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Research Article

Third-Kind Chebyshev Spectral Collocation Method for Solving Models of Two Interacting Biological Species

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Abstract: This paper develops numerical methods for solving a system of two nonlinear integro-differential equations that arise in biological modeling. A spectral collocation method utilizing third-kind Chebyshev polynomials forms the basis of the solution methodology, which efficiently converts the integro-differential system into a collection of nonlinear algebraic equations. To guarantee precise and effective calculation, these algebraic equations are subsequently numerically solved using Newton's method. In comparison to current methods, the suggested approach offers notable gains in computational efficiency and precision. The spectral collocation method's accuracy is confirmed by contrasting the outcomes with those derived from other numerical methods that are published in the literature. To further illustrate the applicability, dependability, and computational efficiency of the suggested approach in resolving complicated biological systems, a number of illustrative instances are provided. The ability of spectral collocation techniques based on third-kind Chebyshev polynomials to solve integro-differential equations in a variety of scientific and engineering applications is highlighted by this work.

***Keywords*:** Chebyshev polynomial, integro-differential equations, spectral collocation method

MSC: 65M70, 34A12, 92D25, 41A10

1. Introduction

Integral equations have become crucial in various areas of applied mathematics due to their wide applicability in modeling real-world phenomena. These equations naturally emerge in many disciplines, including dynamics of fluid, biological living systems, and reaction kinetics [1]. Among these, integro-differential equations are particularly significant, as they combine the features of both differential and integral equations and appear in glass processing [2], hydrodynamics of nano-materials [3].

The system of integro-differential equations that follows is examined in this paper as

$$\frac{dp(\eta)}{d\eta} = p(\eta)\left[c_1 - \nu_1 z(\eta) - \int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau\right] + q_1(\eta), \quad 0 \le \eta \le l, \quad \nu_1, c_1 > 0, \tag{1}$$

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$$\frac{dz(\eta)}{d\eta} = z(\eta)\left[-c_2 + \nu_2 p(\eta) + \int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau\right] + q_2(\eta), \quad 0 \le \eta \le l, \quad \nu_2, c_2 > 0, \tag{2}$$

with initial conditions

$$p(0) = \alpha_1, \quad z(0) = \alpha_2,$$

where q_1, q_2, w_1, w_2 are given functions and $p(\eta)$, $z(\eta)$ are unknown functions. The number of two distinct species at time η is represented by the variables $p(\eta)$ and $z(\eta)$, where the first species is increasing and the second is decreasing. When both species are combined, and it is assumed that the second will feed on the first, the rate of the second species, $\frac{dz}{d\eta}$, will increase. This rate depends on all past values of the first species and the current populations, $p(\eta)$. The following pair of integro-differential equations describe what happens when equilibrium shifts between these species.

$$\frac{dp(\eta)}{d\eta} = p(\eta)\left[c_1 - \nu_1 z(\eta) - \int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau\right], \quad c_1 > 0, \tag{3}$$

$$\frac{dz(\eta)}{d\eta} = z(\eta)\left[-c_2 + \nu_2 p(\eta) + \int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau\right], \quad c_2 > 0. \tag{4}$$

Where the coefficients of raising and reducing the first and second species are, respectively, c_1 and $-c_2$. The corresponding species determines the values of ν_1, w_1, and ν_2, w_2. Let T_0 represent the finite heredity period for each species. The Integro-derivational system with $p(0) = \alpha_1$, $z(0) = \alpha_2$, Eqs. (3)-(4) represent a particular instance of Eqs. (1)-(2). You may find the comprehensive formulations of Eqs. (3)-(4) in [4].

Many analytical and numerical methods have been tamed to solve integro-differential problems [5]. Techniques like the variational iteration method have been used to address these equations and systems [6]. Biazar et al. tackled them using He's homotopy perturbation method [7], while the homotopy analysis method has also been effective for solving high-order equations [8]. Lately, the ADM has been employed to solve such systems [9].

Many numerical methods have been used to solve systems of integro-differential equations. Maleknejad et al. addressed linear integro-differential equations using Bernstein polynomials [10] and the Galerkin method [11]. Other methods, like the differential transform method [12] and the tau method [13], have also been applied. For nonlinear integro-differential equations in population dynamics and growth models [14], Dehghan et al. used the pseudospectral Legendre-Galerkin approach [15] and the rational pseudospectral approximation [16]. Optimization method [17].

Numerical methods for interacting population modeling in predator-prey systems and nonlinear age-structured population models have been improved in [18, 19]. Additionally, the current authors have lately solved numerically the fuzzy integral-differential equations [20], the system of integral equations [21], and the system of nonlinear Volterra integro-differential equations [22, 23]. Spectral methods have been increasingly appearing in the literature due to their remarkable convergence properties, which make them highly effective for solving a wide range of mathematical and physical problems. These methods offer exponential or high-order accuracy, especially when applied to problems with smooth solutions, for recent advances in spectral methods, see [24–35].

Several researchers have worked on solving the biological species coexisting model. Biazar et al. applied the Adomian decomposition method (ADM) to solve this model [36]. Shakeri and Dehghan used the variational iteration method (VIM) and the pseudospectral Legendre method for the same purpose [37]. Their findings showed that VIM provided more accurate results than both the Legendre-spectral method and ADM.

In the study of biological species living together, differential-integral equations play a key role in describing the interaction between species. Traditionally, methods such as the Legendre spectral collocation method have been employed to solve these complex equations [38]. However, in recent times, alternative spectral methods have gained popularity due to their efficiency and accuracy.

