Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

On the Queueing Time Analysis for State-Dependent Fixed-Cycle Traffic Light Queues

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Received: 22 October 2024; **Revised:** 4 January 2025; **Accepted:** 10 February 2025

Abstract: We analyze a Fixed-Cycle Traffic Light (FCTL) intersection model. Vehicles arrive according to a Poisson process and must wait for a green signal. Each signal period (red or green) consists of a number of phases. Exactly one waiting vehicle is released (passes through the intersection) per green signal period phase, while vehicles remain waiting during red signal periods phases. The lengths of red and green signal periods are not constants, rather they depend on the number of vehicles in the queue. That is, we propose a state-dependent scheduling mechanism for green and red signal periods in an FCTL intersection. The number of green phases increases if the number of vehicles waiting in the intersection is greater than or equal to a threshold $N(>0)$. The number of green phases increases from $g(>0)$ to $g_1(\geq g)$ and the number of red phases decreases from $r(>0)$ to $r_1(\leq r)$ in such a way that the total length of a cycle period, $c = g + r = g_1 + r_1$, is fixed. This mechanism allows one to control the waiting time of vehicles through the FCTL intersection. We analyze the distributions of queue length and vehicle waiting time during each phase of the green signal period. We provide several numerical examples to gain insight into the performance of our proposed FCTL scheduling mechanism. The proposed state-dependent FCTL queueing model dynamically adjusts green and red light durations based on the volume of traffic in queues. This FCTL model with state-dependent scheduling is ideal for smart city traffic optimization, improves traffic flow, reduces delays, and minimizes fuel consumption in busy urban areas.

Keywords: fixed-cycle traffic light queue, probability generating function, root finding, waiting-time analysis

MSC: 60K25, 68M20, 90B22

1. Introduction

Fixed-Cycle Traffic Light (FCTL) queue is a commonly analyzed model in traffic engineering. It models vehicles arriving and waiting at an intersection regulated by a traffic light. The traffic light alternates between green and red signal period. Most of the existing literature on FCTL queueing models assumes that vehicles reach the traffic light randomly, following a Poisson process (for continuous-time models) or being geometrically (or Bernoulli) distributed (for discrete-time models). Webster's formula [1] is widely considered as the most prominent result in this area. This formula provides the average waiting time for a vehicle in a straightforward mathematical form. Subsequent work has been performed to

DOI: https://doi.org/10.37256/cm.6420255958

determine performance with greater granularity, such as a complete analytical solution for the distributions of queue length and waiting time for a vehicle. McNeill [2] made progress by offering an exact expression, except for a single unknown factor: the average size of the overflow queue (the mean queue length at the end of a green signal period). Leeuwaarden [3] derived the Probability Generating Function (PGF) of the queue length distribution and the average waiting time of vehicles. This study offers a more realistic version of the FCTL queue, as discussed in, for example, Darroch [4]. This approach enables one to conduct exact analysis as well as numerical computation. For more information on the FCTL queue, see Boon et al. [5], Boon and Leeuwaarden [6], and Timmerman and Boon [7]. Work has been done on relaxing arrival process assumptions, notably by Newell [8], who provides an approximation for average queue length by applying relatively general reasoning while Darroch [4] and McNeill [2] both examine a compound Poisson process. Assuming (compound) Poisson arrivals enables the modeling of the FCTL queue at specific instances, particularly immediately following the departure of a delayed vehicle, as discussed by Darroch [4]. The predominant focus of previous studies have has been on developing formulas for the mean values of performance metrics such as queue length and delay. Meissl [9] independently derived the PGF of the steady-state overflow queue length distribution using nearly identical methodologies. This approach is frequently utilized in queueing theory and heavily depends on the challenge of root finding, which was a significant limitation when these models were first analyzed, but it is no longer a major challenge. In Chaudhry et al. [10], considerable effort is dedicated to address skepticism regarding root finding in queueing theory. Chaudhry and Goswami [11] find the steady state queue-length distribution for the discrete-time $Geo/G/1/N+1$ queueing system using a root-finding approach. Chaudhry and Templeton [12] may be considered as a valuable resource related to the analysis of queues using root-finding techniques. In general, root finding is well structured; for most of the queueing models, distinct roots of characteristic equations are present.

From the above discussion, it is evident that the FCTL queue has been a topic of significant research effort. The queue length and the waiting-time distributions of waiting vehicles have been studied when the traffic light control does not depend on the number of vehicles present in the queue. As far as we know, no studies have been pursued in this direction in the intervening years. No standard procedure has been proposed yet for managing an FCTL queueing system when a large number of vehicles are waiting. In this paper, we deal with a single-server state-dependent FCTL queueing model. In this system, the timing of green and red signal periods depends on the count of vehicles in the queue. The green signal period is divided into uniform time intervals known as phases. For a threshold $N(>0)$, if there are less than N vehicles waiting, we have a fixed number $g(>0)$ phases for the green signal period and a fixed number $r(>0)$ phases for the red signal period making the total cycle length $c = g + r$ phases. However, if the queue has greater than or equal to N waiting vehicles in the g-th phase, then we increase the green signal period duration from g to $g_1 (g_1 \geq g)$ phases and the red signal period immediately following decreases from r to $r_1 (\leq r)$ phases in such a way that the total cycle length is kept at $c = g_1 + r_1$ phases. It is assumed that exactly one vehicle is released (overtaking, jockeying and reneging of vehicles are not allowed) from the intersection on completion of a phase during a green signal period only. This study answers several questions like the following:

a) How to overcome the high traffic situation in an FCTL queue?

b) How are queue-length and waiting-time distributions calculated in the state-dependent FCTL queueing model?

To answer the above questions, this state-dependent FCTL queueing model determines the steady-state queue-length and waiting-time distribution of vehicles. First, we determine a recurrence relation that governs the dynamics of the queue. Then, we use this relation to determine the PGF of the queue-length distribution for the waiting vehicles. We then invert this PGF using a root-finding method (see Chaudhry et al. [10] for details). Finally, we obtain the queue-length distribution for each phase during the green signal period. Using this queue-length distribution, we calculate the waiting time distribution of vehicles for each phase of the green signal period utilizing the distributional form of Little's law; see Keilson and Servi [13] for details.

Finally, it may be remarked here that in Section 2, we propose a state-dependent scheduling mechanism in fixed-cycle traffic light queues. In this scheduling mechanism, the number of green phases increases if the number of vehicles waiting at the FCTL intersection is greater than or equal to N. To the best of our knowledge there are no studies available in the literature that discuss fixed-cycle traffic light queues under such a state-dependent scheduling mechanism. Additionally, we explore a simple method to derive the PGF of the phase completion epoch probabilities for each phase of a green signal period. In Section 2.2, we derive the random epoch probabilities for each phase of a green signal period. In Section 3, we analyze the vehicle waiting time distribution for each phase of a green signal period. Lastly, in Section 4, we validate the method with numerical examples and we draw our conclusions in Section 5.

