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Research Article

Fefferman's Inequality in Weighted Morrey Spaces

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Abstract: We prove Fefferman's inequality in weighted Morrey spaces in this paper. More precisely, for the weight w belonging to booth A_1 and Δ_2, $\kappa = (n-2p)/n$, and f belonging to the weighted Morrey space $L^{p,\,\kappa}(w)$, we prove that there exists a positive constant C such that the following inequality holds

$$\int_{\mathbb{R}^n} |f(x)||\phi(x)|^2 w(x)dx \leq C\|f\|_{L^{p,\,\kappa}(w)} \int_{\mathbb{R}^n} |\nabla\phi(x)|^2 w(x)^{1-\frac{2}{n}} dx,$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$.

***Keywords*:** Morrey spaces, weight, weighted Morrey spaces, Fefferman's inequality, maximal operator

MSC: 26D10, 42B25, 46E30

1. Introduction

Fefferman, in his paper [1], proved that the inequality $\int_{\mathbb{R}^n} |f(x)||\phi(x)|^2 dx \leq C \int_{\mathbb{R}^n} |\nabla\phi(x)|^2 dx$ holds for every compactly supported smooth functions ϕ defined on $\mathbb{R}^n$, that is $\phi \in C_0^\infty(\mathbb{R}^n)$, f belongs to the Morrey space $L^{p,\,n-2p}(\mathbb{R}^n)$, where $1 < p < n/2$, and C is a positive constant depends on n and the Morrey norm of f. Chiarenza and Frasca [2] later simplified the proof of this Fefferman's inequality and used more general assumption on f, that is, f belongs to the Morrey space $L^{p,\,n-qp}(\mathbb{R}^n)$, where $1 < p < n/q$. Recently, Tumalun et al. [3] generalized the result of Chiarenza and Frasca by proving the Fefferman inequality using the assumption that f belongs to the generalized Morrey spaces $L^{p,\,\varphi}(\mathbb{R}^n)$, where p and the function φ satisfy some conditions. In this inequality, the function f is usually called the potential.

The Fefferman inequality is an important tool in investigating the property of the weak solutions of elliptic partial differential equations. In particular, this inequality is applied to investigate the unique continuation property of the elliptic partial differential equations (see [3–5] for example). In 2010, Di Fazio investigated the unique continuation property for positive weak solutions to degenerate elliptic equations [6]. They achieved their result by using a weighted Fefferman inequality, which means the potential of this inequality belongs to some weighted Stummel-Kato clasess, that is proved in [7].

The purpose of this paper is to prove Fefferman's inequality in weighted Morrey spaces. The novelty here is the potential belongs to some weighted Morrey spaces. Our result here recovers the result of [1] and different from the result

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in [3]. We are confident that this inequality can be applied in investigating the unique continuation property and the regularity for positive weak solutions to degenerate elliptic equations.

2. Weighted morrey spaces

Let $a \in \mathbb{R}^n$ and $r > 0$. The set $B = B(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$ is called a ball in $\mathbb{R}^n$ with center a and radius r. When the center of the ball is not necessary, we write $B(a, r) = B_r$.

A function w that is defined on $\mathbb{R}^n$ and takes values in $(0, \infty)$, is called a **weight** if it is locally integrable in $\mathbb{R}^n$, that is $\int_K |w(x)|dx < \infty$, for every compact set K. Let E be a measurable set, we define

$$w(E) = \int_E w(x)dx,$$

and $|E|$ is a Lebesgue measure of E. The weight w satisfies the **doubling condition**, denoted by $w \in \Delta_2$, if there exists a positive constant C such that for every ball B, we have $w(2B) \leq Cw(B)$.

Now let $1 \leq p < \infty$ and w be a weight. The set $L^p_{\text{loc}}(w)$ is the set of all real-valued measurable functions f that is defined on $\mathbb{R}^n$ such that

$$\int_K |f(x)|^p w(x)dx < \infty,$$

for every compact sets K. Meanwhile, for a measurable set E, the set $L^p(E, w)$ is the set of all real-valued measurable functions f that is defined on E such that

$$\int_E |f(x)|^p w(x)dx < \infty.$$

Definition 1 Let $1 \leq p < \infty$, $0 < \kappa < 1$, and w be a weight. The **weighted Morrey space** $L^{p,\kappa}(w)$ is the set of all $f \in L^p_{\text{loc}}(w)$ that satisfies

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x)dx \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all balls $B \subseteq \mathbb{R}^n$.

