{N}$, we denote the sum of m-th powers of the first n positive integers by

$$S_m(n) = 1^m + 2^m + \cdots + n^m, \tag{1}$$

(see [1–6]).

This sum has been studied extensively for several hundred years.

Let $B_n(x)$ be the Bernoulli polynomials given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{2}$$

(see [1–28]).

Then the Faulhaber's formula expresses $S_m(n)$ in terms of Bernoulli polynomials $B_n(x)$, which is given by

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$$S_m(n) = \frac{1}{m+1}\Big(B_{m+1}(n+1) - B_{m+1}\Big). \tag{3}$$

Two well known recursive formulas for $S_m(n)$ are given by

$$(n+1)^m - 1 = \sum_{k=0}^{m} \binom{m+1}{k} S_k(n),$$

and

$$S_m(n) = \frac{n^{m+1}}{m+1} + \sum_{k=0}^{m-1} \binom{m}{k} \frac{(-1)^{m-k+1}}{m-k+1} S_k(n), \tag{4}$$

(see [2]).

Recall that the degenerate falling factorials are given by

$$(x)_{0,\lambda} = 1,\ \ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots\big(x-(n-1)\lambda\big),\ \ (n \geq 1). \tag{5}$$

For m, $n \in \mathbb{N}$, we may consider, as a degenerate version of $S_m(n)$, sums of values of degenerate falling factorials $S_{m,\lambda}(n)$, given by

$$S_{m,\lambda}(n) = (1)_{m,\lambda} + (2)_{m,\lambda} + \cdots + (n)_{m,\lambda}. \tag{6}$$

Note that

$$\lim_{\lambda \to 0} S_{m,\lambda}(n) = S_m(n) = 1^m + 2^m + \cdots + n^m,$$

(see [1–6]),

$$\lim_{\lambda \to 1} S_{m,\lambda}(n) = S_{m,1}(n) = (1)_m + (2)_m + \cdots + (n)_m,$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

The aim of this paper is to derive a recursive formula for sums of values of degenerate falling factorials which is given by

$$S_{m,\lambda}(n) = \sum_{l=0}^{m} S_1(m,l)\lambda^{m-l}\frac{n^{l+1}}{l+1} \tag{7}$$

$$+\sum_{k=0}^{m-1}\sum_{l=0}^{m-k}\binom{m}{k}\frac{S_1(m-k,l)}{l+1}\lambda^{m-k-l}(-1)^{l+1}S_{k,\lambda}(n).$$

Here the (signed) Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \ \ (n \geq 0), \tag{8}$$

(see [1–28]).

This is done by using the identity $(X+Y)_{m,\lambda}\sum_{k=0}^{m}\binom{m}{k}(X)_{m-k,\lambda}(Y)_{k,\lambda}$ (see (17)) and computing the expectations $E\left[(X+Y)_{m,\lambda}\right]$, $E\left[(X)_{m-k,\lambda}\right]$, and $E\left[(Y)_{k,\lambda}\right]$. Here $X \sim \text{Uniform}(-1, 0)$, if its probability density function $f_X(x)$ is given by $f_X(x) = 1$, if $-1 \leq x \leq 0$, and $f_X(x) = 0$, otherwise (see (13)), and Y is the uniform random variable on the integers $1, 2, \ldots, n$.

As to related previous works, we mention the three papers [18–20]. Two expressions for $S_m(n)$ are derived in [19], the one involving the degenerate Bernoulli numbers and the other in terms of the degenerate Stirling numbers of the second kind. In [18], the following three recurrence relations for $S_m(n)$ are obtained: for $m, n \in \mathbb{N}$,

$$S_{m,\lambda}(n) = \frac{(n+1)_{m+1,\lambda}-(1)_{m+1,\lambda}}{m+1} - \frac{1}{m+1}\sum_{r=0}^{m-1}\binom{m+1}{r}(1)_{m+1-r,\lambda}S_{r,\lambda}(n), \tag{9}$$

$$S_{m,\lambda}(n) = \frac{(n)_{m+1,\lambda}}{m+1} + \frac{1}{m+1}\sum_{r=0}^{m-1}\binom{m+1}{r}(-1)^{m+1-r}\langle 1\rangle_{m+1-r,\lambda}S_{r,\lambda}(n), \tag{10}$$

$$S_{m,\lambda}(n) = \frac{n(n+1)_{m,\lambda}}{m+1} - \frac{1}{m+1}\sum_{r=1}^{m-1}(1)_{m+1-r,\lambda}\binom{m}{r-1}S_{r,\lambda}(n) \tag{11}$$

$$-\frac{\lambda}{m+1}\sum_{r=1}^{m-1} r\binom{m}{r}(1)_{m-r,\lambda}S_{r,\lambda}(n),$$

where $\langle x\rangle_{n,\lambda}$ are the degenerate rising factorials given by

$$\langle x\rangle_{0,\lambda} = 1, \ \langle x\rangle_{n,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda), \ \ (n \geq 1).$$