2. State-dependent fixed-cycle traffic light queueing model

In an FCTL queueing model, the durations of green and red signal periods adjust dynamically based on the number of vehicles in the queue. Let us assume that vehicles arrive according to a Poisson process with an average arrival rate of $\lambda(>0)$. Let the duration of each time slot or phase be d time units. During a green signal period, one vehicle is released from the traffic intersection after completion of a phase. Let $q_j,\ j \geq 0$, be the probability that exactly j vehicles enter during a phase, with corresponding PGF $q(z)$. The Laplace-Stieltjes Transform (LST) of the time duration of each phase is denoted by $f_d^*(s)$ which is given by $f_d^*(s) = e^{-ds}$ with $\Re(s) \geq 0$. The probabilities $q_j(j \geq 0)$ can be calculated using properties of the Poisson arrival process. Thus, the PGF of q_j is given by $q(z) = \sum_{j=0}^{\infty} q_j z^j = f_d^*(\lambda - \lambda z) = e^{-d(\lambda - \lambda z)}$; see Chaudhry and Templeton [12]. Similarly, $\widehat{q}_j(j \geq 0)$ denotes the probability that exactly j vehicles arrive in the traffic intersection during a stationary elapsed time duration of a phase. In a similar manner as for $q(z)$, we calculate the PGF of $\widehat{q}_j$ as $\widehat{q}(z) = \mu \dfrac{1 - f_d^*(\lambda - \lambda z)}{\lambda - \lambda z}$, where $\mu = \dfrac{1}{d}$; for more detail, see Chaudhry et al. [14]. Now, we propose a state-dependent scheduling mechanism for green and red signal periods. Here, we assume that if the number of vehicles in the queue is less than N, then the green and red signal periods have fixed numbers of phases g and r, respectively, with total cycle length $c = g + r$ phases. If there are greater than or equal to N vehicles after completion of the g-th phase, then the ongoing green signal period extends from g to $g_1(\geq g)$ phases, while the red signal period immediately following shortens from r to $r_1(\leq r)$ phases, keeping the total cycle time at a constant $c = g_1 + r_1$ phases. According to the above description, we can visualize the traffic signaling system as in Figure 1.

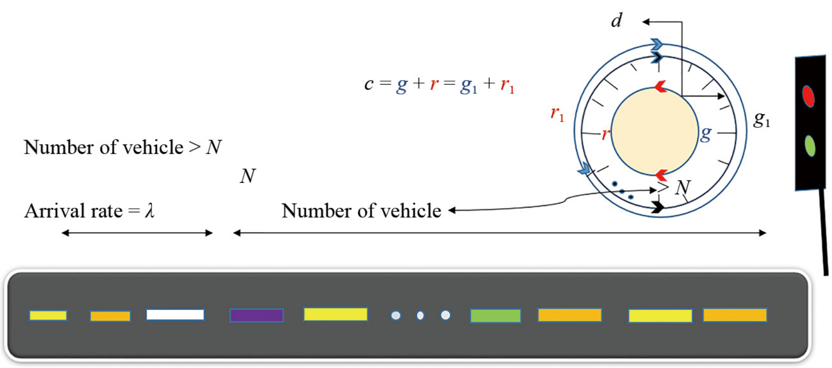

Figure 1. FCTL queueing system

The stability criterion for this state-dependent FCTL queueing system is straightforward. If the average number of vehicles that arrive to the intersection throughout a fixed-cycle time is less than the maximum number of phases of a green signal period, then the FCTL queueing system will be stable. That is, $\lambda cd < g_1$, which implies $\lambda cd < g$. Let us define the traffic intensity $\rho = \dfrac{\lambda cd}{g_1}$, which must less than 1 if the stationary probabilities exist (the state-dependent FCTL queueing system is stable).

2.1 *Phase completion epoch probabilities using root finding*

Let $p^+_{k,\,1,\,n}$ and $p^+_{k,\,0,\,n}$ be the phase completion epoch probabilities that there are n (that is, $n = 0,\ 1,\ 2,\ \ldots$) vehicles waiting in the queue at the completion epoch of the k-th ($k = 1,\ 2,\ \ldots,\ g$) phase of the green and $k = g+1,\ g+2,\ \ldots,\ r$ red signal period, respectively. Also, let $p^{+}_{k,\ 1,\ N+j-(k-g)}$, $(j = 0,\ 1,\ 2,\ \ldots)$ denote the phase completion epoch probabilities for the k-th $(k = g+1,\ g+2,\ \ldots,\ g_1)$ phase while there are $N + j - (k - g)$ vehicles present in the queue. Here, the suffix 1 represents a green signal period and the suffix 0 represents a red signal period. We first calculate the PGF of the probabilities $p^{+}_{k,\ 1,\ n}$ at the end of the k-th phase during a green signal period. In this context, we consider a departure process for delayed vehicles that is slightly more general. In this scenario, a green signal period consists of either g phases (if the number of vehicles in the queue is less than N after completion of the g-th phase) or g_1 phases (if the number of vehicles in the queue is at least N after completion of the g-th phase). We express the number of vehicles during a green signal period using the following recursive relation. For, $k = g,\ g+1,\ \ldots,\ g_1 - 1$ (number of waiting vehicles exceeds N after completion of the g-th phase), we obtain

$$X_{k+1} = \begin{cases} X_k + Y_{k+1} - 1, & X_k \geq 1, \\ \\ 0, & X_k = 0, \end{cases}$$

where X_i denotes the number of vehicles after the end of the i-th phase during a green signal period and Y_i denotes the number of vehicles that arrive during the i-th phase. Also, $X_{k+1} = X_k + Y_{k+1}$ for $k = g_1 + 1,\ \ldots,\ c - 1$, where X_i denotes the number of vehicles after the end of the i-th phase during a red signal period.

Now, we consider the situation when the number of waiting vehicles is at least N, so that the number of phases during the ongoing green signal period extends from g to g_1. Therefore, for every $k = g,\ g+1,\ \ldots,\ g_1 - 1$, we have the following relation between two consecutive phase completion epochs:

$$p^{+}_{k+1,\ 1,\ N+j-(k+1-g)} = \sum_{i=0}^{j} p^{+}_{k,\ 1,\ N+j-i-(k-g)} q_i,\ j = 0,\ 1,\ 2,\ \cdots. \tag{1}$$

The shifted PGF of the phase completion epoch probabilities $p^{+}_{k,\ 1,\ N+j-(k-g)}$ at the end of the k-th phase is $p^{+*}_{k}(z) = \sum_{j=0}^{\infty} p^{+}_{k,\ 1,\ N+j-(k-g)} z^j$. Thus, multiplying both sides of (1) by $z^0,\ z^1,\ z^2,\ \ldots$, and summing, we obtain:

$$p^{+*}_{k+1}(z) = q(z) p^{+*}_{k}(z), \quad k = g+1,\ g+2,\ \ldots,\ g_1 - 1. \tag{2}$$

Now, the PGF of the phase completion epoch probability at the end of the g_1-th phase may be deduced from (2) by successive back substitution for $k = g_1 - 1,\ g_1 - 2,\ \ldots,\ g$. Finally, we obtain

$$p^{+*}_{g_1}(z) = [q(z)]^{g_1 - g - 1} p^{+*}_{g+1}(z). \tag{3}$$

In (3), we require the probability generating function of $p^{+*}_{g+1}(z)$. To calculate this, we first observe the number of waiting vehicles at the end of the g-th phase. If the number of waiting vehicles is at least N, then we increase the number of green phases from g to g_1. The phase completion epoch probabilities of the g-th and $(g+1)$-th phases are related as follows:

$$p^{+}_{g+1,\ 1,\ N+j-1} = \sum_{i=0}^{j} p^{+}_{g,\ 1,\ N+j-i} q_i, \quad j = 0,\ 1,\ 2,\ \ldots. \tag{4}$$