This work introduces a numerical technique using Chebyshev polynomials of the third-kind for solving a system of nonlinear integro-differential equations that arise in biological modeling. Chebyshev polynomials are well-known for their optimal interpolation properties and are particularly advantageous in spectral methods due to their minimization of Runge's phenomenon [39].

By using Chebyshev polynomials of the third-kind, we aim to enhance the accuracy and convergence of the solution, providing an effective alternative to the Legendre spectral collocation method. The current method reduces the system of integro-differential equations to a simple set of nonlinear equations, which are then solved using numerical techniques. It is worthy to report here that, Chebyshev polynomials are essential basis functions in the field of numerical solutions of differential problems, the interested reader is referred to [40–43].

We compare the results obtained through this method with those from existing approaches in the literature, demonstrating the applicability and effectiveness of the Chebyshev polynomial approach.

This work is structured as follows: we introduce the properties of the Chebyshev polynomial in section 2. In section 3, we go into function approximation and the Chebyshev spectral collocation technique. In section 4, verify the convergence and analyze the errors of the suggested polynomial series expansion. Section 5 covers the illustrated examples that demonstrate the correctness and efficiency of the current approach. Section 6 concludes by summarizing the research results.

2. Chebyshev polynomial of third-kind and its properties

Third-kind Chebyshev polynomials, or $V(\iota)$, are a class of orthogonal polynomials having unique properties that make them useful for certain uses in spectral methods, approximation theory, and numerical analysis. While the first and second kinds of Chebyshev polynomials are more widely recognized and applied, the third-kind is essential to solving some boundary value issues because it provides special benefits for managing particular boundary conditions and numerical difficulties.

Third-kind Chebyshev polynomials are orthogonal to the weight function $(1-\iota)^{-1/2}(1+\iota)^{1/2}$ and defined across the interval $[-1, 1]$. One of the main characteristics that make these polynomials helpful for solving differential equations and completing polynomial approximations is their orthogonality. An explicit expression of a polynomial in terms of powers of the variable ι can be found in its power form. The recursive definition can be used to construct the power form for Chebyshev polynomials of the third-kind, $V(\iota)$ [44]:

$$V_n(\iota) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta} \qquad \iota \in [-1, 1], \quad n \geq 0; \quad \theta = \cos^{-1}\iota.$$

$$V_0(\iota) = 1, \quad V_1(\iota) = 2\iota - 1,$$

$$V_n(\iota) = 2\iota V_{n-1}(\iota) - V_{n-2}(\iota) \quad \text{for} \quad n \geq 2.$$

Higher-order polynomials $V_n(\iota)$ for any n can be calculated using this recurrence relation and given as powers of ι. As an illustration:

$$V_2(\iota) = 4\iota^2 - 2\iota - 1$$

$$V_3(\iota) = 8\iota^3 + 4\iota^2 - 4\iota + 1$$

Generally, the process of obtaining the power form of $V(\iota)$ for any n involves expanding the recurrence relation and expressing the result as a sum of terms containing powers of ι. An essential characteristic of third-kind Chebyshev polynomials is their orthogonality with respect to a given weight function. The expression for this orthogonality is:

$$\int_{-1}^{1} (1-\iota)^{-1/2}(1+\iota)^{1/2} V_m(\iota)V_n(\iota)d\iota = 0 \quad \text{for} \quad m \neq n.$$

This characteristic is crucial for solving differential equations and approximating functions with orthogonal polynomials, as in spectral collocation and other numerical approaches.

3. Chebyshev spectral collocation method

In this section, we present the basic idea of the Chebyshev spectral collocation method for solving integro-differential equations defined in (1)-(2). The procedure of approximation involves using Chebyshev-Gauss nodes, which are the zeros of Chebyshev polynomials of the third-kind. To use the Chebyshev spectral collocation method, we consider the collocation nodes are given by the zeros of the Chebyshev polynomial of the third-kind $V_{M+1}(s) = 0$ where M is the degree of the polynomial. Let us approximate the unknown functions $p(\eta)$ and $z(\eta)$ as:

$$p(\eta) = \sum_{k=0}^{M} p_k V_k(\eta), \tag{5}$$

$$z(\eta) = \sum_{k=0}^{M} z_k V_k(\eta), \tag{6}$$

Now, approximate the integro-differential system using the above approximations. The system of integro-differential equations (1)-(2) is reduced as follows:

$$\sum_{k=0}^{M} p_k \frac{d}{d\eta} V_k(\eta) = \left(\sum_{k=0}^{M} p_k V_k(\eta) \right) \left[c_1 - \nu_1 \sum_{k=0}^{M} z_k V_k(\eta) - \int_{\eta - T_0}^{\eta} w_1(\eta - \tau) \sum_{k=0}^{M} z_k V_k(\eta) d\tau \right] + q_1(\eta), \tag{7}$$

$$\sum_{k=0}^{M} z_k \frac{d}{d\eta} V_k(\eta) = \left(\sum_{k=0}^{M} z_k V_k(\eta) \right) \left[-c_2 + \nu_2 \sum_{k=0}^{M} p_k V_k(\eta) + \int_{\eta - T_0}^{\eta} w_2(\eta - \tau) \sum_{k=0}^{M} p_k V_k(\eta) d\tau \right] + q_2(\eta). \tag{8}$$

