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Research Article

On the Edge Irregularity Strength of Finite Graphs

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Abstract: In this paper, we state the value of edge irregularity strength for the complete graphs K_n of order $n \geq 3$, wheel graphs W_n where $n \geq 3$ and the union of disjoint graphs. Also we state a lower bound for edge irregularity strength for the complete sun graph of order $2n$ and size $\frac{n}{2}(n+3)$.

***Keywords*:** simple graph, k-labelling, irregularity strength, wheel graph, disjoint graphs

MSC: 05C78, 05C38

1. Introduction

Our notions are fairly standard, as can be found in many sources, for instance [1–3]. To have a self-contained paper we list the next main concepts and terminologies that used in the current paper.

A graph $G(V, E)$ consists of a non-empty finite set V of elements called *vertices*, and a finite family E of unordered pairs of (not necessarily distinct) elements of V called *edges*. The edge $e = (u, v) \in E$ joining the vertices u and v in V can be written as $e = uv = vu$. Replacing the set E with a set of ordered pairs of vertices, we obtain a directed graph, or digraph. A graph is usually undirected, unless otherwise stated. The *order* of a graph $G(V, E)$ is $|V|$ and is denoted by $O(G)$, and the *size* is $|E|$ and is denoted by $S(G)$. A *simple* graph is a graph that has no edge of the same ends vertices and it has at most one edge joining any two different vertices. Two graphs are called *disjoint* if there is no common vertex between them.

A path is a fundamental concept on graph theory, which is a graph whose vertices can be ordered as $v_1, v_2, \cdots, v_n$ and the edges are v_iv_{i+1}. A path of n vertices is denoted by P_n. A graph G is *connected* if and only if there is at least one path between any two different vertices.

In this article we consider undirected graphs that are connected and simple. For a graph $G(V, E)$, the *degree* of a vertex $v \in V$ is defined to be the number of edges that have their ends in v. The *maximal* degree of a graph G is defined as $\Delta(G) = \max\{\deg(v) \mid v \in V\}$, and the *minimal degree* of a graph G is defined as $\delta(G) = \min\{\deg(v) \mid v \in V\}$.

Recall that, the sequence $0, 1, 1, 2, 3, 5, \ldots$ in which any term is the sum of the previous two terms is called a *sequence of Fibonacci numbers* where each number is referred to as F_n. To find the Fibonacci number F_n one can use the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $n > 2$ and $F_1 = 1, F_2 = 2$. The ratio of two consecutive Fibonacci numbers is

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an irrational number equals to $\frac{1+\sqrt{5}}{2}$. This ratio is called the *golden ratio*. It also can be used to represent Fibonacci numbers such that $F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$, $\varphi = \frac{1+\sqrt{5}}{2}$. For more about Fibonacci numbers and the golden ratio see [4].

By graph *vertex labelling*, it is meant to assign each vertex to an element from a selective set; for example, a set of letters, numbers or colours. Many researches consider the set to be non-negative integers, as we do in our current research.

Recently, the edge irregular strength of a graph is introduced by Ahmad et al. [5]. As, if G is an undirected, simple and connected graph with the vertex set V, and the edge set E. Consider the vertex k-labelling map $\phi : V \to \{1, 2, \cdots, k\}$. Corresponding to this map ϕ, each edge $e = vu \in E$ has the weight $w_\phi(e) = w_\phi(uv) = \phi(u) + \phi(v)$, if such a function maps the distinct edges to distinct weights, then it is called an *irregular k-labelling* of G. The *edge irregularity strength* of a graph G is the smallest k embedding the irregularity of ϕ, denoted by $es(G)$. Recently, the edge irregularity strength of some graphs has been considered, such as paths P_n, Cartesian product of two paths and the star graph $K_{1,\,n}$, see [5]. Edge irregularity strength for the sun graph S_n, or equivalently $C_n \odot K_1$, is investigated by Ahmad in [6]. Some classes of Toeplitz graphs have a calculated edge irregularity strength that found by Ahmad et al. [7]. Further results on edge irregularity strength of graphs have been considered in [7–10]. Recently, in [11, 12] the author stated $es(G)$ of some finite graphs G as $K_{n,\,m}$, $P_n \odot P_m$ and $P_n \odot C_m$. Among other investigations, Mushayt in [13] stated the edge irregularity strength of the Cartesian product of some graphs such as stars, cycles and paths.