Now, multiplying both sides of (4) by $z^0,\ z^1,\ z^2,\ \ldots$ and summing, we obtain the PGF $p_{g+1}^{+*}(z)$:

$$p_{g+1}^{+*}(z) = \frac{q(z)}{z^N}\left[p_g^{+*}(z) - \sum_{i=0}^{N-1} p_{g,\,1,\,i}^{+} z^i\right], \tag{5}$$

where $p_g^{+*}(z) = \sum_{i=0}^{\infty} p_{g,\,1,\,i}^{+} z^i$. The relation (5) is between a normal PGF and a shifted PGF. Here, one may note that at least $N-1$ vehicles remain after the completion of the $(g+1)$-th phase, but there may be any number of vehicles remaining after completion of the g-th phase. Now, for $z = 1$, the right-hand side of (5) must be equal to 1. As a result, we must add the term $\sum_{i=0}^{N-1} p_{g,\,1,\,i}^{+} z^i$ to the right-hand side of (5). Therefore, the final PGF corresponding to the $(g+1)$-th phase is given by:

$$p_{g+1}^{+*}(z) = \frac{q(z)}{z^N}\left[p_g^{+*}(z) - \sum_{i=0}^{N-1} p_{g,\,1,\,i}^{+} z^i\right] + \sum_{i=0}^{N-1} p_{g,\,1,\,i}^{+} z^i. \tag{6}$$

Substituting the PGF $p_{g+1}^{+*}(z)$ in (3), yields, after some algebraic simplification:

$$p_{g_1}^{+*}(z) = \frac{[q(z)]^{g_1-g}}{z^N}\left[p_g^{+*}(z) - \sum_{i=0}^{N-1} p_{g,\,1,\,i}^{+} z^i\right] + [q(z)]^{g_1-g-1}\sum_{i=0}^{N-1} p_{g,\,1,\,i}^{+} z^i. \tag{7}$$

In (7), we require the PGF of $p_g^{+*}(z)$, which is provided by Leeuwaarden [3, Equation (8)]. Thus, $p_g^{+*}(z)$ can be expressed as:

$$p_g^{+*}(z) = \left(\frac{q(z)}{z}\right)^g p_0^{+*}(z) + \left(1 - \frac{z}{q(z)}\right)\sum_{k=0}^{g-1} p_{k,\,1,\,0}^{+}\left(\frac{q(z)}{z}\right)^{g-k-1}, \tag{8}$$

where $p_1^{+*}(z) = \sum_{j=0}^{\infty} p_{1,\,1,\,j}^{+} z^j$ and $p_{1,\,1,\,j}^{+}$ denotes the probability of $j(\geq 0)$ waiting vehicles at the end of a red signal period. Note that, $p_1^{+*}(z) = p_g^{+*}(z)\,(q(z))^r A + p_{g_1}^{+*}(z)\,(q(z))^{r_1}\,(1-A)$, where $A = \sum_{i=0}^{N+g-1} \dfrac{e^{-cd\lambda}(cd\lambda)^i}{i!}$ (since vehicles arrive according to a Poisson process with arrival rate λ). Now, after some algebraic simplification, we can express (8) in the following form:

$$p_g^{+*}(z) = \frac{(\zeta(z)-1)[q(z)]^g z^N \sum_{i=0}^{g-1} p_{i,\,1,\,0}^{+}\zeta(z)^i}{z^{g+N} - Aq(z)^c z^N - (1-A)q(z)^c} + \frac{(1-A)(z^N - q(z))q(z)^{c-1}\sum_{i=0}^{N-1} p_{g,\,1,\,i}^{+} z^i}{z^{g+N} - Aq(z)^c z^N - (1-A)q(z)^c}, \tag{9}$$

where $\zeta(z) = \dfrac{z}{q(z)}$. The first term on the right-hand side of (9) corresponds to the PGF of the phase completion epoch probabilities for the FCTL queueing model without state-dependent scheduling of green and red signal periods (i.e., $N = \infty$); see Leeuwaarden [3]. Here, it is important to note that $q(z) = f_d^*(\lambda - \lambda z) = f_d^*(s)|_{s=\lambda-\lambda z}$, where $f_d^*(s)$ may be approximated as a rational function in s. We choose to approximate $f_d^*(s)$ using the Padé approximation method; see Botta et al. [15] and Singh et al. [16]. Now, we consider the LST of the phase duration as a rational analytic function in

$s(\Re(s) \geq 0)$ as follows: $f_d^*(s) \simeq \dfrac{P(s)}{Q(s)}$, where $Q(s)$ is a polynomial of degree m and $P(s)$ is a polynomial of degree less than or equal to m. Thus, we obtain $f_d^*(\lambda - \lambda z) \simeq \dfrac{P(\lambda - \lambda z)}{Q(\lambda - \lambda z)} = \dfrac{f(z)}{g(z)}$. From (9), we have:

$$p_g^{+*}(z) = \frac{g(z)^c(\zeta(z)-1)[q(z)]^g z^N \sum_{i=0}^{g-1} p_{i,\,1,\,0}^+ \zeta(z)^i}{z^{g+N}g(z)^c - Af(z)^c z^N - (1-A)f(z)^c} + \frac{(1-A)g(z)^c(z^N - q(z))q(z)^{c-1}\sum_{i=0}^{N-1} p_{g,\,1,\,i}^+ z^i}{z^{g+N}g(z)^c - Af(z)^c z^N - (1-A)f(z)^c}. \tag{10}$$

Here, $p_g^{+*}(z)$ is a rational analytic function of z with $|z| \leq 1$. The denominator of the right hand side of (10) is given by $z^{g+N}g(z)^c - Af(z)^c z^N - (1-A)f(z)^c$, which is a polynomial of degree $g+N+cm$, as we have assumed that $g(z)$ is a polynomial of degree m. Thus, one can express the characteristic equation (CE) as follows:

$$z^{g+N}g(z)^c - Af(z)^c z^N - (1-A)f(z)^c = 0. \tag{11}$$

It can be observed that $z^{g+N}g(z)^c - Af(z)^c z^N - (1-A)f(z)^c$ has precisely $g+N$ zeros inside and on the unit circle $|z| = 1$, (for a proof sketch, see Appendix; also see Gail et al. [17, Lemma 1] for a formal proof) and cm zeros outside the unit circle $|z| = 1$. We denote the roots of the CE Equation (11) by $\beta_1,\ \beta_2,\ \beta_3,\ \ldots,\ \beta_{g+N} = 1$, which are inside and on the unit circle, along with the roots. $\beta_{g+N+1},\ \beta_{g+N+2},\ \ldots,\ \beta_{g+N+cm}$ that lie outside the unit circle.

