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Research Article

Homotopy Perturbation ρ-Laplace Transform Approach for Numerical Simulation of Fractional Navier-Stokes Equations

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Abstract: In this study, we tackle the time fractional discrete Navier-Stokes equation by employing the homotopy perturbation ρ-Laplace transform method (HPLTM), utilizing the Caputo-Katugampola fractional derivative of time. Additionally, we present graphical representations of the solution generated using Matlab software, comparing it with the exact solution for $\alpha = 1$. We perform two test problems to verify and demonstrate the effectiveness of our approach. Our numerical findings and graphical analyses indicate that the proposed approach exhibits remarkable efficiency and simplicity, rendering it suitable for addressing a diverse array of challenges encountered in engineering and the sciences.

Keywords: Caputo-Katugampola fractional derivative, HPLTM, Mittag-Leffler function, fractional Navier-Stokes equation

MSC: 35R11, 35Q30, 65R10

1. Introduction

As a fact of matter, mathematicians, physicists and engineers are very interested in the fractional partial differential equations [1]. Nowadays, in many fields, including physics, engineering, electromagnetics, acoustics, viscoelasti-city, electrochemistry and material science, nonlinear fractional partial differential equations are utilized to describe significant phenomena and dynamic processes [2–12]. Besides, the solutions of fractional partial differential equations are of great importance for a better understanding of the processes described by fractional partial differential equations. In addition, the study of explicit and numerical solutions for these equations has attracted the attention of mathematicians and physicists in large part due to advancements emerged in computer technology and symbolic-numerical computing tools, as numerous analytical and numerical techniques that work well have been proposed, mentioning herein some of these methods: Adomian Decomposition Method (ADM) [13, 14], Finite Difference Method (FDM) [15], Homotopy Analysis Method [16–19], Homotopy Perturbatio Method (HPM) [20–22], and Differential transform method (DTM) [23–25]. However, for further information on fractional differential equations, including their types and history, existence and uniqueness of solutions, applications and techniques of solution, the reader is referred to [26–31]. More to the point, the homotopy perturbation approach, which was initially put out by Chinese scholar J. H. He in 1998, has shown to be the most significant of all the methods mentioned above, as this is caused by the fact that it tackles the issue head-on without requiring

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any kind of transformation, linearization or discriminating. Nevertheless, the Laplace transform stands for a popular method in mathematics for resolving differential equations. Definitely, transformations are useful in the solution of many mathematical problems, whereat the concept involves redefining the issue as a different and more manageable problem. Conversely, the inverse transform is useful for computing the solution to the given issue. Moreover, multiple authors [32]. solved fractional nonlinear differential equations using a combination of LT and ADM. The Navier-Stokes (NS) equation, a well-known governing equation of motion for viscous fluid flow, was developed in 1822 [33]. The equation, which combines the momentum, continuity and energy equations, can be thought of as Newton's second law of motion for fluid substances. More and more, numerous physical phenomena, including blood flow, liquid flow in pipes, ocean currents and air flow over an aircraft's wings, are all described by this equation. Likewise, NS equations were fractionally modeled in 2005 by El-Shahed and Salem [34]. The authors used the Laplace transform, finite Hankel transforms and finite Fourier Sine transforms to generalize the conventional NS equations. Several writers Rajarama Mohan et al. [35]. have solved fractional nonlinear differential equations (FNS) using the combination of ET and HPM. Nonetheless, by combining the iterative technique with Elzaki transform ET, the authors solved Navier-Stokes (NS) equation by Kumar et al. [36]. by (HPTM). Additionally, other publications [37, 38], use modified HPM to solve non-linear partial differential equations. A nonlinear fractional model of the NS problem was analytically solved by Kumar et al. [39]. by linking HPM and LTA. In this respect, HAM was used to solve the nonlinear time-fractional NS problem by Ragab et al. [40], and Ganji et al. [41]. For the numerical computation of the time-fractional NS equation, Momani and Odibat [42], and Birajdar [43], used ADM. the analytical solution of the time-fractional NS equation by connecting ADM and LTA. While Chaurasia and Kumar [44], solved the identical equation by coupling the Laplace transform and the finite Hankel transform, Kumar et al. As a result, the objective of this study is to provide an analytical approach known as the Fractional Modified Homotopy Perturbation Laplace Transform Method (HPLTM), which is developed to handle fractional differential equations prevalent in science and engineering. This method, HPLTM, combines the Homotopy Perturbation Method (HPM) and the Laplace transform, exploiting the characteristics of both techniques. The main advantage is its capacity to generate rapidly convergent series solutions for fractional partial differential equations by combining the power of these two methods.