The main aim of this article is to study $es(G)$ of a given graph G by using a suitable map $\phi : V \to \{1, 2, \cdots, k\}$ that associate different edges weights where the value of k is chosen to be the smallest. The next theorems give recommended bounds (lower and upper bounds) of $es(G)$.

Theorem 1 [5] Let G be a graph of order n. Let the sequence F_m of Fibonacci numbers be defined by the recurrence relation $F_m = F_{m-1} + F_{m-2}$, $m \geq 3$, with seed values $F_1 = 1$ and $F_2 = 2$. Then $es(G) \leq F_n$.

Theorem 2 [5] For a simple graph G of size m and maximum degree Δ,

$$es(G) \geq \max\left\{\left\lceil \frac{m+1}{2} \right\rceil, \Delta\right\}$$

As we move forward, in Section 2, we will present some new results regarding the edge irregularity strength of disjoint graph unions. In Section 3, we will show a complete estimation of $es(G)$ for some known graphs, such as complete graphs K_n, wheel graphs W_n, and complete sun graphs KS_n.

2. Edge irregularity strength of disjoint union of n copies of graphs

Recall that, the union of the graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is the graph $G(V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Clearly, the order of $G = G_1 \cup G_2$ is $n \leq |V_1| + |V_2|$ and the size of G is $m \leq |E_1| + |E_2|$. In particular, if G_1 and G_2 are disjoint graphs, then $n = |V_1| + |V_2|$ and $m = |E_1| + |E_2|$.

Our next step is to determine the edge irregularity strength of the union of disjoint graphs. Noting that, $es(P_n) = \left\lceil \frac{n}{2} \right\rceil$, see [5].

Remark 1 Let P_n and P_m be two disjoint paths for $n, m \geq 2$. Then

$$es(P_n \cup P_m) = \left\lceil \frac{n+m-1}{2} \right\rceil.$$

Proof. Suppose that P_n and P_m be two disjoint paths for $n, m \geq 2$, where $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{u_1, u_2, \ldots, u_m\}$ are the sets of vertices of P_n and P_m, respectively. Let $G = P_n \cup P_m$. Then, $S(G) = S(P_n) + S(P_m) = (n-1) + (m-1) = n + m - 2$ and $\Delta(G) = 2$. Using Theorem 2 we get:

$$es(G) \geq \max\left\{\left\lceil\frac{S(G)+1}{2}\right\rceil, \Delta(G)\right\}$$

$$\geq \max\left\{\left\lceil\frac{n+m-2+1}{2}\right\rceil, 2\right\}$$

$$= \left\lceil\frac{n+m-1}{2}\right\rceil$$

Then,

$$es(G) \geq \left\lceil\frac{n+m-1}{2}\right\rceil \tag{1}$$

Define the vertex labelling map ϕ on G by $\phi:\ V_1 \cup V_2 \to \{1,\ 2,\ \ldots,\ k\}$ (as $k = \dfrac{n+m-1}{2}$) where $\phi(V_1)$ and $\phi(V_2)$ are given by:

$$\underbrace{1,\ 1,\ 2,\ 2,\ \cdots,\ \frac{n}{2},\ \frac{n}{2},}_{\phi(V_1)}$$

$$\underbrace{\frac{n}{2},\ \frac{n}{2}+1,\ \frac{n}{2}+1,\ \cdots,\ \frac{n}{2}+\left\lceil\frac{m-1}{2}\right\rceil,}_{\phi(V_2)} \text{ if } n \text{ is even}$$