Remark The PGF $p_{g_1}^{+*}(z)$ is utilized in the calculation of the waiting-time distribution. In the past, it was difficult to find the roots of a CE (as discussed by Neuts [18], Kendall [19]). But our procedure is dependent on finding the roots of the CE (11) (which may be done using MAPLE or MATHEMATICA; see Chaudhry [20, 21]). Additionally, we may also refer to Chaudhry et al. [10] in this context. There are multiple ways to solve the CE, but here we present one approach using the MAPLE software package:

```
> restart :
> with (RootFinding):
> f := (x-5).(x-7).(x-11)^3.(x-13)^5,
  f := (x-5)(x-7)(x-11)^3(x-13)^5,
> fsolve (f, x, complex),
  5., 7., 11., 11., 11., 13., 13., 13., 13., 13.
```

Note that determining the roots in this example is a trivial exercise; the example is provided simply to demonstrate the appropriate MAPLE commands.

The right-hand side of (9) involves $g+N$ unknowns in the numerator,denoted by $p_{0,\,1,\,0}^+,\ p_{1,\,1,\,0}^+,\ p_{2,\,1,\,0}^+,\ \cdots,\ p_{g-1,\,1,\,0}^+$ and $p_{g,\,1,\,0}^+,\ p_{g,\,1,\,1}^+,\ p_{g,\,1,\,2}^+,\ \ldots,\ p_{g,\,1,\,N-1}^+$. These unknowns can be obtained using the following procedure. Since $p_g^{+*}(z)$ is convergent/analytic in $|z| \leq 1$, the zeros of the denominator $z^{g+N} - Aq(z)^c z^N - (1-A)q(z)^c$ inside and on the unit circle $|z| = 1$ must coincide with the zeros of the numerator $(\zeta(z)-1)[q(z)]^g z^N \sum_{i=0}^{g-1} p_{i,\,1,\,0}^+ \zeta(z)^i + (1-A)[z^N - q(z)]q(z)^{c-1}\sum_{i=0}^{N-1} p_{g,\,1,\,i}^+ z^i$. Thus, using the roots inside and on the unit circle and (9), we obtain for the numerator:

$$(\zeta(\beta_j)-1)[q(\beta_j)]^g \beta_j^N \sum_{i=0}^{g-1} p_{i,\,1,\,0}^+ \zeta(\beta_j)^i + (1-A)(\beta_j^N - q(\beta_j))q(\beta_j)^{c-1}\sum_{i=0}^{N-1} p_{g,\,1,\,i}^+ \beta_j^i$$

$$j = 1,\ 2,\ 3,\ \cdots,\ g+N-1, \tag{12}$$

so equating the numerator and denominator for each j leads to $g+N-1$ linear equations with $g+N$ unknowns: $p^{+}_{0,\,1,\,0},\, p^{+}_{1,\,1,\,0},\, p^{+}_{2,\,1,\,0},\, \cdots,\, p^{+}_{g-1,\,1,\,0}$ and $p^{+}_{g,\,1,\,0},\, p^{+}_{g,\,1,\,1},\, p^{+}_{g,\,1,\,2},\, \cdots,\, p^{+}_{g,\,1,\,N-1}$. These equations are linearly dependent and thus to determine a unique solution, we incorporate a normalizing condition, which is described as follows:

$$p_g^{+*}(1) = \frac{\left[\dfrac{d}{dz}[(\zeta(z)-1)[q(z)]^g z^N \sum_{i=0}^{g-1} p^{+}_{i,\,1,\,0}\zeta(z)^i + (1-A)(z^N - q(z))(q(z))^{c-1}\sum_{i=0}^{N-1} p^{+}_{g,\,1,\,i} z^i]\right]_{z=1}}{\left[\dfrac{d}{dz}[z^{g+N} - Aq(z)^c z^N - (1-A)q(z)^c]\right]_{z=1}}$$

$$= 1 \tag{13}$$

Therefore, combining (12) and the normalizing condition (13) results in a linearly independent system of $g+N$ equations with $g+N$ unknowns: $p^{+}_{0,\,1,\,0},\, p^{+}_{1,\,1,\,0},\, p^{+}_{2,\,1,\,0},\, \cdots,\, p^{+}_{g-1,\,1,\,0}$ and $p^{+}_{g,\,1,\,0},\, p^{+}_{g,\,1,\,1},\, p^{+}_{g,\,1,\,2},\, \cdots,\, p^{+}_{g,\,1,\,N-1}$. One may note that before finding the phase completion epoch probabilities, first we ensure the stability criteria ($\lambda cd < g_1$) for the FCTL queueing system. Thereafter, one can express the partial fraction for the PGF $p_g^{+*}(z)$ using the roots lying outside of the unit circle of the CE (11) as follows:

$$p_g^{+*}(z) = \frac{(\zeta(z)-1)[q(z)]^g z^N \sum_{i=0}^{g-1} p^{+}_{i,\,1,\,0}\zeta(z)^i + (1-A)(z^N - q(z))(q(z))^{c-1}\sum_{i=0}^{N-1} p^{+}_{g,\,1,\,i} z^i}{z^{g+N} - Aq(z)^c z^N - (1-A)q(z)^c}$$

$$= \sum_{j=g+N+1}^{g+N+cm} \frac{a_j}{z-\beta_j}, \tag{14}$$

where a_j represents the coefficient associated with the corresponding partial fraction. Then, extracting the coefficients of z^n from the right-hand side of (14), one can obtain the phase completion epoch probabilities as follows:

$$p^{+}_{g,\,1,\,j} = -\sum_{i=g+N+1}^{g+N+mc} \frac{a_i}{\beta_i^{j+1}}, \quad j = 0,\, 1,\, 2,\, 3,\, \cdots, \tag{15}$$

where

$$a_j = \frac{(\zeta(\beta_j)-1)[q(\beta_j)]^g \beta_j^N \sum_{i=0}^{g-1} p^{+}_{i,\,1,\,0}\zeta(\beta_j)^i + (1-A)(\beta_j^N - q(\beta_j))(q(\beta_j))^{c-1}\sum_{i=0}^{N-1} p^{+}_{g,\,1,\,i}\beta_j^i}{\left[\dfrac{d}{dz}[z^{g+N} - Aq(z)^c z^N - (1-A)q(z)^c]\right]_{z=\beta_j}},$$

$$j = g+N+1,\, g+N+2,\, \cdots,\, g+N+cm.$$

Now, using the g-th phase completion epoch probability in (15), we get the corresponding PGF $p_g^{+*}(z) = \sum_{j=0}^{\infty} p^{+}_{g,\,1,\,j} z^j$. Thus, substituting $p_g^{+*}(z)$ in (7), we get the PGF $p_{g_1}^{+*}(z)$. Now, we find the phase completion epoch probability at the g_1-th phase by expanding the function $p_{g_1}^{+*}(z)$ using a Taylor series about the point $z=0$, and then extracting the coefficient of z^j as the phase completion epoch probabilities $p^{+}_{g_1,\,1,\,N+j-(g_1-g)}$. Thereafter, using (1), one can find the probabilities at each phase completion epoch $p^{+}_{k,\ 1,\ N+j-(k-g)}$, for $k = g_1,\ g_1-1,\ \dots,\ g+1$ and $j = 0,\ 1,\ 2,\ \dots$. Also, one can find the phase completion epoch probabilities $p^{+}_{k,\ 1,\ j}$ when the phase varies from $k = 1,\ 2,\ \dots,\ g$ and $j = 0,\ 1,\ \dots$ using Leeuwaarden [3, Equation (4), (5)] and (15). Lastly, we normalize the phase completion epoch probabilities in each phase, according to $\sum_{j=0}^{\infty} p^{+}_{k,\ 1,\ N+j-(k-g)} = 1$, for $k = g+1,\ \dots,\ g_1$, and $\sum_{j=0}^{\infty} p^{+}_{k,\ 1,\ j} = 1$, for $k = 1,\ \dots,\ g$.