The manuscript is structured as follows: Section 2 delves into fundamental concepts surrounding the Caputo-Katugampola fractional derivative and the ρ-Laplace transform, pertinent to the discussed issues. In Section 3, we employ the HPLTM approach to derive solutions for the time fractional Navier-Stokes equation. Section 4 applies our approach to a time fractional Navier-Stokes model to validate its effectiveness and accuracy. We present the results through graphical and numerical analyses. Finally, Section 5 offers conclusions drawn from our findings.

2. Basic definitions of fractional calculus

In this section, we provide the necessary definitions, properties, and lemmas of FC theory which will be used throughout this work.

Definition 1 [45] Consider a finite or infinite interval on the positive half-axis, denoted by $(0 < a < b < \infty)$, within the real numbers $\mathbb{R}^+$. The left-sided and right-sided Katugampola fractional integrals of order α, with $(\alpha > 0,\ \rho \in \mathbb{R}^+,\ n = [\alpha]+1)$ are defined by:

$$\mathfrak{I}_a^{\alpha,\,\rho}\ \psi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho\ - x^\rho}{\rho}\right)^{\alpha-1} \psi(x)\frac{dx}{x^{1-\alpha}} \tag{1}$$

$$\mathfrak{I}_b^{\alpha,\,\rho}\ \psi(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} \psi(x)\frac{dx}{x^{1-\alpha}} \tag{2}$$

respectively, if the integrals exist.

The left-sided and right-sided Katugampola fractional derivatives of order α, with $(\alpha > 0,\ \rho \in \mathbb{R}^+,\ n = [\alpha]+1)$, are defined by the following:

$$ {}_t^K\mathbb{D}_a^{\alpha,\,\rho}\,\psi(t) = (\delta)^n \mathfrak{I}_a^{n-\alpha;\,\rho}\ \psi(t) = \frac{(\delta)^n}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho\ - x^\rho}{\rho}\right)^{n-\alpha-1} \psi(x)\frac{dx}{x^{1-\rho}} \tag{3} $$

$$ {}_t^K\mathbb{D}_b^{\alpha,\,\rho}\,\psi(t) = (-\delta)^n \mathfrak{I}_b^{n-\alpha;\,\rho}\ \psi(t) = \frac{(-\delta)^n}{\Gamma(n-\alpha)} \int_t^b \left(\frac{x^\rho\ - t^\rho}{\rho}\right)^{n-\alpha-1} \psi(x)\frac{dx}{x^{1-\rho}} \tag{4} $$

respectively, where $\delta = t^{1-\rho}\dfrac{d}{t}$, $\Gamma(.)$ represents the Gamma function.

$$ \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt. $$

Remark 1 The number α can be taken as a complex number with a positive real part in the previous definition.

2.1 *Caputo-type Katugampola fractional derivatives*

We define the Caputo-type modification of left-sided and right-sided Katugampola fractional derivatives as:

Theorem 1 [46] Let $(0 < a < b < \infty)$, $(\alpha > 0,\ \ n = [\alpha]+1)$. If $\psi \in AC_\delta^n([a,\ b])$, where

$$ AC_\delta^n([a,\ b]) = \left\{\psi : [a,\ b] \to \mathbb{C};\ \delta^{n-1}\psi(x) \in AC[a,\ b],\ \delta = x^{1-\rho}\frac{d}{x}\right\}, $$

and $AC[a,\ b]$: be the spaces of absolutely continuous functions on $[a,\ b]$, then ${}_t^{CK}\mathbb{D}_a^{\alpha,\,\rho}\,\psi(t)$, ${}_t^{CK}\mathbb{D}_b^{\alpha,\,\rho}\,\psi(t)$ exists everywhere on $[a,\ b]$ and Caputo-type Katugampola fractional derivatives are defined by the following:

$$ {}_t^{CK}\mathbb{D}_a^{\alpha,\,\rho}\,\psi(t) = \mathfrak{I}_a^{n-\alpha;\,\rho}\ (\delta)^n\psi(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho\ - x^\rho}{\rho}\right)^{n-\alpha-1} (\delta)^n\psi(x)\frac{dx}{x^{1-\rho}} \tag{5} $$

$$ {}_t^{CK}\mathbb{D}_b^{\alpha,\,\rho}\,\psi(t) = \mathfrak{I}_b^{n-\alpha;\,\rho}\ (-\delta)^n\psi(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\frac{x^\rho\ - t^\rho}{\rho}\right)^{n-\alpha-1} (-\delta)^n\psi(x)\frac{dx}{x^{1-\rho}} \tag{6} $$