The weighted Morrey spaces were introduced by [8]. By setting $w \equiv 1$ and $\kappa = \lambda/n$ with $0 < \lambda < n$, then $L^{p,\kappa}(w) = L^{p,\lambda}(\mathbb{R}^n)$, the classical Morrey space that was introduced by [9].

Definition 2 Let w be a weight and $f \in L^p_{\text{loc}}(w)$. The Maximal operator M_w with respect to the weight w is defined by

$$M_w f(x) = \sup_{B \ni x} \frac{1}{w(B)} \int_B |f(y)|w(y)dy.$$

By setting $w \equiv 1$, then $M_w = M$ is the classical or the standar maximal operator. The weight w satisfies the condition A_1, if there exists a positive constant C such that for almost every x, we have $Mw(x) \leq Cw(x)$. Let g be in $L^p_{\text{loc}}(w)$.

The function g satisfies the condition $A_1(w)$, if there exists a positive constant C such that for almost every x, we have $M_w g(x) \leq C g(x)$.

Let $C_0^\infty(\mathbb{R}^n)$ be a space of compactly supported smooth functions. For $\phi \in C_0^\infty(\mathbb{R}^n)$, the gradient vector of ϕ is defined by

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n} \right);$$

and its magnitude is defined by

$$|\nabla \phi| = \left(\left(\frac{\partial \phi}{\partial x_1} \right)^2 + \left(\frac{\partial \phi}{\partial x_2} \right)^2 + \cdots + \left(\frac{\partial \phi}{\partial x_n} \right)^2 \right)^{\frac{1}{2}}.$$

Let B be a ball such that ϕ has compact support in B. It is known that the following **subrepresentation formula** also holds

$$|\phi(x)| \leq C \int_B \frac{|\nabla \phi(x)|}{|x-y|^{n-1}} dx,$$

where C is a postive constant depends only on n [10, p.419-420]. This subrepresentation formula will be used in proving the main result in this paper.

We need to note that the positive constant $C = C_{\alpha, \beta, \ldots, \gamma}$, which appears throughout the proofs in this paper, denotes that it is dependent only on α, β, $\ldots$, and γ. The value of this constant may be vary from line to line whenever it appears.

3. Fefferman's inequality

In this section, we will state and prove the Fefferman inequality. We will start this section by proving some properties that will be used to prove the inequality.

Lemma 1 Let $0 < p < \infty$ and E be a measurable set. If $f \in L^p(E, w)$, then

$$\int_E |f(x)|^p w(x) dx = p \int_0^\infty t^{p-1} w(\{x \in E : |f(x)| > t\}) \, dt.$$

Proof. Let $S(f, t) = \{x \in E : |f(x)| > t\}$. We have

$$\int_E |f(x)|^p w(x) dx = \int_E \left(p \int_0^{|f(x)|} t^{p-1} dt \right) w(x) dx$$

$$= p \int_0^\infty \int_E \chi_{(0, |f(x)|)}(t) t^{p-1} w(x) dx dt.$$

Since

$$
\begin{aligned}
p\int_0^\infty \int_E \chi_{(0,\,|f(x)|)}(t)t^{p-1}w(x)dxdt &= p\int_0^\infty t^{p-1}\int_E \chi_{S(f,\,t)}(x)w(x)dxdt \\
&= p\int_0^\infty t^{p-1}\int_{S(f,\,t)} w(x)dxdt \\
&= p\int_0^\infty t^{p-1}w(S(f,\,t))dt,
\end{aligned}
$$

then

$$
\int_E |f(x)|^p w(x)dx = p\int_0^\infty t^{p-1}w(S(f,\,t))dt = p\int_0^\infty t^{p-1}w\left(\{x\in E : |f(x)|>t\}\right)dt.
$$

This proves the lemma. □

The Lemma 1 is used to prove the following property.

Lemma 2 Let $w \in A_1 \cap \Delta_2$ and $f \in L^{p,\,\kappa}(w)$. If $1<\gamma<p$ and $g=(M_w|f|^\gamma)^{\frac{1}{\gamma}}$, then $g\in A_1(w)\cap L^{p,\,\kappa}(w)$.

Proof. Let $x\in\mathbb{R}^n$ and $B=B_r$ be a ball (with radius r) that contains x. We will show that

$$
\frac{1}{w(B)}\int_B |g(y)|w(y)dy \le Cg(x),
$$

where C is a positive constant that does not depend on B.