In [20], formulas analogous to Faulhaber's are derived for the poly-Bernoulli polynomials $B_n^{(k)}(x)$ (see [20] (1.6)), and the type 2 poly-Bernoulli polynomials $\beta_n^{(k)}(x)$ (see [20] (1.8)). Indeed, the following two formulas are obtained: for $n, x \in \mathbb{N}$,

$$\sum_{i=0}^{x-1}\sum_{m=1}^{n}\sum_{j=1}^{m}\binom{n}{m}\frac{(j-1)!}{j^{k-1}}(-1)^{m-j}S_2(m,\ j)i^{n-m}=B_n^{(k)}(x)-B_n^{(k)},$$

$$\sum_{i=0}^{x-1}\sum_{l=1}^{n}\sum_{m=1}^{l}\binom{n}{l}\frac{S_1(l,\ m)}{m^{k-1}}i^{n-l}=\beta_n^{(k)}(x)-\beta_n^{(k)},$$

where $S_2(m,\ j)$ are the Stirling numbers of the second kind given by

$$\frac{1}{j!}(e^t-1)^j=\sum_{m=j}^{\infty}S_2(m,\ j)\frac{t^m}{m!}.$$

For any nonzero $\lambda\in\mathbb{R}$, the degenerate exponentials are defined by

$$e_\lambda^x(t)=\sum_{k=0}^{\infty}(x)_{k,\ \lambda}\frac{t^k}{k!},\quad e_\lambda(t)=e_\lambda^1(t), \tag{12}$$

(see [3, 5, 9, 13, 14, 16–19, 21–24, 28]), where $(x)_{n,\ \lambda}$ is the sequence of degenerate falling factorials in (5). Note that $\lim_{\lambda\to 0}e_\lambda^x(t)=e^{xt}$.

Let X be a continuous random variable. A probability density function $f_X(x)$ of X is an integrable function such that

$$\int_a^b f_X(x)dx=P\{a\leq X\leq b\},$$

(see [3, 5, 16, 17, 26, 29]).

A continuous random variable X is said to have the uniform distribution over the interval $[a,\ b]$, denoted by $X\sim\text{Uniform}(a,\ b)$, if its probability density function $f_X(x)$ is given by

$$f_X(x)=\begin{cases}\dfrac{1}{b-a}, & \text{if } a<x<b,\\[2ex] 0, & \text{if } x\leq a \text{ or } x\geq b,\end{cases} \tag{13}$$

(see [26]).

2. Proof of a recursive formula for $S_{m,\ \lambda}(n)$

Let X be a continuous random variable with probability density function $f_X(x)$. Then the n-th moment of X is defined by

$$E\left[X^n\right]=\int_{-\infty}^{\infty}x^n f_X(x)dx, \tag{14}$$

(see [5, 16, 18, 23, 26, 29]).

While, if X is a discrete random variable, then the n-th moment of X is given by

$$E\left[X^n\right] = \sum_k P\{X = k\}k^n, \tag{15}$$

(see [5, 16, 18, 23, 26, 29, 30]).

Let X be a continuous random variable with probability density function $f_X(x)$, and let Y be a discrete random variable. Assume that X and Y are independent. Then the probability density function $f_Z(z)$ of the sum $Z = X + Y$ is given by

$$f_Z(z) = \sum_y f_X(z - y)P\{Y = y\}. \tag{16}$$

A common example of a sum involving both a continuous and a discrete random variables is calculating the total cost of a purchase including shipping fees, where the price of the item itself is a continuous variable (can take any value within a range) and the shipping cost is a discrete variable (fixed amount based on the delivery option chosen, for example 'standard' or 'expedited' or 'overnight').