Specifically, when $\rho = 1$ in this definition, we obtain the Caputo fractional derivative. Furthermore, and as $\rho \to 0$, we obtain the Caputo Hadamard fractional derivative as defined in [32].

Theorem 2 [46] Let $\psi \in C([a,\ b])$, $\alpha > \beta > 0$ and $\rho > 0$, $a < t < b$ then

$$ {}_t^K\mathbb{D}_{a+}^{\alpha}\mathfrak{I}_{a+}^{\alpha}\psi(t) = \psi(t) \tag{7} $$

Nevertheless, as demonstrated below, it becomes evident that the Katugampola fractional derivative does not serve as the true inverse of the Katugampola fractional integral.

Theorem 3 [46] Let $\mathfrak{I}_{a+}^{n-\alpha}\psi \in AC^n([a,\ b])$, and $n-1 < \alpha\ \le n,\ \rho > 0,\ n \in \mathbb{N}$, then

$$\mathfrak{I}_{a+}^{\alpha,\ \rho}\, {}_{t}^{K}\mathbb{D}_{a}^{\alpha,\ \rho}\psi(t)=\psi(t)-\sum_{k=0}^{n-1}c_k\left(\frac{t^{\rho}\ -x^{\rho}}{\rho}\right)^{k-n+\alpha}, \tag{8}$$

where c_k are real constants.

Definition 2 A two-parameter Mittag-Leffler function $E_{\alpha,\ m}(t)$, is defined by the following series:

$$E_{\alpha,\ m}(t)=\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m\alpha+1)},\ \alpha,\ m,\ t\in\mathbb{R},\ \text{with}\ \alpha,\ m>\ 0. \tag{9}$$

If $\beta\ =\ 1$, we have the one-parameter Mittag-Leffler function:

$$E_{\alpha}(t)=E_{\alpha,\ 1}(t)=\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m\alpha+1)}.$$

Theorem 4 [47] The ρ-Laplace transform of a function ψ defined on $[0,\ \infty)$ is given as:

$$G(s)=\mathscr{L}_{\rho}\left[\psi(t)\right]=\int_{0}^{+\infty}e^{-s\frac{t^{\rho}}{\rho}}\psi(t)\frac{dt}{t^{1-\rho}},\ \rho>0 \tag{10}$$

The inverse amended ρ-Laplace transform is given by:

$$\psi(t)=\mathscr{L}_{\rho}{}^{-1}\left[G(s)\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{-s\frac{t^{\rho}}{\rho}}G(s)\frac{ds}{s},\ c=\Re(s). \tag{11}$$

Theorem 5 [47] Let the function ψ be continuous and of exponential order $e^{-c\frac{t^{\rho}}{\rho}}$ such that $\delta\psi$ is piecewise continuous over every finite interval $[0,\ T]$. Then ρ-Laplace transform of $\delta\psi$ exists for $s\ >\ c$:

$$\mathscr{L}_{\rho}\left[\delta\psi(t)\right](s)=s\mathscr{L}_{\rho}\left[\psi(t)\right](s)\times\psi(0). \tag{12}$$

Theorem 6 [47] ρ-convolution of ψ and g is given as:

$$\psi(t)*g(t)=\int_{0}^{t}\psi\left(\left(t^{\rho}-s^{\rho}\right)^{\frac{1}{\rho}}\right)g(s)\frac{ds}{s^{1-\rho}} \tag{13}$$