Let $|f|^\gamma = f_1+f_2$, where $f_1=|f|^\gamma\chi_{B_{3r}}$ and $f_2=|f|^\gamma\chi_{B_{3r}^c}$. For every $y\in B$, we have $M_w|f|^\gamma(y)\le M_wf_1(y)+M_wf_2(y)$. Since $0<1/\gamma<1$, then $(M_w|f|^\gamma(y))^{1/\gamma}\le(M_wf_1(y))^{1/\gamma}+(M_wf_2(y))^{1/\gamma}$.

Let R be a positive real number that will be chosen latter. We will first estimate the integral of $(M_wf_1(y))^{1/\gamma}w(y)$ over the B. Using Lemma 1, we obtain

$$
\begin{aligned}
\int_B (M_wf_1(y))^{\frac{1}{\gamma}}w(y)dy &= \frac{1}{\gamma}\int_0^\infty t^{\frac{1}{\gamma}-1}w\left(\{y\in B : M_wf_1(y)>t\}\right)dt \\
&= \frac{1}{\gamma}\int_0^R t^{\frac{1}{\gamma}-1}w\left(\{y\in B : M_wf_1(y)>t\}\right)dt \\
&\quad + \frac{1}{\gamma}\int_R^\infty t^{\frac{1}{\gamma}-1}w\left(\{y\in B : M_wf_1(y)>t\}\right)dt \\
&= I+II. \qquad (1)
\end{aligned}
$$

For I, we get

$$I = \frac{1}{\gamma}\int_0^R t^{\frac{1}{\gamma}-1} w(\{y \in B : M_w f_1(y) > t\})\, dt \le \frac{1}{\gamma} w(B) \int_0^R t^{\frac{1}{\gamma}-1} dt = \frac{1}{\gamma} w(B)\gamma R^{\frac{1}{\gamma}} = w(B)R^{\frac{1}{\gamma}}.$$

Since $w \in A_1$, we can use the weak type estimation $(1,\, 1)$ for the $M_w f_1$ (see [11, p.507]) to obtain

$$II = \frac{1}{\gamma}\int_R^\infty t^{\frac{1}{\gamma}-1} w(\{y \in B : M_w f_1(y) > t\})\, dt$$

$$\le C_{n,\, w}\|f_1\|_{L^1(B,\, w)} \frac{1}{\gamma}\int_R^\infty t^{\frac{1}{\gamma}-2} dt$$

$$= C_{n,\, w}\|f_1\|_{L^1(B,\, w)} \frac{1}{\gamma - 1} R^{\frac{1}{\gamma}-1}.$$

Substituting the estimations of I and II to the (1) gives us

$$\frac{1}{w(B)}\int_B (M_w f_1(y))^{\frac{1}{\gamma}} w(y)dy \le \frac{1}{w(B)} w(B)R^{\frac{1}{\gamma}} + \frac{1}{w(B)} C_{n,\, w,\, \gamma}\|f_1\|_{L^1(B,\, w)} R^{\frac{1}{\gamma}-1}$$

$$= R^{\frac{1}{\gamma}} + \frac{1}{w(B)} C_{n,\, w,\, \gamma}\|f_1\|_{L^1(B,\, w)} R^{\frac{1}{\gamma}-1}.$$

Choose $R = \|f_1\|_{L^1(B,\, w)}/w(B)$. We have

$$\frac{1}{w(B)}\int_B (M_w f_1(y))^{\frac{1}{\gamma}} w(y)dy \le \frac{\|f_1\|_{L^1(B,\, w)}^{\frac{1}{\gamma}}}{w(B)^{\frac{1}{\gamma}}}(1 + C_{n,\, w,\, \gamma})$$

$$\le C_{n,\, w,\, \gamma} \frac{1}{w(B)^{\frac{1}{\gamma}}} \left(\int_{B_{3r}} |f(y)|^\gamma w(y)dy\right)^{\frac{1}{\gamma}}$$

$$\le C_{n,\, w,\, \gamma} \left(\frac{1}{w(B_{3r})}\int_{B_{3r}} |f(y)|^\gamma w(y)dy\right)^{\frac{1}{\gamma}}$$