and

$$\underbrace{1,\ 1,\ 2,\ 2,\ \cdots,\ \left\lceil\frac{n}{2}\right\rceil-1,\ \left\lceil\frac{n}{2}\right\rceil-1,\ \left\lceil\frac{n}{2}\right\rceil,}_{\phi(V_1)}$$

$$\underbrace{\left\lceil\frac{n}{2}\right\rceil,\ \left\lceil\frac{n}{2}\right\rceil,\ \left\lceil\frac{n}{2}\right\rceil+1,\ \left\lceil\frac{n}{2}\right\rceil+1,\ \cdots,\ \left\lceil\frac{n}{2}\right\rceil+\left\lceil\frac{m-1}{2}\right\rceil,}_{\phi(V_2)} \text{ if } n \text{ is odd}$$

Corresponding to the previous labelling, there are $n+m-2$ distinct edge weights which can be listed by $W = \{2,\ 3,\ 4,\ \cdots,\ n+m-1\}$, hence ϕ is an edge irregular labelling of G. Using Equation (1), $k \leq es(G) \leq k$ and the proof is completed. □

Generalizing the previous lemma for the union of n disjoint paths, the following theorem results.

Theorem 3 Let $G = \bigcup_{i=1}^{n} P_{k_i}$ be the union of disjoint n paths P_{k_i}, $k_i \geq 2$ for all i. Then

$$es(G) = \left\lceil \frac{\left(\sum_{i=1}^{n} k_i\right) - (n-1)}{2} \right\rceil. \tag{2}$$

Proof. Let P_{k_i}, $i = 1, \ldots, n$ be n disjoint paths, and $V_{k_1}, \ldots, V_{k_n}$ be the vertices sets of $P_{k_1}, \ldots, P_{k_n}$ respectively. Let $G = \bigcup_{i=1}^{n} P_{k_i}$. Then $\Delta(G) = 2$, $|V| = O(G) = \sum_{i=1}^{n} k_i$ and $|E| = S(G) = S(P_{k_1}) + \ldots, S(P_{k_n}) = (k_1 - 1) + \cdots + (k_n - 1) = \sum_{i=1}^{n} k_i - n$. By Theorem 2, we have

$$es(G) \geq \max \left\{ \left\lceil \frac{\left((\sum_{i=1}^{n} k_i) - n \right) + 1}{2} \right\rceil, 2 \right\} = \left\lceil \frac{\left(\sum_{i=1}^{n} k_i\right) - (n-1)}{2} \right\rceil.$$

Now, define the vertex labelling map ϕ on G by $\phi : \bigcup_{i=1}^{n} V_{k_i} \to \{1, 2, \ldots, k\}$ where $k = \left\lceil \frac{\left(\sum_{i=1}^{n} k_i\right) - (n-1)}{2} \right\rceil$ and the labelling is as follows

• if k_1 is even then we label the vertices by

$$\underbrace{1, 1, 2, 2, \cdots, \frac{k_1}{2}, \frac{k_1}{2}}_{\phi(V_1)}, \underbrace{\frac{k_1}{2}, \frac{k_1}{2}+1, \frac{k_1}{2}+1, \cdots, \left\lceil \frac{\sum_{i=1}^{n} k_i - (n-1)}{2} \right\rceil}_{\phi(V_{k_2} \cup \cdots \cup V_{k_n})},$$

• if k_1 is odd then we label the vertices by

$$\underbrace{1, 1, 2, 2, \cdots, \left\lceil \frac{k_1}{2} \right\rceil - 1, \left\lceil \frac{k_1}{2} \right\rceil - 1, \left\lceil \frac{k_1}{2} \right\rceil}_{\phi(V_{k_1})},$$

$$\underbrace{\left\lceil \frac{k_1}{2} \right\rceil, \left\lceil \frac{k_1}{2} \right\rceil, \left\lceil \frac{k_1}{2} \right\rceil + 1, \left\lceil \frac{k_1}{2} \right\rceil + 1, \cdots, \left\lceil \frac{\sum_{i=1}^{n} k_i - n}{2} \right\rceil}_{\phi(V_{k_2} \cup \cdots \cup V_{k_n})}.$$