Let $X \sim \text{Uniform}(-1,\, 0)$ (see (13)), and let a random variable Y have the uniform distribution on the integers $1,\, 2,\, \ldots,\, n$. Then, by (16), X and Y are independent and $Z = X + Y \sim \text{Uniform}(0,\, n)$. By using (12), we can show that

$$(X+Y)_{m,\,\lambda} = \sum_{k=0}^{m} \binom{m}{k} (X)_{m-k,\,\lambda} (Y)_{k,\,\lambda}, \quad (m \geq 0). \tag{17}$$

Thus, by (17), we have

$$\begin{aligned} E\left[(X+Y)_{m,\,\lambda}\right] &= \sum_{k=0}^{m} \binom{m}{k} E\left[(X)_{m-k,\,\lambda}(Y)_{k,\,\lambda}\right] \\ &= \sum_{k=0}^{m} \binom{m}{k} E\left[(X)_{m-k,\,\lambda}\right] E\left[(Y)_{k,\,\lambda}\right]. \end{aligned} \tag{18}$$

We observe from (8) that

$$\begin{aligned} (z)_{m,\,\lambda} &= z(z-\lambda)(z-2\lambda)\cdots\left(z-(m-1)\lambda\right) \\ &= \lambda^m \frac{z}{\lambda}\left(\frac{z}{\lambda}-1\right)\left(\frac{z}{\lambda}-2\right)\cdots\left(\frac{z}{\lambda}-(m-1)\right) \\ &= \lambda^m \left(\frac{z}{\lambda}\right)_m = \sum_{l=0}^{m} S_1(m,\, l)\lambda^{m-l} z^l. \end{aligned} \tag{19}$$

From (13), (14) and (19), we note that

$$E\left[(X+Y)_{m,\lambda}\right]=E\left[(Z)_{m,\lambda}\right]=\int_{0}^{n}(z)_{m,\lambda}f_{Z}(z)dz \tag{20}$$

$$=\frac{1}{n}\sum_{l=0}^{m}S_{1}(m,l)\lambda^{m-l}\int_{0}^{n}z^{l}dz$$

$$=\frac{1}{n}\sum_{l=0}^{m}S_{1}(m,l)\lambda^{m-l}\frac{n^{l+1}}{l+1}.$$

Thus, by (18) and (20), we have

$$\frac{1}{n}\sum_{l=0}^{m}S_{1}(m,l)\lambda^{m-l}\frac{n^{l+1}}{l+1}=E\left[(X+Y)_{m,\lambda}\right]=\sum_{k=0}^{m}\binom{m}{k}E\left[(X)_{m-k,\lambda}\right]E\left[(Y)_{k,\lambda}\right]. \tag{21}$$

On the one hand, as $X\sim\text{Uniform}(-1,0)$, we get

$$E\left[(X)_{m-k,\lambda}\right]=\int_{-1}^{0}(x)_{m-k,\lambda}f_{X}(x)dx=\int_{-1}^{0}(x)_{m-k,\lambda}dx \tag{22}$$

$$=\sum_{l=0}^{m-k}S_{1}(m-k,l)\lambda^{m-k-l}\int_{-1}^{0}x^{l}dx$$

$$=\sum_{l=0}^{m-k}\frac{S_{1}(m-k,l)}{l+1}\lambda^{m-k-l}(-1)^{l}.$$

On the other hand, since Y has the uniform distribution on the integers $1, 2, 3, \ldots, n$ (see (15)), we get

$$E\left[(Y)_{k,\lambda}\right]=\sum_{l=0}^{n}(l)_{k,\lambda}P\{Y=l\}=\frac{1}{n}\sum_{l=1}^{n}(l)_{k,\lambda}=\frac{1}{n}S_{k,\lambda}(n). \tag{23}$$

By (21), (22) and (23), we derive

$$\frac{1}{n}\sum_{l=0}^{m}S_{1}(m,l)\lambda^{m-l}\frac{n^{l+1}}{l+1}=\sum_{k=0}^{m-1}\binom{m}{k}E\left[(X)_{m-k,\lambda}\right]E\left[(Y)_{k,\lambda}\right]+E\left[(Y)_{m,\lambda}\right] \tag{24}$$

$$=\frac{1}{n}\sum_{k=0}^{m-1}\binom{m}{k}\sum_{l=0}^{m-k}\frac{S_{1}(m-k,l)}{l+1}\lambda^{m-k-l}(-1)^{l}S_{k,\lambda}(n)+\frac{1}{n}S_{m,\lambda}(n).$$

Thus, from (24), we obtain the following theorem.