$$\le C_{n,\, w,\, \gamma}(M_w|f|^\gamma(x))^{\frac{1}{\gamma}}$$

$$= C_{n,\, w,\, \gamma} g(x). \tag{2}$$

Now, we will estimate the integral of $(M_w f_2(y))^{1/\gamma} w(y)$ over the B. Let $y \in B = B_r$ and $B' = B'_s$ be a ball that contains y such that $B' \cap (B_{3r})^c \neq \emptyset$. Then $B \subseteq B'_{3s}$. Therefore,

$$\frac{1}{w(B')}\int_{B'} f_2(z)w(z)dz \le C\frac{1}{w(B'_{3s})}\int_{B'_{3s}} f_2(z)w(z)dz$$

$$\le CM_w f_2(x) \qquad (x\in B'_{3s}),$$

since $w\in\Delta_2$. We may ignore the case $B'\cap(B_{3r})^c=\emptyset$, since $\frac{1}{w(B')}\int_{B'} f_2(z)w(z)dz=0$. Moreover, since $y\in B'$, then

$$M_w f_2(y)\le CM_w f_2(x).$$

Hence,

$$\frac{1}{w(B)}\int_B (M_w f_2(y))^{\frac{1}{\gamma}} w(y)dy \le C\frac{1}{w(B)}\int_B (M_w f_2(x))^{\frac{1}{\gamma}} w(y)dy$$

$$= C(M_w f_2(x))^{\frac{1}{\gamma}}\frac{1}{w(B)}\int_B w(y)dy$$

$$= C(M_w f_2(x))^{\frac{1}{\gamma}}$$

$$\le Cg(x). \tag{3}$$

According to the (2) and (3), we obtain

$$\frac{1}{w(B)}\int_B |g(y)|w(y)dy \le \frac{1}{w(B)}\int_B (M_w f_1(y))^{\frac{1}{\gamma}} w(y)dy + \int_B (M_w f_2(y))^{\frac{1}{\gamma}} w(y)dy$$

$$\le C_{n,\,w,\,\gamma}\, g(x). \tag{4}$$

From (4), we conclude that

$$\frac{1}{w(B)}\int_B |g(y)|w(y)dy \le C_{n,\,w,\,\gamma}\, g(x).$$

Thus, $M_w g(x)\le C_{n,\,w,\,\gamma}\, g(x)$. This means $g\in A_1(w)$.
By using the property [8, Theorem 3.1], we have

$$\|g\|_{L^{p,\,\kappa}(w)} = \|\,(M_w|f|^{\gamma})^{\frac{1}{\gamma}}\,\|_{L^{p,\,\kappa}(w)} = \|M_w|f|^{\gamma}\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma},\,\kappa}(w)} \le C\||f|^{\gamma}\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma},\,\kappa}(w)} = C\|f\|_{L^{p,\,\kappa}(w)} < \infty.$$

Therefore $g \in L^{p,\kappa}(w)$. $\square$

Lemma 3 Let $w \in A_1$ and $g \in L^{p,\kappa}(w)$. Then

$$\int_{\mathbb{R}^n} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx \leq C_{n,\kappa,p} M_w g(y)^{\frac{1}{2}} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} w(y)^{1-\frac{1}{n}}.$$

Proof. Let $\delta > 0$. We have

$$\int_{\mathbb{R}^n} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx = \int_{|x-y|<\delta} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx + \int_{|x-y|\geq\delta} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx$$

$$= I + II. \tag{5}$$

For I, we use the assumption $w \in A_1$ to obtain

$$I = \int_{|x-y|<\delta} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx$$

$$= \sum_{k=0}^{\infty} \int_{\frac{\delta}{2^{k+1}} \leq |x-y| < \frac{\delta}{2^k}} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx$$

$$\leq \sum_{k=0}^{\infty} \frac{(2^{k+1})^{n-1}}{\delta^{n-1}} \frac{w\left(B\left(y, \frac{\delta}{2^k}\right)\right)}{w\left(B\left(y, \frac{\delta}{2^k}\right)\right)} \int_{B\left(y, \frac{\delta}{2^k}\right)} |g(x)| w(x)dx$$

$$\leq M_w g(y) \sum_{k=0}^{\infty} \frac{(2^{k+1})^{n-1}}{\delta^{n-1}} w\left(B\left(y, \frac{\delta}{2^k}\right)\right)$$

$$= C_n M_w g(y) \delta \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \frac{1}{\left|B\left(y, \frac{\delta}{2^k}\right)\right|} \int_{B\left(y, \frac{\delta}{2^k}\right)} w(x)dx$$