Certainly, this map is irregular labelling map, where $W = \left\{ 2, 3, 4, \ldots, \sum_{i=1}^{n} k_i - (n-1) \right\}$ is the set of distinct edge weights. Thus $es(G) \leq k$, which completes the proof. □

Remark 2 Consider the disjoint graphs G and H. Then

$$\max\{es(G),\ es(H)\} \leq es(G \cup H) \leq es(G) + es(H).$$

Proof. Suppose that G and H be two disjoint graphs for which $\phi_G : V_G \to \{1,\ 2,\ \cdots,\ k_1\}$ and $\phi_H :\ V_H \to \{1,\ 2,\ \cdots,\ k_2\}$ are edge irregular labelings of G and H (respectively) with to $k_1 = es(G)$ and $k_2 = es(H)$. Without loss of generality assume that $k_1 \leq k_2$, then define $\phi : V_G \cup V_H \to \{1,\ 2,\ \cdots,\ k = k_1 + k_2\}$ by

$$\phi(v) = \begin{cases} \phi_G(v),\ v \in G \\ \\ \phi_H(v) + k_1,\ v \in H \end{cases}$$

To check the irregularity of ϕ, one needs to find the edge weight corresponding to every edge in $G \cup H$. Since G and H are disjoint, then every edge in the union should be only in G or only in H. Let $e_1,\ e_2$ be two distinct edges in $G \cup H$. To show that $w_\phi(e_1) \neq w_\phi(e_2)$, we have the following cases:

• If $e_1,\ e_2 \in G$, then $w_\phi(e_1) = w_{\phi_G}(e_1)$ and $w_\phi(e_2) = w_{\phi_G}(e_2)$, which are distinct, because ϕ_G is an irregular vertex labelling on G.

• If $e_1,\ e_2 \in H$, then $w_\phi(e_1) = w_{\phi_H}(e_1) + 2k_1$ and $w_\phi(e_2) = w_{\phi_H}(e_2) + 2k_1$ and since ϕ_H is irregular vertex labelling map, then $w_{\phi_H}(e_1) \neq w_{\phi_H}(e_2)$. So $w_{\phi_H}(e_1) + 2k_1 \neq w_{\phi_H}(e_2) + 2k_1$. This implies that $w_\phi(e_1) \neq w_\phi(e_2)$.

• If $e_1 \in G$ and $e_2 \in H$, then $2 \leq w_\phi(e_1) \leq 2k_1 < 2 + 2k_1 \leq w_\phi(e_2) \leq k_1 + 2k_2$ and this shows that the weights of two distinct edges each is in one of the component graphs are distinct.

From the previous list, we see that ϕ produces different weights for the different edges. Thus, the map ϕ is an edge irregular k-labelling of $G \cup H$. This shows that $es(G \cup H) \leq k = k_1 + k_2$.

On the other hand, $S(G \cup H) = S(G) + S(H) \geq \max\{S(G),\ S(H)\}$, implies that $es(G \cup H) \geq \max\{es(G),\ es(H)\}$. This completes the proof. □

3. Edge irregularity strength of certain graphs

The considered graphs in this part are: the complete graph, wheel graph and complete sun graph.

It is worth recalling that, a *regular* graph is a graph in which all the vertices have the same degree. A *complete* graph is a simple graph that contains every possible edge between all the vertices. A complete graph with n vertices is denoted by K_n.

Certainly, $G = K_n$ is an $(n-1)$-regular graph. Therefore, the degrees sequence of G is $\underbrace{n-1,\ n-1,\ \cdots,\ n-1}_{n-terms}$.

Indeed, $\delta(G) = \Delta(G) = n-1$ and $S(G) = \frac{1}{2}n(n-1)$. The next theorem finds the edge irregularity strength of the complete graphs.