Theorem 7 [47] (ρ-Convolution theorem)

$$\mathscr{L}_{\rho}\left\{\psi(t)*g(t)\right\}=\mathscr{L}_{\rho}\left[\psi(t),\ s\right]\mathscr{L}_{\rho}\left[g(t),\ s\right]=F(s)G(s), \tag{14}$$

or equivalently,

$$\mathscr{L}_\rho{}^{-1}\{F(s)G(s),\, t\} = \psi(t) * g(t). \tag{15}$$

Theorem 8 [47] Let $\alpha > 0$ and ψ be a piecewise continuous function on each interval $[0, t]$ and of ρ-exponential order $e^{-c\frac{t^\rho}{\rho}}$. Then:

$$\mathscr{L}_\rho\left\{\mathfrak{I}_0^{\alpha;\, \rho}\ \psi(t),\, s\right\} = \frac{\mathscr{L}_\rho\left[\psi(t)\right]}{s^\alpha},\ s > c. \tag{16}$$

Theorem 9 [47] Let $\alpha > 0$ and $\psi \in AC_\delta^{n-1}[0, a]$ for any $a > 0$ and $\delta^k\psi, k = 0, 1, 2, \ldots, n-1$ be of ρ-exponential order $e^{-c\frac{t^\rho}{\rho}}$. Then

$$\mathscr{L}_\rho\left\{\left({}_t^C\mathbb{D}_0^{\alpha,\, \rho}\psi(t),\, s\right)\right\} = s^\alpha\left\{\mathscr{L}_\rho\{\psi(t)\} - \sum_{k=0}^{n-1}(s)^{-k-1}\left(\delta^k\psi\right)(0)\right\},\ s > c. \tag{17}$$

Applying ρ-Laplace transform on both sides of Equation (5), we find ρ-Laplace transform of the derived Caputo-Katugampola fraction is given as follows:

$$\mathscr{L}_\rho\left\{\left({}_t^{CK}\mathbb{D}_0^{\alpha,\, \rho}\psi(t),\, s\right)\right\} = s^\alpha\left\{\mathscr{L}_\rho\{\psi(t)\}\right\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}\left(\delta^k\psi\right)(0). \tag{18}$$

Lemma 1 [47, 48] $\Re(\alpha) > 0$ and $\left|\frac{\lambda}{s^\alpha}\right| < 1$, the ρ-Laplace transform of some special functions are as below:

- $\mathscr{L}_\rho\{1\} = \frac{1}{s},\ s > 0.$
- $\mathscr{L}_\rho\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$
- $\mathscr{L}_\rho\left\{E_\alpha\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right\} = \frac{s^{\alpha-1}}{(s^\alpha-\lambda)}.$
- $\mathscr{L}_\rho\left\{\left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\, \alpha}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right\} = \frac{1}{(s^\alpha-\lambda)}.$

3. Homotopy perturbation ρ-Laplace transform method (HPLTM)

To elucidate the concept of HPLTM, we consider the following fractional-order nonlinear non-homogeneous partial differential equation along with its initial condition (IC):

$${}_t^{CK}\mathbb{D}_0^{\alpha,\, \rho}\psi(x, y, z, t) + R\psi(x, y, z, t) + N\psi(x, y, z, t) = g(x, y, z, t),\ 0 < \alpha \le 1 \tag{19}$$

$$\psi(x, y, z, 0) = h(x, y, z). \tag{20}$$

The expression ${}_{t}^{CK}\mathbb{D}_{0}^{\alpha}\psi$ represents the derivative of ψ in the Caputo sense, where R and N denote linear and nonlinear differential operators, respectively, and g stands for the source term. Upon applying the Laplace Transform to both sides of Equation (19), we obtain:

$$\mathscr{L}_{\rho}\left\{{}_{t}^{CK}\mathbb{D}_{0}^{\alpha,\,\rho}\psi(x,\,y,\,z,\,t)+R\psi(x,\,y,\,z,\,t)+N\psi(x,\,y,\,z,\,t)\right\}=\mathscr{L}_{\rho}\left\{g(x,\,y,\,z,\,t)\right\}, \tag{21}$$

$$\mathscr{L}_{\rho}\left\{{}_{t}^{CK}\mathbb{D}_{0}^{\alpha,\,\rho}\psi(x,\,y,\,z,\,t)\right\}=\mathscr{L}_{\rho}\left\{-R\psi(x,\,y,\,z,\,t)-N\psi(x,\,y,\,z,\,t)\right\}+\mathscr{L}_{\rho}\left\{g(x,\,y,\,z,\,t)\right\}. \tag{22}$$