To handle the integral terms, we use the Chebyshev-Gauss quadrature for the third-kind. We change the interval $[\eta - T_0, \, \eta]$ to $[-1, \, 1]$ by the substitution

$$s = 1 + 2\left(\frac{\tau - \eta}{T_0}\right).$$

Thus, the integral becomes

$$\int_{\eta - T_0}^{\eta} w_1(\eta - \tau) \sum_{k=0}^{M} z_k V_k(\eta) d\tau = \frac{T_0}{2} \int_{-1}^{1} w_1\left(-\frac{T_0}{2}(s-1)\right) \left(\sum_{k=0}^{M} z_k V_k\left(\eta + \frac{T_0}{2}(s-1)\right)\right) ds,$$

$$= \frac{T_0}{2} \sum_{j=0}^{M} B_j w_1\left(-\frac{T_0}{2}(s_j - 1)\right) \left(\sum_{k=0}^{M} z_k V_k\left(\eta + \frac{T_0}{2}(s_j - 1)\right)\right). \tag{9}$$

$$\int_{\eta - T_0}^{\eta} w_2(\eta - \tau) \sum_{k=0}^{M} p_k V_k(\eta) d\tau = \frac{T_0}{2} \int_{-1}^{1} w_2\left(-\frac{T_0}{2}(s-1)\right) \left(\sum_{k=0}^{M} p_k V_k\left(\eta + \frac{T_0}{2}(s-1)\right)\right) ds,$$

$$= \frac{T_0}{2} \sum_{j=0}^{M} B_j w_2\left(-\frac{T_0}{2}(s_j - 1)\right) \left(\sum_{k=0}^{M} p_k V_k\left(\eta + \frac{T_0}{2}(s_j - 1)\right)\right). \tag{10}$$

Here, s_j for $j = 0, 1, \ldots, M$ are the Chebyshev-Gauss nodes, which are the zeros of the Chebyshev polynomial of the third-kind $V_{M+1}(t) = 0$, and B_j are the corresponding weights defined as

$$B_j = \frac{\pi}{M+1} \cdot \frac{(1 + t_j)}{(1 - t_j^2)}, \quad j = 0, 1, \ldots, M.$$

Substituting (9)-(10) into (7)-(8), we derive the following equations:

$$\sum_{k=0}^{M} p_k \frac{d}{dt} V_k(t) = \left(\sum_{k=0}^{M} p_k V_k(t)\right) \left[c_1 - \nu_1 \sum_{k=0}^{M} z_k V_k(t)\right.$$

$$\left. - \frac{T_0}{2} \sum_{j=0}^{M} B_j w_1\left(-\frac{T_0}{2}(s_j - 1)\right) \left(\sum_{k=0}^{M} z_k V_k\left(t + \frac{T_0}{2}(s_j - 1)\right)\right)\right] + q_1(t), \tag{11}$$

$$\sum_{k=0}^{M} z_k \frac{d}{dt} V_k(t) = \left(\sum_{k=0}^{M} z_k V_k(t)\right) \left[-c_2 + \nu_2 \sum_{k=0}^{M} p_k V_k(t)\right.$$

$$\left. + \frac{T_0}{2} \sum_{j=0}^{M} B_j w_2\left(-\frac{T_0}{2}(s_j - 1)\right) \left(\sum_{k=0}^{M} p_k V_k\left(t + \frac{T_0}{2}(s_j - 1)\right)\right)\right] + q_2(t). \tag{12}$$

Next, we apply the Chebyshev-Gauss collocation nodes t_i for $i = 0, 1, \ldots, M$ to discretize the equations (11)-(12):

$$\sum_{k=0}^{M} p_k \frac{d}{dt_i} V_k(t_i) = \left(\sum_{k=0}^{M} p_k V_k(t_i)\right) \left[c_1 - \nu_1 \sum_{k=0}^{M} z_k V_k(t_i)\right.$$

$$\left. - \frac{T_0}{2} \sum_{j=0}^{M} B_j w_1 \left(-\frac{T_0}{2}(s_j - 1)\right) \left(\sum_{k=0}^{M} z_k V_k \left(t_i + \frac{T_0}{2}(s_j - 1)\right)\right)\right] + q_1(t_i), \tag{13}$$

$$\sum_{k=0}^{M} z_k \frac{d}{dt_i} V_k(t_i) = \left(\sum_{k=0}^{M} z_k V_k(t_i)\right) \left[-c_2 + \nu_2 \sum_{k=0}^{M} p_k V_k(t_i)\right.$$

$$\left. + \frac{T_0}{2} \sum_{j=0}^{M} B_j w_2 \left(-\frac{T_0}{2}(s_j - 1)\right) \left(\sum_{k=0}^{M} p_k V_k \left(t_i + \frac{T_0}{2}(s_j - 1)\right)\right)\right] + q_2(t_i). \tag{14}$$

Equations (13)-(14) form a system of nonlinear equations with dimension $2M+2$. Continuing from the given initial conditions, we end-up with

$$\sum_{k=0}^{M} p_k V_k(0) = \alpha_1,$$

$$\sum_{k=0}^{M} z_k V_k(0) = \alpha_2.$$

Therefore, for $k = 0, 1, \ldots, M$, the set of nonlinear algebraic equations formed by equations (13)-(14) has $2M+2$ unknowns for each of the variables p_k and z_k. By applying a numerical solution to this system, we may determine the values of the unknowns p_k and z_k, where $k = 0, 1, \ldots, M$. Therefore, we use Equations (5)-(6) to derive the approximate/semi-analytic solutions of the integro-differential equations (1)-(2).

4. Error estimate

The convergence and error analysis of the suggested polynomial approximation are thoroughly examined in this section. Therefore, this research uses a number of necessary lemmas.