Theorem 4 Consider the complete graph $G(V,\ E) = K_n$ of order $n \geq 3$. Then $es(G) = \left\lfloor \frac{\varphi^{n+1}}{\sqrt{5}} + \frac{1}{2} \right\rfloor$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Proof. Let $G = K_n$ be the complete graph of order $n \geq 3$, where the set of vertices is V and the set of edges of G is E. Define on the graph G the vertex labelling map $\phi : V \to \{1,\ 2,\ \cdots,\ k\}$ as follows, $\phi(v_1) = 1$, $\phi(v_2) = 2$ and $\phi(v_m) = \phi(v_{m-1}) + \phi(v_{m-2})$ for all $m = 3,\ 4,\ \cdots,\ n$. Then, the vertices labelings $\phi(v_1),\ \phi(v_2),\ \phi(v_3),\ \cdots,\ \phi(v_n)$ is the Fibonacci sequence of the terms $1,\ 2,\ 3,\ 5,\ 8,\ \cdots,\ k = \left\lfloor \frac{\varphi^{n+1}}{\sqrt{5}} + \frac{1}{2} \right\rfloor$, and the corresponding edge weights are

3, 4, 5, $\cdots$, $\phi(v_{n-1})+\phi(v_n)$ which are all distinct. Thus, ϕ is an edge irregular k-labelling of G, and if $es(G) \leq k$ then the used labelling map is not one-to-one, for which it will be not irregular (every vertices are adjacent). Therefore $es(G) \geq k$. Using Theorem ,lwhich indicates that $es(G) \leq F_n = k$. Hence, the claim follows.

Example 1 Let $G = K_5$ be the complete graph of order 5, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ is the set of vertices of G. Noting that, $\left\lfloor \frac{\varphi^{n+1}}{\sqrt{5}} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\varphi^{5+1}}{\sqrt{5}} + \frac{1}{2} \right\rfloor = 8$. To find the edge irregularity strength of G, define $\phi : V \to \{1, 2, 3, 5, 8 = k\}$ (the first 5-terms of the Fibonacci sequence), such that $\phi(v_1) = F_1 = 1$, $\phi(v_2) = F_2 = 2$ and $\phi(v_m) = \phi(v_{m-1}) + \phi(v_{m-2}) = F_{m-1} + F_{m-2}$, $m = 3, 4, 5$. Then, the Figure 1 shows the edges weights assigned by such map ϕ:

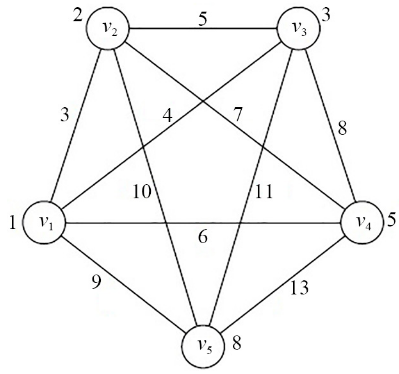

Figure 1. Edges weights assigned by map

Clearly, there are 10 distinct edges weights $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 13\}$. That is to say ϕ is an irregular k-labelling map, implise that $es(G) \leq k = 8$. Moreover ϕ is a bijective map. If not then there are at least two vertices v_i and v_j in V for which $\phi(v_i) = \phi(v_j)$ and so $\phi(v) + \phi(v_i) = \phi(v) + \phi(v_j)$ for any $v \in V$, thus the map will not produce different edges weights. Finally, suppose that $es(G) < 8$, for such an assumption, the only available options are 4, 6 or 7, but $4+1 = 2+3$, $6+2 = 5+3$ and $7+1 = 5+3$ and so neither of these options can produce different edges weights. This implies $es(G) \geq 8$. Hence, $es(G) = 8 = \left\lfloor \frac{\varphi^{5+1}}{\sqrt{5}} + \frac{1}{2} \right\rfloor$.