Next, by leveraging the differential property of the ρ-Laplace transform for the fractional derivative:

$$\begin{aligned}\mathscr{L}_{\rho}\left\{{}_{t}^{CK}\mathbb{D}_{0}^{\alpha,\,\rho}\psi(x,\,y,\,z,\,t)\right\}&=s^{\alpha}\left\{\mathscr{L}_{\rho}\left\{\psi(t)\right\}\right\}-\sum_{k=0}^{n-1}s^{\alpha-k-1}\left(\delta^{k}\psi\right)(0)\\&=\mathscr{L}_{\rho}\left\{-R\psi(x,\,y,\,z,\,t)-N\psi(x,\,y,\,z,\,t)\right\}+\mathscr{L}_{\rho}\left\{g(x,\,y,\,z,\,t)\right\}.\end{aligned} \tag{23}$$

On simplifying of Equation (23), we have:

$$\mathscr{L}_{\rho}\left\{\psi(x,\,y,\,z,\,t)\right\}=\sum_{k=0}^{n-1}s^{-k-1}\left(\delta^{k}\psi\right)(0)+\frac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{g(x,\,y,\,z,\,t)\right\}-\mathscr{L}_{\rho}\left\{R\psi(x,\,y,\,z,\,t)+N\psi(x,\,y,\,z,\,t)\right\}\right\}. \tag{24}$$

Operating the inverse ρ- Laplace transform on both sides in Equation (24), we get:

$$\psi(x,\,y,\,z,\,t)=G(x,\,y,\,z,\,t)-\mathscr{L}_{\rho}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{R\psi(x,\,y,\,z,\,t)+N\psi(x,\,y,\,z,\,t)\right\}\right\}\right]. \tag{25}$$

Here, $G(x,\,y,\,z,\,t)$ denotes the term derived from the initial condition and source term.
Upon employing the HPM to Equation (25), we obtain:

$$\psi(x,\,y,\,z,\,t)=G(x,\,y,\,z,\,t)-p\left(\mathscr{L}_{\rho}^{-1}\left[\frac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{R\psi(x,\,y,\,z,\,t)+N\psi(x,\,y,\,z,\,t)\right\}\right\}\right]\right). \tag{26}$$

The homotopy parameter p is employed to expand the solution as follows:

$$\psi(x,\,y,\,z,\,t)=\sum_{n=0}^{\infty}p^{n}\psi_{n}(x,\,y,\,z,\,t). \tag{27}$$

The nonlinear term is decomposed as follows:

$$N\psi(x,\,y,\,z,\,t)=\sum_{n=0}^{\infty}p^{n}H_{n}(\psi). \tag{28}$$

where $H_n(\psi)$ is He's polynomials and is given by:

$$H_n(\psi_0,\ \psi_1,\ \psi_2,\ \dots\ \psi_n) = \frac{1}{n!}\frac{\partial}{\partial p^n}\left[N\sum_{n=0}^{\infty} p^n \psi_n\right]. \tag{29}$$

Substituting Equations (27) and (28) in Equation (26), we get:

$$\sum_{n=0}^{\infty} p^n \psi_n(x,\ y,\ z,\ t) = G(x,\ y,\ z,\ t)$$

$$-p\left(\mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{R\sum_{n=0}^{\infty} p^n\psi_n(x,\ y,\ z,\ t)+N\sum_{n=0}^{\infty} p^n H_n(\psi)\right\}\right\}\right]\right). \tag{30}$$

By comparing the coefficients of identical powers of p on both sides of the aforementioned equation, we derive the following equations:

$$p^0 : \psi_0(x,\ y,\ z,\ t) = G(x,\ y,\ z,\ t), \tag{31}$$

$$p^1 : \psi_1(x,\ y,\ z,\ t) = \mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{R\psi_0(x,\ y,\ z,\ t)+H_0(\psi)\right\}\right\}\right], \tag{32}$$

$$p^2 : \psi_2(x,\ y,\ z,\ t) = \mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{R\psi_1(x,\ y,\ z,\ t)+H_1(\psi)\right\}\right\}\right], \tag{33}$$