$$\leq C_n \delta M_w g(y) w(y).$$

For II, we set $s = n - (3/2)p$ and use Hölder's inequality to obtain

$$II = \int_{|x-y|\geq\delta} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx$$

$$= \int_{|x-y|\geq\delta} \frac{|g(x)||x-y|^{\frac{s}{p}+1-n}}{|x-y|^{\frac{s}{p}}} w(x)dx$$

$$\leq \left(\int_{|x-y|\geq\delta} \frac{|g(x)|^p}{|x-y|^s} w(x)dx\right)^{\frac{1}{p}} \left(\int_{|x-y|\geq\delta} |x-y|^{\left(\frac{s}{p}+1-n\right)p'} w(x)dx\right)^{\frac{1}{p'}}$$

$$= III^{1/p} \cdot IV^{1/p'},$$

where $1/p' = 1 - 1/p$.

We estimate III by the following computations

$$III = \int_{|x-y|\geq\delta} \frac{|g(x)|^p}{|x-y|^s} w(x)dx$$

$$= \sum_{j=0}^{\infty} \int_{2^j\delta\leq|x-y|<2^{j+1}\delta} \frac{|g(x)|^p}{|x-y|^s} w(x)dx$$

$$\leq \sum_{j=0}^{\infty} \frac{1}{(2^j\delta)^s} \frac{w(B(y,\, 2^{j+1}\delta))^\kappa}{w(B(y,\, 2^{j+1}\delta))^\kappa} \int_{B(y,\, 2^{j+1}\delta)} |g(x)|^p w(x)dx$$

$$\leq \|g\|^p_{L^{p,\,\kappa}(w)} \sum_{j=0}^{\infty} \frac{1}{(2^j\delta)^s} w(B(y,\, 2^{j+1}\delta))^\kappa$$

$$= \|g\|^p_{L^{p,\,\kappa}(w)} \sum_{j=0}^{\infty} \frac{1}{(2^j\delta)^s} \left(\frac{|B(y,\, 2^{j+1}\delta)|}{|B(y,\, 2^{j+1}\delta)|} \int_{B(y,\, 2^{j+1}\delta)} w(x)dx\right)^\kappa$$

$$\leq C_{n,\,\kappa} \|g\|^p_{L^{p,\,\kappa}(w)} Mw(y)^\kappa \sum_{j=0}^{\infty} \frac{(2^j\delta)^{n\kappa}}{(2^j\delta)^s}$$

$$\leq C_{n,\,\kappa} \|g\|^p_{L^{p,\,\kappa}(w)} w(y)^\kappa \delta^{n\kappa-s},$$

since $w \in A_1$. For IV, we also compute

$$
\begin{aligned}
IV &= \int_{|x-y|\geq\delta} |x-y|^{\left(\frac{s}{p}+1-n\right)p'} w(x)dx \\
&= \sum_{j=0}^{\infty} \int_{2^j\delta\leq|x-y|<2^{j+1}\delta} |x-y|^{\left(\frac{s}{p}+1-n\right)p'} w(x)dx \\
&\leq \sum_{j=0}^{\infty} (2^j\delta)^{\left(\frac{s}{p}+1-n\right)p'} \frac{|B(y,\, 2^{j+1}\delta)|}{|B(y,\, 2^{j+1}\delta)|} \int_{B(y,\, 2^{j+1}\delta)} w(x)dx \\
&\leq C_n Mw(y) \sum_{j=0}^{\infty} (2^j\delta)^{\left(\frac{s}{p}+1-n\right)p'} (2^j\delta)^n \\
&\leq C_n w(y) \delta^{n+\left(\frac{s}{p}+1-n\right)p'} \sum_{j=0}^{\infty} \left(2^{n+\left(\frac{s}{p}+1-n\right)p'}\right)^j \\
&= C_{n,\,p} w(y) \delta^{n+\left(\frac{s}{p}+1-n\right)p'}.
\end{aligned}
$$

By using the estimations of III and IV, we get

$$
\begin{aligned}
II &\leq \left(C_{n,\,\kappa} \|g\|^p_{L^{p,\,\kappa}(w)} w(y)^{\kappa} \delta^{n\kappa-s}\right)^{\frac{1}{p}} \left(C_n w(y) \delta^{n+\left(\frac{s}{p}+1-n\right)p'}\right)^{\frac{1}{p'}} \\
&= C_{n,\,\kappa,\,p} \|g\|_{L^{p,\,\kappa}(w)} w(y)^{\frac{\kappa}{p}+\frac{1}{p'}} \delta^{\frac{n\kappa-s}{p}+\frac{n+\left(\frac{s}{p}+1-n\right)p'}{p'}} \\
&= C_{n,\,\kappa,\,p} \|g\|_{L^{p,\,\kappa}(w)} w(y)^{1-\frac{2}{n}} \delta^{-1}.
\end{aligned}
$$