Recall that, the wheel graph $G = W_n$ is obtained from a cycle graph C_n and a new vertex v called hub connected to all $u \in C_n$. Therefore, the order of the wheel graph $G = W_n$ is $n+1$ and the size is $n+n = 2n$. Furthermore, $\Delta(G) = \deg(v) = n$ and $\delta(G) = 3$. Since each $u \in C_n$ is of degree 3, then $\underbrace{3, 3, \ldots, 3}_{n}$, n is the degrees sequence of G. So, the following theorem can be shown.

Theorem 5 Let W_n, $n \geq 3$ be the wheel graph. Then

$$es(W_n) = n + 2 + \left\lfloor \frac{n-3}{5} \right\rfloor$$

Proof. Let $G(V, E) = W_n$, $n \geq 3$ be the wheel graph, and $V = \{v_1, v_2, \cdots, v_n\} \cup \{v\}$, where v is the hub of G. Then $|E| = 2n$ and $\Delta(G) = n$. Using Theorem 2, it follows that

$$es(G) \geq n+1 \tag{3}$$

For $n = 3$, we have $es(G) \geq 3+1 = 4$, and define a vertex labelling map $\phi : \{v_1, v_2, v_3, v\} \to \{1, 3, 2, 5 = k\}$ by $\phi(v_i) = i$, $\phi(v) = 5$. Then ϕ is irregular vertex labelling, and so $es(G) \leq k = 5$. On the other hand, any vertex labelling

$\alpha : V \to \{1,\ 2,\ 3,\ 4\}$, will be not irregular, which implies that $es(G) \geq 5$. Thus $5 \leq es(W_3) \leq 5$, then $es(W_3) = 5$. Similarly, for $n = 4$ and $n = 5$, one has $es(G) = 6$ and $es(G) = 7$ respectively.

For $n \geq 6$ define the map $\phi : V \to \{1,\ 2,\ 3,\ \cdots,\ k\}$, which should be injective to avoid similar edge weights and surjective to get the smallest such k. Without lose of generality set $\phi(v) = 6$ (v is the hub of the wheel) $\phi(v_1) = 1,\ \phi(v_2) = 3,\ \phi(v_3) = 2,\ \phi(v_{n-1}) = 7$ and $\phi(v_n) = 5$. Then, the smallest weight that can be produced by ϕ is 4. Moreover, there is no $v_i \in V$ for which $\phi(v_i) = 4$ using $\phi(v) = 6$, therefore $\phi(v_i) \in \{1,\ 2,\ 3,\ 5,\ 7,\ 8,\ 9,\ \cdots\}$, implies that $k \geq n+2$. In particular, the remaining labels $\{\phi(v_4),\ \phi(v_5),\ \cdots,\ \phi(v_{n-2})\}$ of $n-5$ vertices should be selected from $H = \{8,\ 9,\ 10,\ \cdots,\ k\}$ which has $k-8+1 = k-7$ elements but not all could be used, so $\left\lfloor \frac{n-3}{5} \right\rfloor$ elements of H will be excluded using ϕ. That is H includes only $(k-7) - \left\lfloor \frac{n-3}{5} \right\rfloor$ acceptable labels for $n-5$ vertices, which implies that $(k-7) - \left\lfloor \frac{n-3}{5} \right\rfloor = n-5$, thus $k = n+2+\left\lfloor \frac{n-3}{5} \right\rfloor$. Therefore $es(G) \leq k$. Suppose on the contrary that $k < n+2+\left\lfloor \frac{n-3}{5} \right\rfloor$ and let $n = 6$. Then $k = 7 < 8$, which implies that there is an irregular vertex labelling $\alpha : V \to \{1,\ 2,\ 3,\ \cdots,\ 7 < k\}$ that produces 12 different weights, using $\phi(v) = 2$ or 6, removing such label (2 or 6) from $\{1,\ 2,\ 3,\ \cdots,\ 7 < k\}$. Then we have 2,520 arrangements (labelling maps) none of these maps produce different edge weights (such calculations have been done using computer software). Hence $es(W_6) > 7$. □

Most of our calculations (as for wheel graphs W_n) have been done by building certain algorithms using GAP (Groups, Algorithm, and Programming). This is a programming language and system designed specifically for computational discrete algebra. For more about it consult, for instance, see [14].