Theorem 2.1 For $m,\, n \in \mathbb{N}$, we have the recurrence relation

$$S_{m,\,\lambda}(n)=\sum_{k=0}^{m-1}\sum_{l=0}^{m-k}\binom{m}{k}\frac{S_1(m-k,\,l)}{l+1}\lambda^{m-k-l}(-1)^{l+1}S_{k,\,\lambda}(n)$$

$$+\sum_{l=0}^{m}S_1(m,\,l)\lambda^{m-l}\frac{n^{l+1}}{l+1}.$$

By taking both $\lambda \to 0$ and $\lambda \to 1$ of the recurrence relation in Theorem 2.1, we obtain the following identities, the first of which is the one in (4).

Corollary 2.2 For $m,\, n \in \mathbb{N}$, we have the recurrence relations

$$S_m(n)=\sum_{k=0}^{m-1}\binom{m}{k}\frac{1}{m-k+1}(-1)^{m-k+1}S_k(n)+\frac{n^{m+1}}{m+1},$$

$$S_{m,\,1}(n)=\sum_{k=0}^{m-1}\sum_{l=0}^{m-k}\binom{m}{k}\frac{S_1(m-k,\,l)}{l+1}(-1)^{l+1}S_{k,\,1}(n)$$

$$+\sum_{l=0}^{m}S_1(m,\,l)\frac{n^{l+1}}{l+1}.$$

Remark 2.3 By taking $\lambda \to 0$ of (10), we obtain the recurrence relation given by

$$S_m(n)=\frac{n^{m+1}}{m+1}+\frac{1}{m+1}\sum_{r=0}^{m-1}\binom{m+1}{r}(-1)^{m+1-r}S_r(n), \tag{25}$$

which is easily seen to be equal to the first recurrence relation in Corollary 2.2. Indeed, we observe that (25) is equal to

$$S_m(n)=\frac{n^{m+1}}{m+1}+\frac{1}{m+1}\sum_{r=0}^{m-1}\binom{m+1}{m+1-r}(-1)^{m+1-r}S_r(n)$$

$$=\frac{n^{m+1}}{m+1}+\frac{1}{m+1}\sum_{r=0}^{m-1}\binom{m}{m-r}\frac{m+1}{m+1-r}(-1)^{m+1-r}S_r(n)$$

$$=\frac{n^{m+1}}{m+1}+\sum_{r=0}^{m-1}\binom{m}{r}\frac{1}{m+1-r}(-1)^{m+1-r}S_r(n).$$

Remark 2.4 Here we observe that the recurrence relation in Theorem 2.1 involves the Stirling numbers of the first kind and no degenerte falling factorials, while all the recurrence relations in (9), (10) and (11) involve degenerate falling factorials and no Stirling numbers of the first kind. The Stirling numbers of the first kind can be easily determined, for example by using $S_2(n+1,\, k) = S_2(n,\, k-1) + kS_2(n,\, k),\ (n \geq k \geq 0)$. However, computing degenerate falling factorials are not so easy. Thus we may say that the recurrence relation in Theorem 2.1 is better than those in (9), (10) and (11).

We now consider more general case. Let $X_{a,\,c} \sim \text{Uniform}(a,\, a+c)$, and let Y be the uniform distribution on the integers $1,\, 2,\, \dots,\, n$, as before, where $a,\, c$ are real numbers with $0 < c \leq 1$. Then, by (16), we see that $X_{a,\,c}$ and Y are independent and $Z_{a,\,b} = X_{a,\,b} + Y$ has the probability density function $f_{Z_{a,\,c}}(x)$ given by

$$f_{Z_{a,\,c}}(x) = \begin{cases} \dfrac{1}{nc}, & \text{if } x \in \cup_{j=1}^{n}\, (j+a,\, j+a+c), \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