2.2 *Derivation of random epoch probabilities using phase completion epoch probabilities*

Let $p_{k,\ 1,\ N+i-(k-g)}$, $i \geq 0$, be the random epoch probabilities for the k-th phase, for $k = g+1,\ g+2,\ \dots,\ g_1$. Random and phase completion epoch probabilities are connected by relating the probabilities corresponding to the g_1-th phase and the (g_1-1)-th phase, for more details see Guha and Banik [22, Equation (14), (15), (16)].

These probabilities are related as follows:

$$p_{g_1,\ 1,\ N+i-(g_1-g)} = \sum_{j=0}^{i} p^{+}_{g_1-1,\ 1,\ N+i-j-(g_1-1-g)} \widehat{q}_j,\ \ i = 0,\ 1,\ 2,\ \dots. \tag{16}$$

Using a similar recursive formula, we can determine the random epoch (or arbitrary epoch) probabilities at each of the phases $g_1,\ g_1-1,\ \dots,\ g+1$. Also, let $p_{g,\ 1,\ i}$, $i \geq 0$, be the random epoch probabilities for the g-th phase. Phase completion and random epoch probabilities are connected by relating the probabilities corresponding to the g-th phase and $(g-1)$-th phase in the following manner; for more details see Guha and Banik [22, Equation (14), (15), (16)]

These probabilities are related as follows:

$$p_{g,\ 1,\ i} = \sum_{j=0}^{i} p^{+}_{g-1,\ 1,\ i-j} \widehat{q}_j,\ \ i = 0,\ 1,\ 2,\ \dots. \tag{17}$$

Using a similar recursive formula, we can determine the random epoch probabilities (or arbitrary epoch) for each phase $1,\ 2,\ 3,\ \dots,\ g$.

Mean number of vehicles at phase completion epoch and random epoch:

The mean number of waiting vehicles can be calculated as follows. Let $L^{+}_{q,\ k}$ and $L_{q,\ k}$ be the mean number of waiting vehicles at phase completion and random epoch, respectively. Therefore, for $k = g+1,\ g+2,\ \dots,\ g_1$:

$$L^{+}_{q,\ k} = \sum_{j=0}^{\infty} [N+j-(k-g)] p^{+}_{k,\ 1,\ N+j-(k-g)}, \tag{18}$$

$$L_{q,\ k} = \sum_{j=0}^{\infty} [N+j-(k-g)] p_{k,\ 1,\ N+j-(k-g)}, \tag{19}$$

and for $k = 1,\ 2,\ \dots,\ g$:

$$L^{+}_{q,\ k} = \sum_{i=0}^{\infty} i p^{+}_{k,\ 1,\ i}, \tag{20}$$

$$L_{q,\,k}=\sum_{j=0}^{\infty} j p_{k,\,1,\,j}, \text{ for } k=1,\,2,\,\ldots,\,g. \tag{21}$$

3. Derivation of waiting-time distribution of a vehicle

The waiting time of a vehicle in a phase, denoted as $D_q^{(n)}$, is the amount of time the n-th arriving vehicle has to wait until it passes the k-th phase. Then, $W_{q,\,k}(t)=\lim_{n\to\infty} P\left(D_q^{(n)}\le t\right)$ and in the limiting case as $n\to\infty$, we denote D_q^n by D_q. The LST of the waiting time distribution of a random vehicle is given by $W^*_{q,\,k}(s)=\int_0^\infty e^{-st} dW_{q,\,k}(t)$ with $\Re(s)\ge 0$. Now, following Chaudhry and Templeton [23], the total probability of the number of arrivals during the waiting time is equal to the total probability of possible number of vehicles waiting in the queue during the k-th phase. Now, using the distributional form of Little's law (see Keilson and Servi [13] for details), we obtain

$$\sum_{i=0}^{\infty} p_{k,\,1,\,N+i-(k-g)} z^i = W^*_{q,\,k}(\lambda-\lambda z), \quad \text{for } k=g+1,\,g+2,\,\ldots g_1, \tag{22}$$

$$\sum_{i=0}^{\infty} p_{k,\,1,\,i} z^i = W^*_{q,\,k}(\lambda-\lambda z), \quad \text{for } k=1,\,2,\,\ldots,\,g. \tag{23}$$

Using the substitution $s=\lambda-\lambda z$ in (22) and (23) and after some algebraic rearrangements, we arrive at the following result:

$$W^*_{q,\,k}(s)=\sum_{j=0}^{\infty} p_{k,\,1,\,N+j-(k-g)}\left(1-\frac{s}{\lambda}\right)^j \text{ for } k=g+1,\,g+2,\,\ldots,\,g_1. \tag{24}$$

$$W^*_{q,\,k}(s)=\sum_{j=0}^{\infty} p_{k,\,1,\,j}\left(1-\frac{s}{\lambda}\right)^j \text{ for } k=1,\,2,\,\ldots,\,g. \tag{25}$$

By employing the Padé approximation on the right-hand side of Equation (24), it is possible to estimate $W^*_{q,\,k}(s)$ using the rational function $\dfrac{\widehat{P}(s)}{\widehat{Q}(s)}$. Let us assume that the degree of the numerator $\widehat{P}(s)$ is R_0 and the degree of the denominator is $R_d(>R_0)$. The validity of the Padé approximation can be verified by considering the fact that $\dfrac{\widehat{P}(0)}{\widehat{Q}(0)}=\sum_{j=0}^{\infty} p_{k,\,1,\,N+j-(k-g)}=1$ for $k=g+1,\,g+2,\,\ldots,\,g_1$ and $-\dfrac{d}{ds}\left(\dfrac{\widehat{P}(s)}{\widehat{Q}(s)}\right)|_{s=0}=\dfrac{L_{q,\,k}}{\lambda}$ for $k=g+1,\,g+2,\,\ldots,\,g_1$. Moreover, it is possible to enhance the Padé approximation by taking into account these two criteria. Considering that $W^*_{q,\,k}(s)$ converges or is analytic in the region $\Re(s)\ge 0$, we can assume that, the R_d zeros of the denominator $\widehat{Q}(s)$ are given by $s_1,\,s_2,\,s_3,\,\ldots,\,s_{R_d}$ with $\Re(s_j)<0,\ j=1,\,\ldots,\,R_d$. By applying partial fractions, we can express the right-hand side of (24) in the following manner:

$$W^*_{q,\,k}(s)\simeq\frac{\widehat{P}(s)}{\widehat{Q}(s)}=\sum_{j=0}^{R_d}\frac{k_j}{(s-s_j)} \text{ for } k=g+1,\,g+2,\,\ldots,\,g_1. \tag{26}$$

Now, by inverting (26), we get the probability density function as follows:

$$w_{q,\ k}(t) = \sum_{j=1}^{R_d} k_j e^{s_j t}, \ \ t > 0 \ \text{ for } \ k = g+1,\ g+2,\ \ldots,\ g_1, \tag{27}$$

where $k_j = \dfrac{\widehat{P}(s)}{\left[\dfrac{d}{ds}[\widehat{Q}(s)]\right]_{s=s_j}}$, $j = 1,\ 2,\ 3,\ \cdots,\ R_d$. Hence, the CDF of the waiting time for a random vehicle is given by

$$W_{q,\ k}(x) = \int_0^x w_{q,\ k}(t)dt, \ \ \text{for} \ \ k = g+1,\ g+2,\ \ldots,\ g_1. \tag{28}$$

Similarly, we find the waiting-time distribution of vehicles at phase $k = 1,\ 2,\ \ldots,\ g$ using (25).