Finally, we find the solution $\psi_n(x,\ y,\ z,\ t)$ as follows, which can be written as follows:

$$\psi(x,\ y,\ z,\ t) = \psi_0(x,\ y,\ z,\ t)+\psi_1(x,\ y,\ z,\ t)+\psi_2(x,\ y,\ z,\ t)+\psi_3(x,\ y,\ z,\ t)+\dots \tag{34}$$

3.1 *Convergence analysis and error estimation*

The convergence of HPM towards a solution for the fractional NS equation, as well as the error estimation of HPM, are established through the following two theorems. Consider $\Omega \in \mathbb{R}^n$ be an opened and bounded domain, and let T be a positive constant with $0 < T \leq \infty$. To illustrate the idea of HPM technique, let us consider the fractional NS equation: for any: $(x,\ y,\ z,\ t) \in \Omega \times [0,\ T]$.

Theorem 10 [49] Let $\psi_n(x,\ y,\ z,\ t)$ be the function in a Banach space $C(\Omega \times [0,\ T])$ defined by Equation (34) for any $n \in \mathbb{N}$. The infinite series $\sum_{k=0}^{\infty} \psi_k(x,\ y,\ z,\ t)$ converges to the solution ψ of Equation (19) if there exists a constant $0 < \mu < 1$ such that $\psi_n(x,\ y,\ z,\ t) \leq \mu\psi_{n-1}(x,\ y,\ z,\ t)$ for all $n \in \mathbb{N}$. Define that $\{S_n\}_{n=0}^{\infty}$ is the sequence of the partial sums of the series $\sum_{k=0}^{\infty} \psi_k(x,\ y,\ z,\ t)$. Thus, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $C(\Omega \times [0,\ T])$, consequently, the solution $\sum_{k=0}^{\infty} \psi_k(x,\ y,\ z,\ t)$ converge to ψ.

Next, we give the theorem to truncate an inaccurate solution as follows:

Theorem 11 [49] The maximum absolute error of the series solution, defined in Equation (34), is estimated as:

$$\left|\psi(x,\, y,\, z,\, t) - \sum_{k=0}^{\infty}\psi_k(x,\, y,\, z,\, t)\right| \leq \left(\frac{\mu^{m+1}}{1-\mu}\right)\|\psi_0\|.$$

4. Application of HPLTM on FNS equation

In this section, we apply the homotopy perturbation ρ-Laplace transform method (HPLTM) for solving time-fractional Navier-Stokes equation. Moreover, the time fractional Navier-Stokes equation, characterized by a constant density ρ_1 and a kinematic viscosity $v = \dfrac{\eta}{\rho_1}$, is expressed as follows [50]:

$$\begin{cases} {}^{CK}_{t}\mathbb{D}^{\alpha,\,\rho}_{0}U(x,\, y,\, z,\, t) + (U(x,\, y,\, z,\, t)\cdot\nabla)U(x,\, y,\, z,\, t) = \rho_0\nabla^2 U(x,\, y,\, z,\, t) - \dfrac{1}{\rho_1}\nabla P,\ \Omega\times(0,\, T), \\ \nabla U(x,\, y,\, z,\, t) = 0, \\ U(x,\, y,\, z,\, 0) = 0.\ 0 < \alpha \leq 1. \end{cases} \tag{35}$$

Here, U represents the velocity vector, P denotes pressure, and v, calculated as $\dfrac{\eta}{\rho_1}$, stands for the kinematic viscosity, where η represents dynamic viscosity, and ρ_1 signifies density. Additionally, the term ${}^{CK}_{t}\mathbb{D}^{\alpha}_{0}U$ corresponds to local acceleration, capturing changes in velocity over time at a fixed point within the flow, while the expression $(U(x,\, y,\, z,\, t)\cdot\nabla)U(x,\, y,\, z,\, t)$ denotes convective acceleration, depicting alterations in velocity as particles transition across infinitesimal space within the flow field. The component $\dfrac{1}{\rho_1}\nabla P$ denotes the pressure term, indicating fluid movement towards areas of greatest pressure change. The term $\rho_0\nabla^2 U$ represents the viscous term, reflecting frictional forces induced by viscosity acting on fluid particles in motion. Notably, both pressure and viscous forces act externally on the fluid particle's surface. The domain, denoted as $\Omega = (-3,\, 3)\times(0,\, 3)$, encompasses the boundary $\partial\Omega$.