Lemma [44] For all $i > 0$, we have:

$$|p_i(\eta)| \leq 2i+1,$$

$$|z_i(\eta)| \leq 2i+1.$$

Theorem 1 [44] For $\mu > 3$, assume that $p(\eta)$ and $z(\eta)$ are C^{μ}-functions. Let $p(\eta)$ and $z(\eta)$ can be approximated as:

$$p(\eta) \approx p_n(\eta) = \sum_{i=0}^{n} u_i p_i(\eta),$$

$$z(\eta) \approx z_n(\eta) = \sum_{i=0}^{n} u_i p_i(\eta),$$

then, the following estimate can be obtained

$$|u_i| \leq i^{-\mu}.$$

If $p(\eta)$ and $z(\eta)$ meet the assumptions of Theorem 1, the following two theorems apply.

Theorem 2 [44] This estimate of the truncation error is accurate:

$$|p - p_n| \leq n^{2-\mu}, \tag{15}$$

$$|z - z_n| \leq n^{2-\mu}. \tag{16}$$

Where $\mu > 3$.

We now derive an upper bound for the global error associated with the two residuals of the system of integral equations. These residuals are defined as the difference between the left-hand side (LHS) and right-hand side (RHS) of the equation, obtained by substituting the approximate solution into the integral equation. Using the collocation method, an error is introduced, and we establish a dominant term for this error in the following theorem.

Theorem 3 [44] If we define

$$R_1 = p_n'(\eta) - p_n(\eta)\left[c_1 - \nu_1 z_n(\eta) - \int_{\eta-T_0}^{\eta} w_1(\eta-\tau) z_n(\tau) d\tau\right] - q_1(\eta), \tag{17}$$

$$R_2 = z_n'(\eta) - z_n(\eta)\left[-c_2 + \nu_2 p_n(\eta) + \int_{\eta-T_0}^{\eta} w_2(\eta-\tau) p_n(\tau) d\tau\right] - q_2(\eta) \tag{18}$$

We will substitute q_1 from (1) into (17) and substitute q_2 from (2) into (18), then we will get

$$R_1 = p_n'(\eta) - p_n(\eta)\left[c_1 - \nu_1 z_n(\eta) - \int_{\eta-T_0}^{\eta} w_1(\eta-\tau) z_n(\tau) d\tau\right]$$

$$- \left[p'(\eta) - p(\eta)\left[c_1 - \nu_1 z(\eta) - \int_{\eta-T_0}^{\eta} w_1(\eta-\tau) z(\tau) d\tau\right]\right],$$

$$R_2 = z_n'(\eta) - z_n(\eta)\left[-c_2 + \nu_2 p_n(\eta) + \int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p_n(\tau)d\tau\right]$$

$$-\left[z'(\eta) - z(\eta)\left[-c_2 + \nu_2 p(\eta) + \int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau\right]\right].$$

Now, we will expand the brackets and collect the same terms together

$$R_1 = (p_n'(\eta) - p'(\eta)) - c_1(p_n(\eta) - p(\eta)) + \nu_1(p_n(\eta)z_n(\eta) - p(\eta)z(\eta))$$

$$+ p_n(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z_n(\tau)d\tau - p(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau,$$

$$R_2 = (z_n'(\eta) - z'(\eta)) + c_2(z_n(\eta) - z(\eta)) - \nu_2(z_n(\eta)p_n(\eta) - z(\eta)p(\eta))$$

$$-z_n(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p_n(\tau)d\tau - z(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau.$$

Now, we will use the triangle inequality

$$|R_1| \leq |p_n'(\eta) - p'(\eta)| - |c_1||p_n(\eta) - p(\eta)| + |\nu_1||(p_n(\eta)z_n(\eta) - p(\eta)z(\eta)|$$

$$+\left|p_n(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z_n(\tau)d\tau - p(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau\right|, \quad (19)$$

$$|R_2| \leq |z_n'(\eta) - z'(\eta)| + |c_2||z_n(\eta) - z(\eta)| - |\nu_2||z_n(\eta)p_n(\eta) - z(\eta)p(\eta)|$$

$$-\left|z_n(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p_n(\tau)d\tau - z(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau\right|. \quad (20)$$

We will make some simplifications

$$|z_n(\eta)p_n(\eta) - z(\eta)p(\eta)| = |z_n(\eta)p_n(\eta) - p(\eta)z_n(\eta) + p(\eta)z_n(\eta) - z(\eta)p(\eta)|$$

$$\leq |z_n||p_n(\eta) - p(\eta)| + |p(\eta)||z_n(\eta) - z(\eta)| \quad (21)$$

$$\left|p_n(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z_n(\tau)d\tau - p(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau\right| = \left|p_n(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z_n(\tau)d\tau\right.$$

$$\left. - p_n(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau + p_n(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau - p(\eta)\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau\right|$$

$$\leq |p_n(\eta)|\int_{\eta-T_0}^{\eta} |w_1(\eta-\tau)||z_n(\tau)-z(\tau)|d\tau + |p_n(\eta)-p(\eta)|\left|\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau\right| \tag{22}$$

$$\left|z_n(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p_n(\tau)d\tau - z(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau\right| = \left|z_n(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p_n(\tau)d\tau\right.$$

$$\left. - z_n(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau + z_n(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau - z(\eta)\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau\right|$$

$$\leq |z_n(\eta)|\int_{\eta-T_0}^{\eta} |w_2(\eta-\tau)||p_n(\tau)-p(\tau)|d\tau + |z_n(\eta)-z(\eta)|\left|\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau\right|. \tag{23}$$