Example 2 Let $G = W_6$ be the wheel graph of order 7, for which the set of vertices is $V = \{v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6\} \cup \{v\}$ where v is the hub. So we have 7 vertices each should be labeled by an injective labelling map ϕ. Otherwise, if there exists v_i and v_j in V for which $\phi(v_i) = \phi(v_j)$, then $\phi(v) + \phi(v_i) = \phi(v) + \phi(v_j)$, or if there exists v_i in V such that $\phi(v_i) = \phi(v)$, then $\phi(v) + \phi(v_m) = \phi(v_i) + \phi(v_m)$ where v_m is any adjacent vertex of v_i. Therefore, we need to define $\phi : V \to L = \{1,\ 2,\ \ldots,\ k\}$, for which $|L| = |V|$ as 6 distinct labels for the cycle vertices and one for the hub. That is $k \geq n+1$, since 4 can not be used as a vertex label for which there exists similar edges weights. then $k \geq n+1+1 = 6+1+1 = 8$. Considering that, in the set $L = \{1,\ 2,\ 3,\ \ldots,\ n \geq 4\}$ there are $\left\lfloor \frac{n-3}{5} \right\rfloor + 1$ numbers that can not be used as vertex label. So, for $n = 6$ set $L = \{1,\ 2,\ 3,\ 5,\ 6,\ 7,\ k\}$, where k is $6+1$ (hub label) $+1$ (the label 4 should be removed) $+\ 0$ (no other removed label for $k < 13$) $= 8$ and define ϕ on V as shown in the Figure 2:

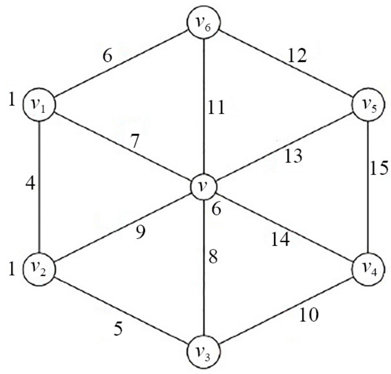

Figure 2. Define ϕ on V

As shown in the previous figure, the set of the edges weights produced by ϕ is $W = \{4,\ 5,\ 6, \ldots,\ 15\}$ which consists of 12 distinct sequential values. Implies that ϕ is an irregular k-labelling map on G. Thus $es(G) \leq k = 8$, recalling that $es(G) \geq k$. Hence, $es(G) = k$.

The complete n-sun graph ($n \geq 3$) is a graph consists of a complete graph K_n as a center of the n-sun graph and an outer ring of n vertices, where every vertex is joined to the closest edge of the complete graph. In this context, we will denote the complete n-sun graph by KS_n and write the set of vertices of KS_n as $V = \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ and the set of edges $E = \{u_i u_j \mid i, j = 1, 2, \cdots, n \text{ and } i \neq j\} \cup \{v_1 u_1, v_1 u_2, v_2 u_2, v_2 u_3, \cdots, v_n u_n, v_n u_1\}$.

So, the degree sequence of KS_n is $\underbrace{2, 2, \cdots, 2}_{n-terms}, \underbrace{n+1, n+1, \cdots, n+1}_{n-terms}$. Observe that, the order of the graph KS_n is $2n$ and the size is $\frac{1}{2}(2n + n(n+1)) = \frac{1}{2}n(n+3) = \frac{1}{2}n(n+1) + n$. Using this description one can proof the next results.