From (14), (19) and (26), we note that

$$\begin{aligned} E\Big[(X_{a,\,c}+Y)_{m,\,\lambda}\Big] &= E\Big[(Z_{a,\,c})_{m,\,\lambda}\Big] = \int_{-\infty}^{\infty} (z)_{m,\,\lambda} f_{Z_{a,\,c}}(z)dz \\ &= \frac{1}{nc}\sum_{j=1}^{n}\int_{j+a}^{j+a+c} (z)_{m,\,\lambda}dz \\ &= \frac{1}{nc}\sum_{j=1}^{n}\sum_{l=0}^{m} S_1(m,\, l)\lambda^{m-l}\int_{j+a}^{j+a+c} z^l dz \\ &= \frac{1}{nc}\sum_{j=1}^{n}\sum_{l=0}^{m} S_1(m,\, l)\lambda^{m-l}\frac{1}{l+1}\big((j+a+c)^{l+1} - (j+a)^{l+1}\big) \\ &= \frac{1}{nc}\sum_{l=0}^{m} S_1(m,\, l)\lambda^{m-l}\frac{1}{l+1}\sum_{j=1}^{n}\big((j+a+c)^{l+1} - (j+a)^{l+1}\big). \end{aligned} \tag{27}$$

In addition, as $X_{a,\,c} \sim \text{Uniform}(a,\, a+c)$, we get

$$\begin{aligned} E\Big[(X_{a,\,c})_{m-k,\,\lambda}\Big] &= \int_{a}^{a+c} (x)_{m-k,\,\lambda} f_{X_{a,\,c}}(x)dx = \frac{1}{c}\int_{a}^{a+c} (x)_{m-k,\,\lambda}dx \\ &= \frac{1}{c}\sum_{l=0}^{m-k} S_1(m-k,\, l)\lambda^{m-k-l}\int_{a}^{a+c} x^l dx \\ &= \frac{1}{c}\sum_{l=0}^{m-k} \frac{S_1(m-k,\, l)}{l+1}\lambda^{m-k-l}\big((a+c)^{l+1} - a^{l+1}\big). \end{aligned} \tag{28}$$

As $E\Big[(X_{a,\,c}+Y)_{m,\,\lambda}\Big] = \sum_{k=0}^{m-1}\binom{m}{k}E\Big[(X_{a,\,c})_{m-k,\,\lambda}\Big]E\Big[(Y)_{k,\,\lambda}\Big] + E\Big[(Y)_{m,\,\lambda}\Big]$, from (23), (27) and (28), we get the following result. Notice here that Theorem 2.1 corresponds to the $a = -1,\, c = 1$ case of the following.

Theorem 2.5 Let $m,\ n\in\mathbb{N}$, and let $a,\ c$ be any real numbers with $0<c\leq 1$. Then we have the recurrence relation

$$cS_{m,\,\lambda}(n)=\sum_{k=0}^{m-1}\sum_{l=0}^{m-k}\binom{m}{k}\frac{S_1(m-k,\,l)}{l+1}\lambda^{m-k-l}\left(a^{l+1}-(a+c)^{l+1}\right)S_{k,\,\lambda}(n)$$

$$+\sum_{l=0}^{m}S_1(m,\,l)\lambda^{m-l}\frac{1}{l+1}\sum_{j=1}^{n}\left((j+a+c)^{l+1}-(j+a)^{l+1}\right).$$

By taking both $\lambda\to 0$ and $\lambda\to 1$ of the identity in Theorem 2.3, we obtain the following corollary.
Corollary 2.6 Let $m,\ n\in\mathbb{N}$, and let $a,\ c$ be any real numbers with $0<c\leq 1$. Then we have the recurrence relations

$$cS_m(n)=\sum_{k=0}^{m-1}\binom{m}{k}\frac{1}{m-k+1}\left(a^{m-k+1}-(a+c)^{m-k+1}\right)S_k(n)$$

$$+\frac{1}{m+1}\sum_{j=1}^{n}\left((j+a+c)^{m+1}-(j+a)^{m+1}\right),$$

$$cS_{m,\,1}(n)=\sum_{k=0}^{m-1}\sum_{l=0}^{m-k}\binom{m}{k}\frac{S_1(m-k,\,l)}{l+1}\left(a^{l+1}-(a+c)^{l+1}\right)S_{k,\,1}(n)$$

$$+\sum_{l=0}^{m}S_1(m,\,l)\frac{1}{l+1}\sum_{j=1}^{n}\left((j+a+c)^{l+1}-(j+a)^{l+1}\right).$$