Now, we will substitute (21) into (19) and (20) then substitute (22) into (19) and substitute (23) into (20) then, we will get

$$|R_1| \leq |p_n'(\eta)-p'(\eta)| - |c_1||p_n(\eta)-p(\eta)| + |\nu_1|\big||z_n||p_n(\eta)-p(\eta)| + |p(\eta)||z_n(\eta)-z(\eta)|\big|$$

$$+\left||p_n(\eta)|\int_{\eta-T_0}^{\eta} |w_1(\eta-\tau)||z_n(\tau)-z(\tau)|d\tau + |p_n(\eta)-p(\eta)|\left|\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)z(\tau)d\tau\right|\right|,$$

$$|R_2| \leq |z_n'(\eta)-z'(\eta)| + |c_2||z_n(\eta)-z(\eta)| - |\nu_2|\big||z_n||p_n(\eta)-p(\eta)| + |p(\eta)||z_n(\eta)-z(\eta)|\big|$$

$$-\left||z_n(\eta)|\int_{\eta-T_0}^{\eta} |w_2(\eta-\tau)||p_n(\tau)-p(\tau)|d\tau + |z_n(\eta)-z(\eta)|\left|\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)p(\tau)d\tau\right|\right|.$$

We will suppose that

$$|p(\eta)| \leq B_1, \tag{24}$$

$$|z(\eta)| \leq B_2, \tag{25}$$

$$\int_{\eta-T_0}^{\eta} w_1(\eta-\tau)d\tau = A_1, \tag{26}$$

$$\int_{\eta-T_0}^{\eta} w_2(\eta-\tau)d\tau = A_2, \tag{27}$$

where A_1, A_2, B_1, and B_2 are constants.

Now, we will use (15), (16), (24), (25), (26), and (27) to bound $|R_1|$ and $|R_2|$

$$|R_1| \le |p_n'(\eta)-p'(\eta)| - |c_1||p_n(\eta)-p(\eta)| + |\nu_1|\left[B_2|p_n(\eta)-p(\eta)| + B_1|z_n(\eta)-z(\eta)|\right]$$

$$+\left||p_n(\eta)|\int_{\eta-T_0}^{\eta}|w_1(\eta-\tau)||z_n(\tau)-z(\tau)|d\tau + |p_n(\eta)-p(\eta)|A_1\right|,$$

$$|R_2| \le |z_n'(\eta)-z'(\eta)| + |c_2||z_n(\eta)-z(\eta)| + |\nu_2|\left[B_2|p_n(\eta)-p(\eta)| + B_1|z_n(\eta)-z(\eta)|\right]$$

$$+\left||z_n(\eta)|\int_{\eta-T_0}^{\eta}|w_2(\eta-\tau)||p_n(\tau)-p(\tau)|d\tau + |z_n(\eta)-z(\eta)|A_2\right|.$$

making simplify

$$|R_1| \le n^{1-\mu} - |c_1|n^{2-\mu} + |\nu_1|\left((2n+1)n^{2-\mu} + B_1 n^{2-\mu} + (2n+1)A_1 n^{2-\mu} + n^{2-\mu}B_2 A_1\right),$$

$$|R_2| \le n^{1-\mu} + |c_2|n^{2-\mu} - |\nu_2|\left((2n+1)n^{2-\mu} + B_1 n^{2-\mu} + (2n+1)A_2 n^{2-\mu} + n^{2-\mu}A_2 B_1\right).$$

$$|R_1| = \mathscr{O}(n^{1-\mu}),$$

$$|R_2| = \mathscr{O}(n^{1-\mu}).$$

with $1-\mu < 0$, and $\mathscr{O}$ denotes the lagrange notations.

5. Illustrative examples

We demonstrate the application of the Chebyshev spectral collocation method on several biological models and compare the results with other existing methods to verify the accuracy and efficiency of the proposed method.

Example 1 [37] We examine the system specified in Eqs. (1)-(2) with

$$w_1(\eta) = 1, \quad w_2(\eta) = e^{-\eta}, \quad c_1 = 1, \quad c_2 = 2, \quad \nu_1 = \frac{1}{3}, \quad \nu_2 = 1,$$

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad T_0 = \frac{1}{2},$$

$$q_1(\eta) = -\frac{5}{2}\eta^3 + \frac{49}{12}\eta^2 + \frac{17}{12}\eta - \frac{23}{6},$$

$$q_2(\eta) = \frac{15}{8}\eta^3 - \frac{1}{4}\eta^2 + \frac{3}{8}\eta - 1.$$

the smooth solution to this system is $P(\eta) = -3\eta + 1$ and $z(\eta) = \eta^2 - \eta$. The computed results obtained using the Chebyshev spectral collocation method for $M = 6$ have been compared with those from the Legendre spectral collocation method (LSCM) [38] and the variational iteration method (VIM) [37] at $n = 6$. These comparisons are shown in Table 1. Newton's method was used to solve the resulting system of nonlinear algebraic equations for this system.

Table 1. Comparison of numeric results for Example 1

η	Absolute error (AE) for $p(\eta)$			AE for $z(\eta)$		
	LSCM [38]	VIM [37]	Our method	LSCM [38]	VIM [37]	Our method
0.1	$1.41e-11$	$3.15e-4$	$5.4173e-16$	$1.22e-12$	$3.34e-5$	$3.42346e-17$
0.2	$2.15e-11$	$4.27e-4$	$3.40208e-16$	$2.96e-12$	$8.54e-5$	$3.12711e-17$
0.3	$2.58e-11$	$4.72e-4$	$1.29306e-16$	$4.16e-12$	$1.33e-4$	$1.21995e-16$
0.4	$2.85e-11$	$4.85e-4$	$2.27676e-15$	$5.12e-12$	$1.79e-4$	$2.00528e-16$
0.5	$2.98e-11$	$4.74e-4$	$2.99641e-14$	$6.58e-12$	$2.22e-4$	$2.31485e-16$
0.6	$2.84e-11$	$4.45e-4$	$1.64593e-13$	$9.19e-12$	$2.37e-4$	$2.00528e-16$
0.7	$2.56e-11$	$4.36e-4$	$6.02949e-13$	$1.29e-11$	$1.62e-4$	$1.21995e-16$
0.8	$1.22e-11$	$4.35e-4$	$1.73236e-12$	$1.69e-11$	$1.07e-4$	$3.12711e-17$
0.9	$3.33e-12$	$9.10e-4$	$4.22502e-12$	$1.92e-11$	$7.31e-4$	$3.42346e-17$
1.0	$2.12e-11$	$1.82e-3$	$9.15113e-12$	$1.69e-11$	$1.90e-3$	$5.55112e-17$