Lemma 1 Let $G = KS_n$ be the complete sun graph of order $2n$ and size $\frac{1}{2}n(n+3)$ and let F_m be the Fibonacci sequence of m-terms with seed values $F_1 = 1$ and $F_2 = 2$. Then

$$F_{n+1} \leq es(G)$$

Proof. Given that $H = K_n$ is the central core of $G = KS_n$, then $es(H) \leq es(G)$. Furthermore, $|V_G| = |V_H| + n$. Set $v_1, v_2, \cdots, v_n$ to be the vertices of G not in H. So, any irregular labelling map on G should not assign any of $1, 2, 3, \cdots, F_n$ for any v_i. Therefore, $es(G) \geq F_{n+1}$.

Note that for $G = KS_n$, the maximal degree is $\Delta(G) = n+1$. Using Theorem 2, one has:

$$es(G) \geq \max\left\{ \left\lceil \frac{|E|+1}{2} \right\rceil, \Delta(G) \right\}$$

$$= \max\left\{ \left\lceil \frac{\frac{1}{2}n(n+3)+1}{2} \right\rceil, n+1 \right\}$$

$$= \max\left\{ \left\lceil \frac{n(n+3)+2}{4} \right\rceil, n+1 \right\}$$

$$= \max\left\{ \left\lceil \frac{1}{4}(n+1)(n+2) \right\rceil, n+1 \right\}$$

and since $n \geq 3$, then $n+2 > 4$, and so $\frac{1}{4}(n+2) > 1$ implies that $\left\lceil \frac{1}{4}(n+1)(n+2) \right\rceil > n+1$. Thus

$$es(G) \geq \left\lceil \frac{1}{4}(n+1)(n+2) \right\rceil \tag{4}$$

Using the previous facts and some computer calculations, we have the following assumption.

Conjecture 1 Let $G = KS_n$ be the complete n-sun graph. Then $es(G) = \frac{1}{2}n(n+1) + 3$.

Example 3 Figures 3-5 show selective irregular labelling maps for the complete sun graphs KS_3, KS_4 and KS_5. Such maps produced distinct edge weights. That is to say $es(KS_3) \leq 9$, $es(KS_4) \leq 13$ and $es(KS_5) \leq 18$. All these values agree with our conjecture.

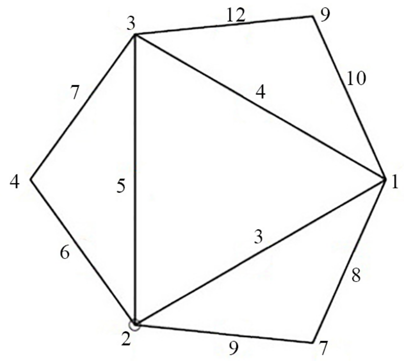

Figure 3. Irregular vertex labeling of KS_3

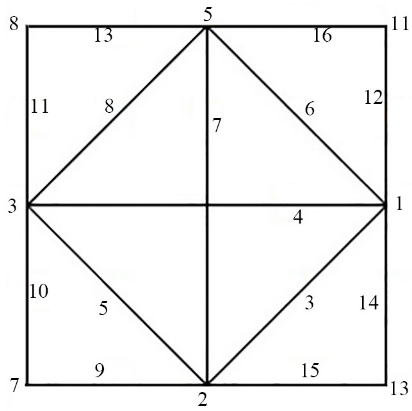

Figure 4. Irregular vertex labeling of KS_4

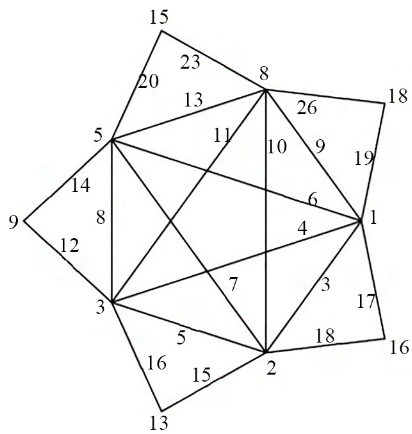

Figure 5. Irregular vertex labeling of KS_5

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Conflict of interest

The authors declare no competing financial interest.

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