Therefore

$$
\int_{\mathbb{R}^n} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx \leq C_n \delta M_w g(y) w(y) + C_{n,\,\kappa,\,p} \|g\|_{L^{p,\,\kappa}(w)} w(y)^{1-\frac{2}{n}} \delta^{-1}.
$$

By choosing $\delta = M_w g(y)^{-\frac{1}{2}} \|g\|^{\frac{1}{2}}_{L^{p,\,\kappa}(w)} w(y)^{-\frac{1}{n}}$, we have

$$\int_{\mathbb{R}^n} \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx \le C_n \left(M_w g(y)^{-\frac{1}{2}} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} w(y)^{-\frac{1}{n}} \right) M_w g(y) w(y)$$

$$+ C_{n,\kappa,p} \|g\|_{L^{p,\kappa}(w)} w(y)^{1-\frac{2}{n}} \left(M_w g(y)^{-\frac{1}{2}} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} w(y)^{-\frac{1}{n}} \right)^{-1}$$

$$= C_n M_w g(y)^{\frac{1}{2}} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} w(y)^{1-\frac{1}{n}} + C_{n,\kappa,p} M_w g(y)^{\frac{1}{2}} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} w(y)^{1-\frac{1}{n}}$$

$$= C_{n,\kappa,p} M_w g(y)^{\frac{1}{2}} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} w(y)^{1-\frac{1}{n}}.$$

The lemma is proved. □

Now we are ready to prove our main result in this paper, namely Fefferman's inequality in weighted Morrey spaces.

Theorem 1 (Fefferman's Inequality) Let $w \in A_1 \cap \Delta_2$ and $\kappa = (n-2p)/n$. If $f \in L^{p,\kappa}(w)$, then there exists a positive constant C, such that

$$\int_{\mathbb{R}^n} |f(x)||\phi(x)|^2 w(x)dx \le C\|f\|_{L^{p,\kappa}(w)} \int_{\mathbb{R}^n} |\nabla\phi(x)|^2 w(x)^{1-\frac{2}{n}} dx, \tag{6}$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let $1 < \gamma < p$ and $g = (M_w|f|^\gamma)^{\frac{1}{\gamma}}$. According to Lemma 2, we have $g \in A_1(w) \cap L^{p,\kappa}(w)$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ and B be a ball such that $\phi \in C_0^\infty(B)$. To prove this theorem, it suffices to prove that inequality (6) is satisfied by replacing f with g.

By using the subrepresentation formula, we get

$$|\phi(x)|^2 \le C_n \int_B \frac{|\phi(y)||\nabla\phi(y)|}{|x-y|^{n-1}} dy.$$

This inequality and Lemma 3 yield

$$\int_{\mathbb{R}^n} |g(x)||\phi(x)|^2 w(x)dx = \int_B |g(x)||\phi(x)|^2 w(x)dx$$

$$\le C_n \int_B \left(\int_B \frac{|\phi(y)||\nabla\phi(y)|}{|x-y|^{n-1}} dy \right) |g(x)| w(x)dx$$

$$= C \int_B \left(\int_B \frac{|g(x)|}{|x-y|^{n-1}} w(x)dx \right) |\phi(y)||\nabla\phi(y)|dy$$

$$\le C_{n,p} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} \int_B M_w g(y)^{\frac{1}{2}} w(y)^{1-\frac{1}{n}} |\phi(y)||\nabla\phi(y)|dy. \tag{7}$$