From the result reports for Table 2 we can see that the maximum absolute error is strictly less than the theoretical error at $n = 6$.

Table 2. Error analysis of Example 1

Absolute error of p_i	Absolute error of z_i	Theoretical error
$9.15113e-12$	$2.31485e-16$	0.0277777778

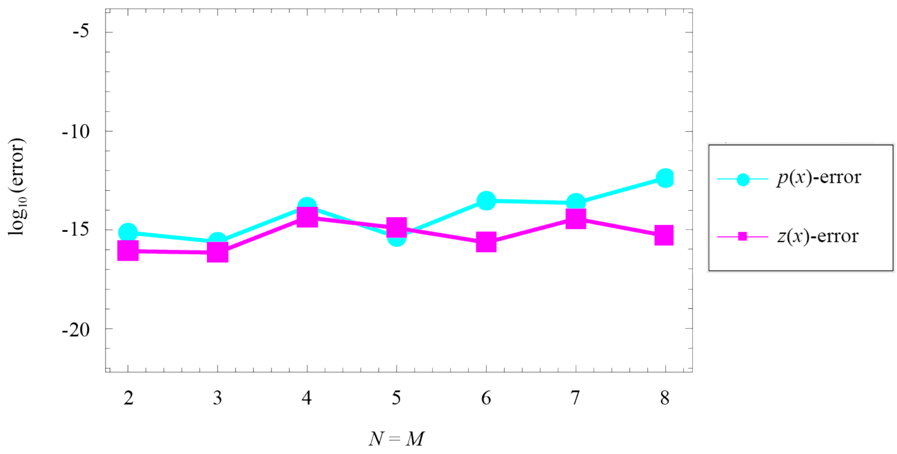

Figure 1. Log errors of Example 1

Example 2 [37] We examine the system specified in Eqs. (1)-(2) with

$$w_1(\eta) = 2\eta - 3, \quad w_2(\eta) = \eta, \quad c_1 = 2, \quad c_2 = 2, \quad \nu_1 = 1, \quad \nu_2 = 1,$$

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad T_0 = \frac{1}{3},$$

$$q_1(\eta) = \eta^2\left(2 - 3\eta e^{-\eta} - \frac{7}{2}e^{-\eta} + \frac{13}{6}\eta e^{\frac{1}{3}-\eta} + \frac{22}{9}e^{\frac{1}{3}-\eta}\right) - 2\eta,$$

$$q_2(\eta) = \frac{1}{648}e^{-\eta}\left(342\eta^3 - 8\eta^2 + 325\eta + 324\right).$$

the smooth solution to this system is $x(\eta) = -\eta^2$ and $y(\eta) = \frac{1}{2}\eta e^{-\eta}$. The numerical results from the Chebyshev spectral collocation method for $M = 6$ have been compared with the results from LSCM [38] (where $n = 6$), and VIM [37]. These comparisons are shown in Table 3. For this system, the resulting system of nonlinear algebraic equations has been tackled using Newton's method.

Table 3. Comparison of numeric results for Example 2

η	AE for $p(\eta)$			AE for $z(\eta)$		
	LSCM [38]	VIM [37]	Our method	LSCM [38]	VIM [37]	Our method
0.1	5.27e−8	4.50e−10	5.1473e−16	1.54e−6	9.80e−8	3.42346e−17
0.2	8.03e−8	4.07e−9	3.40208e−16	1.36e−6	6.93e−8	3.12711e−17
0.3	1.00e−7	4.72e−8	1.29306e−16	1.05e−6	2.69e−7	1.21995e−16
0.4	1.20e−7	3.64e−7	2.27676e−15	8.60e−7	3.55e−7	2.00528e−16
0.5	1.41e−7	2.03e−6	2.99641e−14	7.06e−7	2.49e−6	2.31485e−16
0.6	1.66e−7	8.80e−6	2.41663e−12	5.50e−7	1.08e−5	1.74917e−13
0.7	1.95e−7	3.12e−5	5.0219e−12	4.30e−7	3.85e−5	1.96878e−13
0.8	2.29e−7	9.44e−5	1.03658e−11	3.57e−7	1.14e−4	2.0789e−13
0.9	2.72e−7	2.51e−4	2.11792e−11	2.57e−7	3.00e−4	2.12756e−13
1.0	3.21e−7	6.04e−4	4.2663e−11	1.31e−7	7.11e−4	2.14075e−13

From the result reports for Table 4 we can see that the maximum absolute error is strictly less than the theoretical error at $n = 6$.