Remark 2.7 There are too numerous applications of sums of powers of consecutive integers $S_m(n)$ (see (1)) to mention, which is the limit $\lambda\to 0$ of $S_{m,\,\lambda}(n)$ (see (6)). These include applications to computer science, physics, engineering, statistics, combinatorics, numerical analysis, number theory, finance and geometry. More specifically, sums of powers of consecutive integers can be used to analyze the time complexity of algorithm and optimize algorithms that involve iterating data structures, to calculate the energy levels of atoms and molecules in quantum mechanics and the average energy of a system of particles in statistical mechanics, to model the behavior of the vibration of a bridge in civil engineering and analyze systems with regular patterns in electrical engineering, to compute moments of a distribution in statistics, to encode combinatorial information in generating functions and solve various counting problems, to approximate solutions to numerical problems related to differential equations, to give integer solutions to some diophantine equations, to model the growth of investments over time in finance, and to compute the area of a polygon and the volume of a solid in geometry. In this paper, we derived a recursive formula for $S_{m,\,\lambda}(n)$ (see Theorem 2.1) by utilizing probabilistic methods. The sums of powers of consecutive integers $S_m(n)$ have wide-ranging applications across many fields, and similarly, the sums of values of degenerate falling factorials $S_{m,\,\lambda}(n)$ are expected to play a key role in solving many real-world problems.

3. Further remark

With the degenerate exponentials as in (12), the degenerate Bernoulli polynomials, introduced by Carlitz, are given by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{29}$$

(see [7, 13, 17, 22, 25]).

When $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x)=B_n(x)$, $(n\geq 0)$, where $B_n(x)$ are the ordinary Bernoulli polynomials in (2). From (29), we note that

$$\beta_{n,\lambda}(x)=\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}(x)_{n-k,\lambda}, \quad (n\geq 0). \tag{30}$$

By (30), we get $\beta_{0,\lambda}(x)=\beta_{0,\lambda}$.

Now, we observe that

$$\begin{aligned}\sum_{k=0}^{n}e_\lambda^k(t)&=\frac{1}{e_\lambda(t)-1}\left(e_\lambda^{n+1}(t)-1\right) \\ &=\frac{1}{t}\left(\frac{t}{e_\lambda(t)-1}e_\lambda^{n+1}(t)-\frac{t}{e_\lambda(t)-1}\right) \\ &=\frac{1}{t}\sum_{m=0}^{\infty}\left(\beta_{m,\lambda}(n+1)-\beta_{m,\lambda}\right)\frac{t^m}{m!} \\ &=\sum_{m=0}^{\infty}\left(\frac{\beta_{m+1,\lambda}(n+1)-\beta_{m+1,\lambda}}{m+1}\right)\frac{t^m}{m!}.\end{aligned} \tag{31}$$

On the other hand, by (17), we get

$$\sum_{k=0}^{n}e_\lambda^k(t)=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{n}(k)_{m,\lambda}\right)\frac{t^m}{m!}. \tag{32}$$

Comparing the coefficients on both sides of (31) and (32), we have the following proposition.

Proposition 3.1 For $m, n\in\mathbb{N}$, we have the identity

$$S_{m,\lambda}(n)=\sum_{k=0}^{n}(k)_{m,\lambda}=\frac{1}{m+1}\left(\beta_{m+1,\lambda}(n+1)-\beta_{m+1,\lambda}\right). \tag{33}$$

We note that (33) reduces to the Faulhaber's formula in (3) by letting $\lambda\to 0$.

4. Conclusion

In recent years, certain degenerate versions of many special numbers and polynomials have been investigated by employing different methods. Indeed, they have been explored by using probability theory, combinatorial methods, generating functions, umbral calculus, p-adic calculus, differential equations, special functions and analytic number theory.

In this paper, by computing $E\left[(X+Y)_{m,\lambda}\right]=\sum_{k=0}^{m}\binom{m}{k}E\left[(X)_{m-k,\lambda}\right]E\left[(Y)_{k,\lambda}\right]$, we derived a recursive relation for $S_{m,\lambda}(n)$ (see (6), (7)). Here $X\sim\text{Uniform}(-1,0)$ (see (13)), and Y is the uniform random variable on the integers $1,2,\ldots,n$. We let the reader compare our recursive formula in (7) with the ones in (9), (10) and (11). In addition, we showed an expression for $S_{m,\lambda}(n)$ in (29), which is a degenerate version of the well known expression for $S_m(n)$ (see, (1), (3)).

As one of our future research projects, we would like to continue to study degenerate versions of various things, which include special numbers and polynomials, some transcendental functions, recurrence relations and so on.

Conflict of interest

The authors declare no competing financial interest.

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