We estimate the integrand of the right-hand side of (7) by the following

$$\int_B M_w g(y)^{\frac{1}{2}} w(y)^{1-\frac{1}{n}} |\phi(y)||\nabla\phi(y)|dy \leq \left(\int_B M_w g(y)|\phi(y)|^2 w(y)dy\right)^{\frac{1}{2}} \left(\int_B |\nabla\phi(y)|^2 w(y)^{1-\frac{2}{n}} dy\right)^{\frac{1}{2}}$$

$$\leq \left(\int_B |g(y)|\phi(y)|^2 w(y)dy\right)^{\frac{1}{2}} \left(\int_B |\nabla\phi(y)|^2 w(y)^{1-\frac{2}{n}} dy\right)^{\frac{1}{2}}$$

$$= \left(\int_B |g(x)|\phi(x)|^2 w(x)dx\right)^{\frac{1}{2}} \left(\int_B |\nabla\phi(x)|^2 w(x)^{1-\frac{2}{n}} dx\right)^{\frac{1}{2}}, \tag{8}$$

where we use the fact that $g \in A_1(w)$. Substituting (8) into (7), we obtain

$$\int_{\mathbb{R}^n} |g(x)||\phi(x)|^2 w(x)dx \leq C_{n,p} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} \left(\int_B |g(x)|\phi(x)|^2 w(x)dx\right)^{\frac{1}{2}} \left(\int_B |\nabla\phi(x)|^2 w(x)^{1-\frac{2}{n}} dx\right)^{\frac{1}{2}}$$

$$\leq C_{n,p} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |g(x)|\phi(x)|^2 w(x)dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla\phi(x)|^2 w(x)^{1-\frac{2}{n}} dx\right)^{\frac{1}{2}}.$$

This means

$$\left(\int_{\mathbb{R}^n} |g(x)|\phi(x)|^2 w(x)dx\right)^{\frac{1}{2}} \leq C_{n,p} \|g\|_{L^{p,\kappa}(w)}^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla\phi(x)|^2 w(x)^{1-\frac{2}{n}} dx\right)^{\frac{1}{2}}.$$

Therefore

$$\int_{\mathbb{R}^n} |g(x)|\phi(x)|^2 w(x)dx \leq C_{n,p} \|g\|_{L^{p,\kappa}(w)} \int_{\mathbb{R}^n} |\nabla\phi(x)|^2 w(x)^{1-\frac{2}{n}} dx.$$

The theorem is proved for the case $g = (M_w|f|^\gamma)^{\frac{1}{\gamma}}$. □

4. Conclusions

The Fefferman inequality in weighted Morrey spaces, which is stated in Theorem 1, is proved in this paper. We provide a detailed proof that can be used in many similar situations and may be useful to other researchers studying the unique continuation property and the regularity of positive weak solutions to degenerate elliptic equations.

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Author contributions

Nicky K. Tumalun, Philotheus E. A. Tuerah, Derel F. Kaunang, and Marvel G. Maukar contributed to the conception, design, and review of the manuscript. Nicky K. Tumalun provided and edited the manuscript. All authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Fefferman C. The uncertainty principle. *Bulletin of the American Mathematical Society*. 1983; 9: 129-206.
[2] Chiarenza F, Frasca M. A remark on a paper by C. Fefferman. *Proceedings of the American Mathematical Society*. 1990; 108: 407-409.
[3] Tumalun NK, Hakim DI, Gunawan H. Some function spaces and their application to elliptic partial differential equations. *Matematicki Vesnik*. 2023; 75(2): 71-86.
[4] Tumalun NK, Tuerah PEA, Maukar MG, Tilaar ALF, Runtu PVJ. An application of generalized Morrey spaces to unique continuation property of the quasilinear elliptic equations. *AIMS Mathematics*. 2023; 8(11): 26007-26020.
[5] Zamboni P. Unique continuation for non-negative solutions of quasilinear elliptic equations. *Bulletin of the Australian Mathematical Society*. 2001; 64(1): 149-156.
[6] Di Fazio G, Zamboni P. Unique continuation for positive solutions of degenerate elliptic equations. *Mathematische Nachrichten*. 2010; 283: 994-999.
[7] Di Fazio G, Zamboni P. Regularity for quasilinear degenerate elliptic equations. *Mathematische Zeitschrift*. 2006; 253: 787-803.
[8] Komori Y, Shirai S. Weighted Morrey spaces and a singular integral operator. *Mathematische Nachrichten*. 2009; 282: 219-231.
[9] Morrey CB. On the solutions of quasi-linear elliptic partial differential equations. *Transactions of the American Mathematical Society*. 1938; 43: 126-166.
[10] Wheeden RL, Zygmund A. *Measure and Integral: an Introduction to Real Analysis*. 2nd Ed. Boca Raton, FL, USA: CRC Press; 2015.
[11] Grafakos L. *Classical Fourier Analysis*. 3rd Ed. New York, NY, USA: Springer; 2014.