Table 4. Error analysis of Example 2

Absolute error of p_i	Absolute error of z_i	Theoretical error
4.2663e−11	2.14075e−13	0.0277777778

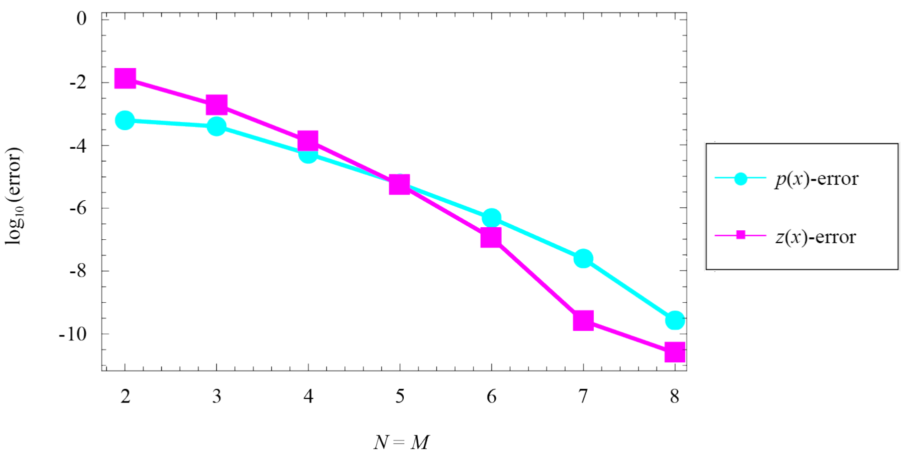

Figure 2. Log errors of Example 2

Example 3 [37] Consider the system of integro-differential equations defined in Eqs. (1)-(2) with

$$w_1(\eta) = 1, \quad w_2(\eta) = e^{-\eta}, \quad c_1 = \frac{1}{3}, \quad c_2 = \frac{1}{2}, \quad \nu_1 = 2, \quad \nu_2 = 1,$$

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad T_0 = \frac{3}{10}.$$

$$q_1(\eta) = \frac{1}{4}\cos\eta - \frac{1}{4}\sin\eta\left(\frac{1}{3} + \frac{1}{2}\sin\eta - \frac{1}{4}\cos\eta + \frac{1}{4}\cos\left(\eta - \frac{3}{10}\right)\right),$$

$$q_2(\eta) = -\frac{1}{4}\cos\eta + \frac{1}{4}\sin\eta\left(-\frac{1}{2} + \frac{3}{8}\sin\eta - \frac{1}{8}\cos\eta + \frac{1}{8}e^{-\frac{3}{10}}\left(\cos\left(\eta - \frac{3}{10}\right) - \sin\left(\eta - \frac{3}{10}\right)\right)\right).$$

the smooth solution to this system is $p(\eta) = \frac{1}{4}\sin\eta$ and $z(\eta) = -\frac{1}{4}\sin\eta$. The numerical results from the Chebyshev spectral collocation method for $M = 6$ have been compared with the results from LSCM [38] (where $n = 6$), and VIM [37]. These comparisons are shown in Table 5. For this system, the resulting system of nonlinear algebraic equations has been tackled using Newton's method.

Table 5. Comparison of numeric results for Example 3

η	AE for $p(\eta)$			AE for $z(\eta)$		
	LSCM [38]	VIM [37]	Our method	LSCM [38]	VIM [37]	Our method
0.1	2.4e−7	5.2e−10	3.3586e−8	2.1e−7	4.6e−10	5.63295e−11
0.2	2.7e−7	6.2e−9	4.80769e−8	2.2e−7	2.9e−9	2.85664e−10
0.3	2.8e−7	1.6e−7	5.0535e−8	2.02e−7	7.5e−8	8.09017e−10
0.4	2.9e−7	1.24e−6	4.63195e−8	2.0e−7	6.1e−7	1.7777e−9
0.5	3.2e−7	6.12e−6	3.91855e−8	2.0e−7	3.0e−6	3.35956e−9
0.6	3.5e−7	2.20e−5	3.15258e−8	1.9e−7	1.06e−5	5.71428e−9
0.7	3.7e−7	6.4e−5	2.46773e−8	2.0e−7	3.1e−5	8.95948e−9
0.8	4.14e−7	1.6e−4	1.92269e−8	2.0e−7	7.5e−5	1.31323e−8
0.9	4.5e−7	3.5e−4	1.52801e−8	2.0e−7	1.6e−4	1.81548e−8
1.0	4.9e−7	7.0e−4	1.26717e−8	2.0e−7	3.2e−4	2.3812e−8

From the result reports for Table 6 we can see that the maximum absolute error is strictly less than the theoretical error at $n = 6$.

Table 6. Error analysis of Example 3

Absolute error of p_i	Absolute error of z_i	Theoretical error
5.0535e−8	2.3812e−8	0.0277777778

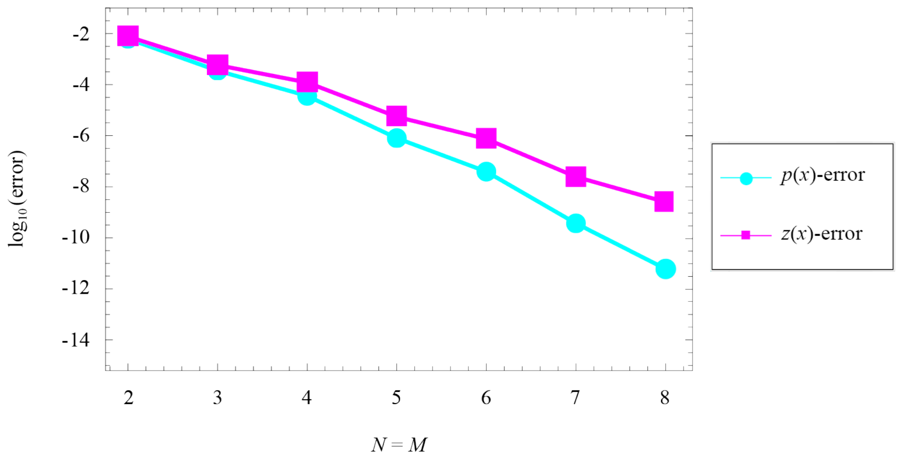

Figure 3. Log errors of Example 3

Example 4 [37] We examine the system specified in Eqs. (1)-(2) with

$$w_1(\eta) = \eta, \quad w_2(\eta) = \eta + 1, \quad c_1 = 1, \quad c_2 = 1, \quad \nu_1 = \frac{1}{2}, \quad \nu_2 = 3,$$