4. Numerical example

This section validates the results from earlier sections with numerical experiments. Several experiments were conducted with different model settings, and an example is provided here. We used MAPLE 2015 on a Windows 11 (64-bit) system for these experiments. The results in this paper are rounded off to six decimal places; we can provide more precise calculations upon request.

Example Let us examine the scenario where each vehicle traverses the traffic intersection with a constant time phase length of d, while the rate of vehicle arrivals is denoted by λ. So, the PGF for vehicles arriving in the queue is given by $q(z) = f_d^*(\lambda - \lambda z) = e^{-d(\lambda - \lambda z)}$. We take the model parameters $d = 1, N = 15, \lambda = 0.3,\ g = 3,\ r = 5,$ $g_1 = 5$, and $r_1 = 3$, so $c = g + r = g_1 + r_1$. In this scenario, (11) possesses eighteen roots inside or on the unit disk. These eighteen roots are as follows: $\beta_1 = -0.214796$, $\beta_2 = 0.023515 - 0.214810\text{i}$, $\beta_3 = 0.023515 + 0.214810\text{i}$, $\beta_4 = 0.107636 - 0.187416\text{i}$, $\beta_5 = 0.107636 + 0.187416\text{i}$, $\beta_6 = -0.198904 - 0.085829\text{i}$, $\beta_7 = -0.198904 + 0.085829\text{i}$, $\beta_8 = 0.175050 - 0.128503\text{i}$, $\beta_9 = 0.175050 + 0.128503\text{i}$, $\beta_{10} = -0.065561 - 0.207807\text{i}$, $\beta_{11} = -0.065561 + 0.207807\text{i}$, $\beta_{12} = 0.213215 - 0.045863\text{i}$, $\beta_{13} = 0.213215 + 0.045863\text{i}$, $\beta_{14} = -0.147581 - 0.162840\text{i}$, $\beta_{15} = -0.147581 + 0.162840\text{i}$, $\beta_{16} = -0.248574 - 0.271702\text{i}$, $\beta_{17} = -0.248574 + 0.271702\text{i}$, $\beta_{18} = 1.000000$. Table 1 presents the sum of phase completion epoch probabilities after completion of the g_1-th phase $\sum_{i=m}^{\infty} p^{+}_{g_1,\ 1,\ N+i-(g_1-g)}$, (for any nonnegative integer $m(\geq 0)$) and at a random epoch $\sum_{i=m}^{\infty} p_{g_1,\ 1,\ N+i-(g_1-g)}$, (for any nonnegative integer $m(\geq 0)$).

Table 1. Sum of phase completion epoch probabilities after completion of the g_1-th phase with $g_1 = 5,\ r_1 = 3,\ g = 3,\ r = 5$, and $d = 1$

λ	$\sum_{i=10}^{\infty} p^{+}_{g_1,\ 1,\ N+i-(g_1-g)}$	$\sum_{i=10}^{\infty} p_{g_1,\ 1,\ N+i-(g_1-g)}$	$\sum_{i=20}^{\infty} p^{+}_{g_1,\ 1,\ N+i-(g_1-g)}$	$\sum_{i=20}^{\infty} p_{g_1,\ 1,\ N+i-(g_1-g)}$
0.30	1.0213E–02	9.4823E–03	5.0401E–10	1.3848E–04
0.32	3.2489E–02	3.1836E–02	3.3306E–09	1.6345E–03
0.34	8.8212E–02	1.0004E–02	1.7563E–08	1.8174E–02
0.36	1.8624E–02	3.3049E–02	6.7733E–08	1.8521E–02

In Figure 2, we present the phase completion epoch probability and random epoch probability at the g-th phase. Here, we take the model parameters to be $N = 10,\ g_1 = 5,\ g = 3,\ \lambda = 0.30$ and the number of waiting vehicles ranges from $i = 0$ to 10. Figure 2 illustrates that, if the number of vehicles increases in the queue in the g-th phase, then the phase completion epoch probability as well as the random epoch probability decrease but the decay for the phase completion epoch probability is faster than for the random epoch probability.

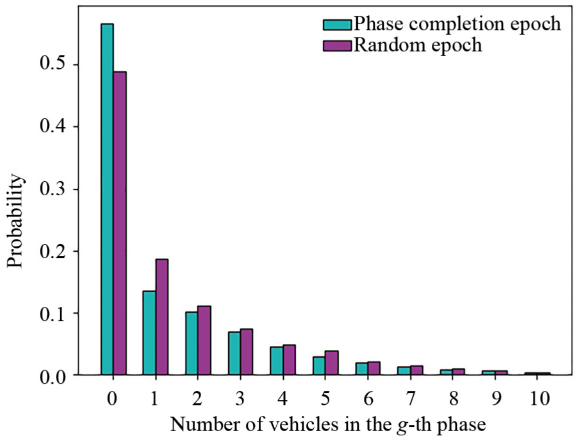

Figure 2. Queue length probabilities at phase completion epoch $p^+_{g,\,1,\,i}$ and random epoch $p_{g,\,1,\,i}$ for $i = 0$ to 10 after completion of the g-th phase

In Figure 3, we present the phase completion epoch probability and random epoch probability at the g_1-th phase, which contains at least $N + i - (g_1 - g) = 8$ vehicles. Here, we take the model parameters to be $N = 10,\ g_1 = 5,\ g = 3$ and $i = 0$ to 10 (so that the number of waiting vehicles ranges from 8 to 18). Figure 3 illustrates that, if the number of waiting vehicles increases in the g_1-th phase, then the phase completion epoch probability as well as the random epoch probability decrease.

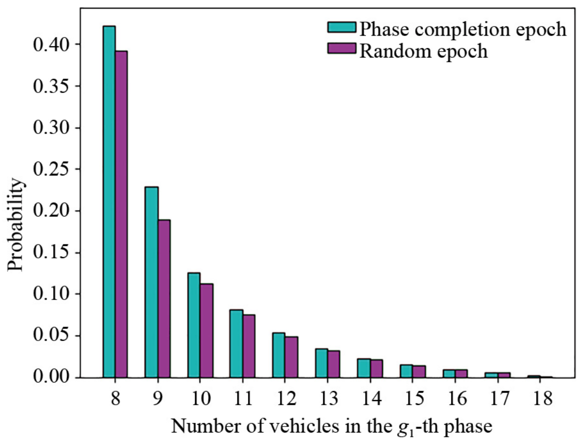

Figure 3. Queue length probabilities at phase completion epoch $p^+_{g_1,\,1,\,N+i-(g_1-g)}$ and random epoch $p_{g_1,\,1,\,N+i-(g_1-g)}$ for $i = 0$ to 10 after completion of the g_1-th phase

Figure 4 shows how the phase completion epoch probabilities improve as d decreases. The distribution becomes more skewed to the left and has a faster decay rate.