$$\alpha_1 = 0, \quad \alpha_2 = -1, \quad T_0 = \frac{1}{4}.$$

$$q_1(\eta) = 2\eta - 1 - (\eta^2 - \eta)\left(1 + \frac{11}{18}e^{-3\eta} - \frac{1}{36}e^{\frac{3}{4}-3\eta}\right),$$

$$q_2(\eta) = \frac{1}{3{,}072}e^{-3\eta}\left(10{,}080\eta^2 - 10{,}304\eta + 6{,}275\right).$$

the smooth solution to this system is $p(\eta) = \eta^2 - \eta$ and $z(\eta) = -e^{-3\eta}$. The numerical results from the Chebyshev spectral collocation method for $M = 6$ have been compared with the results from LSCM [38] (where $n = 6$), and VIM [37]. These comparisons are shown in Table 7. For this system, the resulting system of nonlinear algebraic equations have been tackled using Newton's method.

Table 7. Comparison of numeric results for Example 4

η	AE for $p(\eta)$			AE for $z(\eta)$		
	LSCM [38]	VIM [37]	Our method	LSCM [38]	VIM [37]	Our method
0.1	1.2e−6	3.6e−7	7.19241e−10	4.5e−4	1.1e−5	0.0129161
0.2	4.4e−6	2.7e−7	1.70245e−9	4.2e−4	1.5e−5	0.0016647
0.3	8.4e−6 1	4.7e−7	2.73166e−9	3.4e−4	8.2e−6	0.000264965
0.4	1.4e−5	1.6e−5	3.51544e−9	2.9e−4	9.3e−5	0.0000516318
0.5	2.0e−5	6.9e−5	3.80848e−9	2.4e−4	3.9e−4	0.0000121728
0.6	2.5e−5	1.7e−4	3.51544e−9	2.0e−4	9.4e−4	3.42229e−6
0.7	3.0e−5	3.2e−4	2.73166e−9	1.6e−4	1.6e−3	1.1288e−6
0.8	3.5e−5	4.9e−4	1.70245e−9	1.4e−4	2.3e−3	4.29302e−7
0.9	4.0e−5	6.4e−4	7.19241e−10	1.2e−4	2.6e−3	1.84961e−7
1.0	4.5e−5	7.4e−4	1.0568e−16	1.0e−3	2.6e−3	8.87108e−8

From the result reports for Table 8 we can see that the maximum absolute error is strictly less than the theoretical error at $n = 6$.

Table 8. Error analysis of Example 3

Absolute error of p_i	Absolute error of z_i	Theoretical error
3.80848e−9	3.42229e−6	0.0277777778

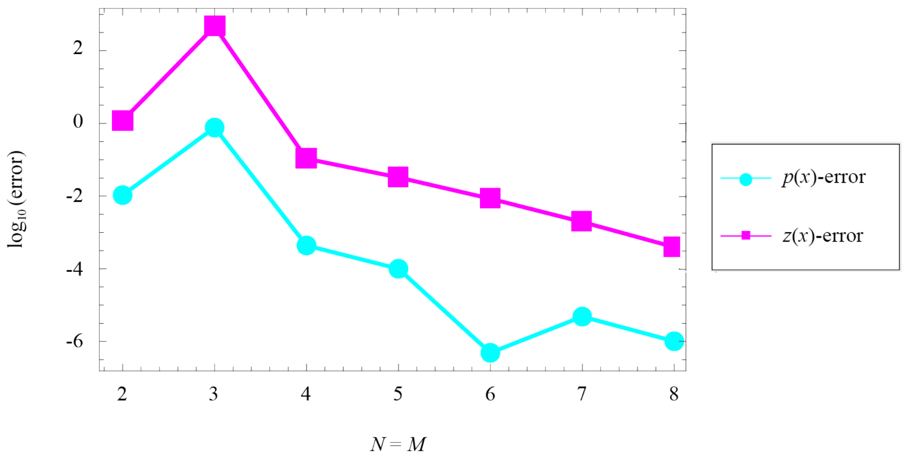

Figure 4. Log errors of Example 4

The results presented in Tables 1-8 clearly demonstrate that the Chebyshev Spectral Collocation Method outperforms other approaches, providing superior accuracy and precision, the log-errors presented in Figures 1-4 clearly demonstrate the exponential convergence of the proposed spectral method.

6. Conclusion

This work showed how well the Chebyshev spectral collocation approach works for solving integro-differential equation systems, emphasizing how accurate and computationally efficient it is in comparison to other methods. The approach produced accurate answers with little computer work by converting these equations into nonlinear algebraic systems. Though there are still issues with implementation complexity, sensitivity to solution smoothness, and scalability, its principal benefits include high accuracy, quick computing, and adaptation to different types of equations. In order to ensure its continuous relevance and usefulness, future research could overcome these constraints by extending its applicability to larger, real-world issues in domains like fluid dynamics and quantum mechanics through hybrid approaches, better management of discontinuities, and parallel computing.

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Conflict of interest

The authors declare no competing interest.

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