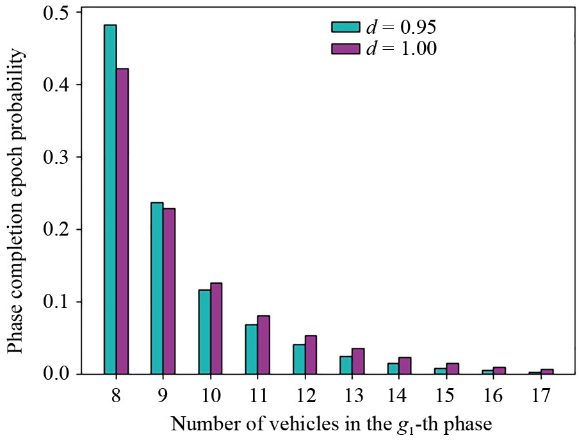

Figure 4. Comparison of phase completion epoch probabilities for different values of d

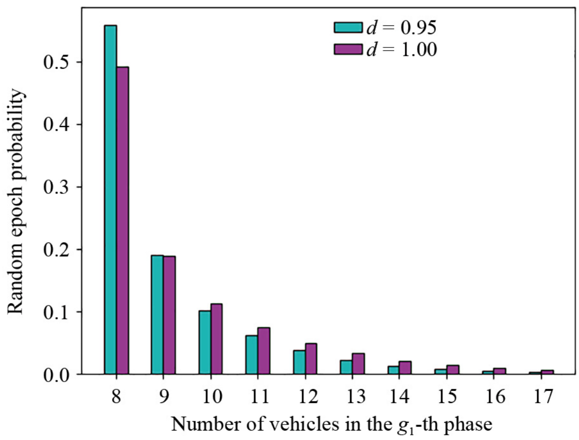

Figure 5. Comparison of random epoch probabilities for different values of d

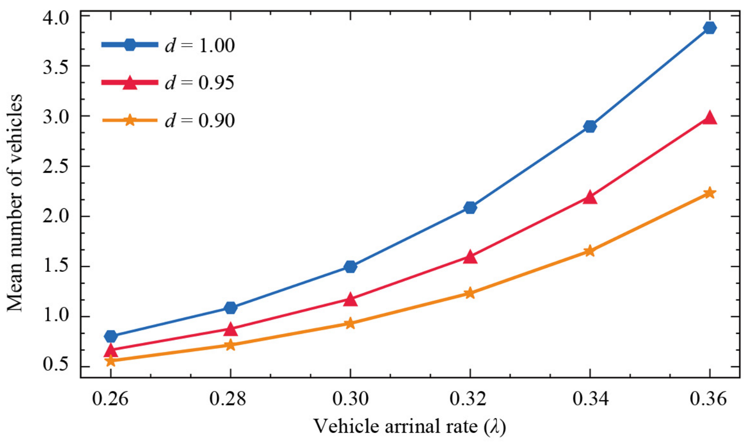

Figure 6. Effect of vehicle arrival rate (λ) on the mean number of vehicles at phase completion epoch, for the g_1-th phase

Figure 5 shows a similar effect as Figure 4, but for the random epoch probabilities. Figure 6 shows that when the vehicle arrival rate (λ) increases, then the mean number of vehicles ($L^+_{q,\, g_1}$) increases for $d = 1$, $d = 0.95$ and $d = 0.90$. Also, Figure 6 illustrates that the mean number of vehicles is increasing, if the time that a vehicle spends in the intersection increases. Both of these increases are nonlinear.

In Figure 7, as we increase the vehicle arrival rate (λ), the mean number of waiting vehicles ($L_{q,\, g_1}$) increases for $d = 1$, $d = 0.95$ and $d = 0.90$. Also, Figure 7 illustrates that the mean number of waiting vehicles increases, if the time that a vehicle spends in the intersection increases. Again, both increases are nonlinear.

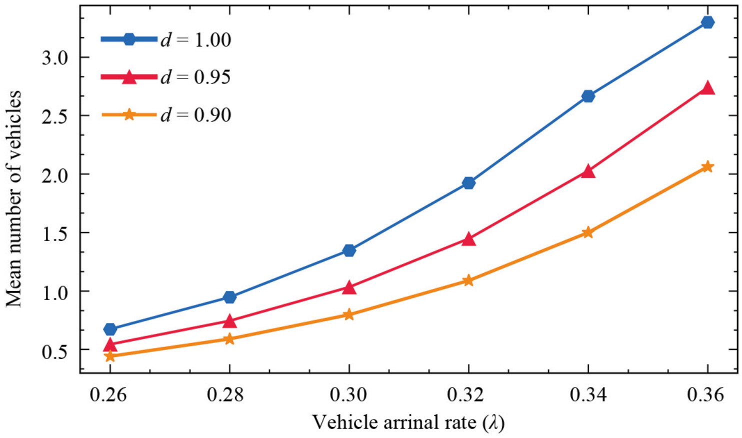

Figure 7. Effect of vehicle arrival rate (λ) on the mean number of vehicles at random epoch, for g_1-th phase

The CDF $W_{q,\, g_1}(t)$ of the waiting-time distribution of the vehicles in the g_1 phase can be calculated for the given model parameters $g = 3,\ r = 5,\ g_1 = 5,\ r_1 = 3,\ d = 1,\ N = 15,\ \lambda = 0.3$ as follows:

$$W_{q,\, g_1}(t) = 0.904354 + 0.402152e^{-0.95829t} - 1.30651e^{-0.177407t}\cos(0.0655691t)$$

$$-0.552572e^{-0.177407t}\sin(0.0655691t),\ t \geq 0. \tag{29}$$

In Table 2, we illustrate the estimated waiting-time distribution experienced by vehicles, showcasing variations across different values of d.

Table 2. Computation of CDF of approximate waiting time of vehicles in the g_1-th phase at different times and for different values of d (in the green phase) with parameters $g = 3,\ r = 5,\ N = 15$ and $g_1 = 5,\ r_1 = 3,\ \lambda = 0.3$

t	$W_{q,\, g_1}(t)$ for $d = 0.97$	$W_{q,\, g_1}(t)$ for $d = 0.98$	$W_{q,\, g_1}(t)$ for $d = 0.99$	$W_{q,\, g_1}(t)$ for $d = 1$
0	0.000000	0.000000	0.000000	0.000000
5	0.100420	0.222873	0.318186	0.324951
10	0.280551	0.465033	0.623578	0.671551
15	0.337002	0.549293	0.741351	0.821633
20	0.351717	0.574844	0.781720	0.879339
25	0.354101	0.580887	0.793472	0.898879
⋮	⋮	⋮	⋮	⋮

In Figure 8 it can be seen that if we increase the time that a vehicle spends in the intersection, then the vehicle waiting times increase in a nonlinear manner.

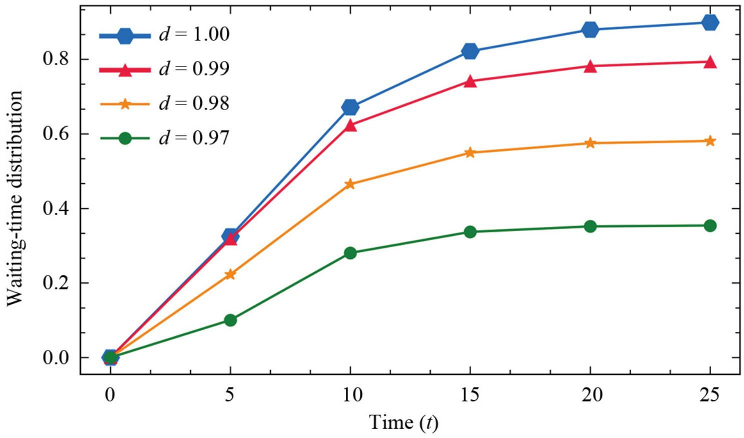

Figure 8. Effect of vehicle time in intersection (d) on the waiting time distribution

In Figure 9, we show a scenario where vehicles cross a traffic intersection within a fixed time phase of $d = 1$, and different vehicle arrival rates at the intersection are represented by λ taking on the values $0.26,\ 0.28,\ \ldots,\ 0.36$. We select the model parameters as $g = 3$, $g_1 = 5$, with the fixed cycle length $c = g + r = g_1 + r_1 = 8$, and $N = 15$. In this situation, a state-dependent scheduling mechanism works well. Figure 9 illustrates that as traffic intensity increases, the state-dependent scheduling mechanism performs better. Specifically, when traffic intensity rises, fewer vehicles are observed in the g_1-th phase compared to the g-th phase. This means our model works better by reducing vehicle waiting time and increasing traffic flow when the traffic intensity is higher in the given range of study.

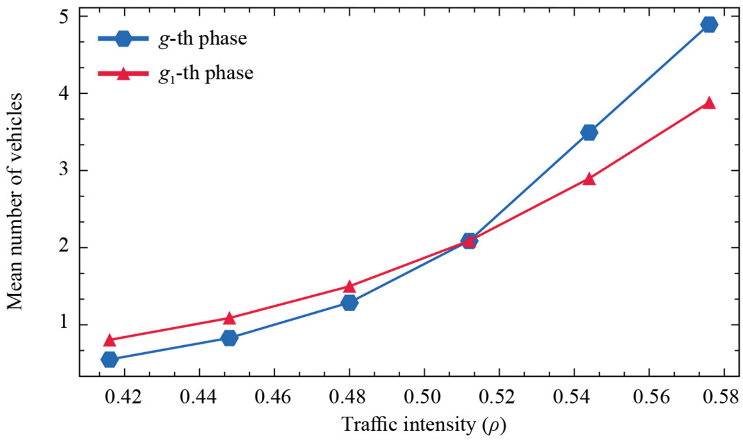

Figure 9. Comparison of mean number of vehicles in g-th phase and g_1-th phase

In Figure 10, a scenario is depicted where vehicles traverse a traffic intersection during a fixed time phase of duration $d = 1$, with different arrival rate of vehicles at the intersection represented by λ varies from $0.26,\ 0.28,\ \ldots,\ 0.36$. We select the model parameters as $g = 3$, $g_1 = 5$, with the fixed cycle length $c = g + r = g_1 + r_1 = 8$, and $N = 15$. In this context, a state-dependent scheduling mechanism demonstrates superior performance. Specifically, as traffic intensity increases, the mean waiting time during the g_1-phase is shorter compared to the g-th phase. This indicates that our model effectively reduces vehicle waiting times and enhances traffic flow when the traffic intensity is higher in the given range of study.

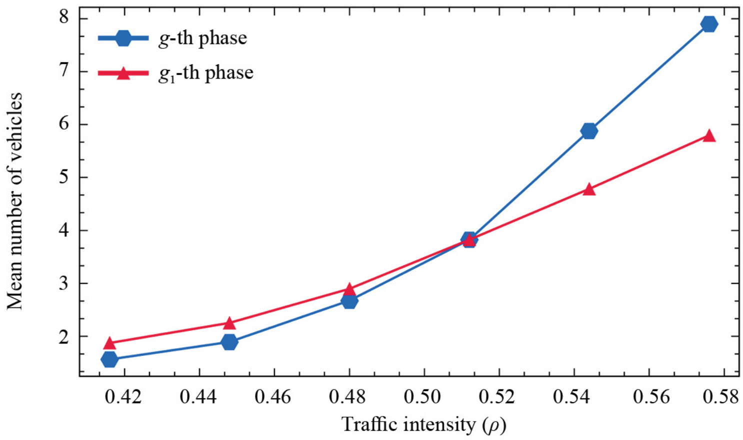

Figure 10. Comparison of mean waiting time of vehicles in g-th phase and g_1-th phase

5. Discussion and conclusion

In this paper, we propose a state-dependent scheduling mechanism in an FCTL queueing system, deriving the stationary probability distributions for the number of vehicles at random and phase completion epochs. We used a functional relationship involving the PGF of the random epoch probabilities and the LST of the queueing-time distribution to find the waiting-time distribution for each phase when the signal is green. Additionally, we derived the Cumulative Distribution Function (CDF) and probability density functions for the waiting time of vehicles for each time phase when the signal is green. While these results are mathematically elegant, they sometimes face skepticism because obtaining performance characteristics involves inverting the transform using root finding, which in the past has been considered challenging. Some might argue that Darroch's solution for the FCTL queue, as discussed in Section 2.1, falls into this category. Nowadays, Darroch's formula is used to calculate the mean overflow queue length and this job is considered to be quite tedious due to its numerical complexity (for more details, refer to [24, p.4]). We tend to disagree. Evaluating an expression numerically is subjective and depends on the prevailing mindset. In the early days of queueing theory (1950s, 1960s), root finding and inverting transforms were rightly seen as challenging, maybe even sometimes prohibitive. However, with better numerical algorithms and increased computational power, both challenges can be addressed effectively today, as discussed in Section 2.1 As stated in the introduction, various approximations have been proposed for the mean waiting-time in the FCTL queue. The results in Section 4 demonstrate that the differences in performance characteristics for different phase lengths d are considerable. This sensitivity should be useful for operations managers to keep in mind. Our model incorporates methods for managing situations when more than N vehicles are waiting. Lastly, we would like to highlight some potential extensions of our model. We believe that the framework demonstrated in this study could effectively be extended to a multi-dimensional model, accommodating scenarios where one traffic stream is green while others are red at an intersection. This approach may also be useful in a production inventory process where items arrive in the queue in batches of random size and pass through several phases of production where each phase duration may be different. This may lead to more complex FCTL queueing scenarios which are of practical interest.

Acknowledgment

The first author acknowledges full financial support from the University Grant Commission (UGC), New Delhi, India under the research grant 1260/(CSIR-UGC NET JUNE 2019). The second author is supported by the Discovery Grant program of the Natural Sciences and Engineering Research Council of Canada. The third author received partial financial support from the Department of Science and Technology (DST), New Delhi, India under the research grant MTR/2021/000287.

Conflict of interest

No potential conflict of interest was reported by the author(s).

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