Formulation of Equation (35) in Cartesian coordinates following on $x,\ y$ and z is written as:

$$\begin{cases} {}^{CK}_{t}\mathbb{D}^{\alpha,\,\rho}_{0}U(x,\, y,\, z,\, t) + U\dfrac{\partial U}{\partial x} + V\dfrac{\partial U}{\partial y} + W\dfrac{\partial U}{\partial z} = -\dfrac{1}{\rho_1}\dfrac{\partial P}{\partial x} + \rho_0\left(\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2}\right), \\ {}^{CK}_{t}\mathbb{D}^{\alpha,\,\rho}_{0}U(x,\, y,\, z,\, t) + U\dfrac{\partial V}{\partial x} + V\dfrac{\partial V}{\partial y} + W\dfrac{\partial V}{\partial z} = -\dfrac{1}{\rho_1}\dfrac{\partial P}{\partial y} + \rho_0\left(\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}\right), \\ {}^{CK}_{t}\mathbb{D}^{\alpha,\,\rho}_{0}U(x,\, y,\, z,\, t) + U\dfrac{\partial W}{\partial x} + V\dfrac{\partial W}{\partial y} + W\dfrac{\partial W}{\partial z} = -\dfrac{1}{\rho_1}\dfrac{\partial P}{\partial z} + \rho_0\left(\dfrac{\partial^2 W}{\partial x^2} + \dfrac{\partial^2 W}{\partial y^2} + \dfrac{\partial^2 W}{\partial z^2}\right), \\ \dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y} + \dfrac{\partial W}{\partial z} = 0. \end{cases} \tag{36}$$

4.1 *Applications: time-fractional navier-stokes equations*

Provided the value of P is known, then all the values of $g_1 = -\dfrac{1}{\rho_1}\dfrac{\partial P}{\partial x}$ $g_2 = \dfrac{1}{\rho_1}\dfrac{\partial P}{\partial y}$ and $g_3 = -\dfrac{1}{\rho_1}\dfrac{\partial P}{\partial z}$ can be determined.

Application 1 From Equation (36), 2d Navier-Stokes equation of fractional order with $g_1 = -g_2$ may be written as:

$$\begin{cases} {}_{t}^{CK}\mathbb{D}_{0}^{\alpha,\ \rho}U(x,\ y,\ t)+U\dfrac{\partial U}{\partial x}+V\dfrac{\partial U}{\partial y}=+g_1+\rho_0\left(\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}\right), \\ {}_{t}^{CK}\mathbb{D}_{0}^{\alpha,\ \rho}V(x,\ y,\ t)+U\dfrac{\partial V}{\partial x}+V\dfrac{\partial V}{\partial y}=-g_2+\rho_0\left(\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}\right), \\ \dfrac{\partial U}{\partial x}+\dfrac{\partial V}{\partial y}=0. \end{cases} \tag{37}$$

with IC: $U(x,\ y,\ 0)=sin(x+y),\ V(x,\ y,\ 0)=-sin(x+y)$.

Applying LT on both sides of Equation (37), we get:

$$\mathscr{L}_\rho\left\{U(x,\ y,\ t)\right\}=\frac{1}{s}\sin(x+y)+\ \frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+g_1-U\frac{\partial U}{\partial x}-V\frac{\partial U}{\partial y}\right\}\right\}, \tag{38}$$

$$\mathscr{L}_\rho\left\{V(x,\ y,\ t)\right\}=-\frac{1}{s}\sin(x+y)+\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)-g_2-U\frac{\partial V}{\partial x}-V\frac{\partial V}{\partial y}\right\}\right\}. \tag{39}$$

The inverse ρ-Laplace transform of Equations (38) and (39) implies that:

$$U(x,\ y,\ t)=sin(x+y)+\mathscr{L}_\rho{}^{-1}\ \left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+g_1-U\frac{\partial U}{\partial x}-V\frac{\partial U}{\partial y}\right\}\right\}\right], \tag{40}$$

$$V(x,\ y,\ t)=-sin(x+y)+\mathscr{L}_\rho{}^{-1}\ \left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)-g_2-U\frac{\partial V}{\partial x}-V\frac{\partial V}{\partial y}\right\}\right\}\right]. \tag{41}$$

Simplifying Equations (40) and (41), we get:

$$U(x,\ y,\ t)=sin(x+y)+g_1\frac{\left(\dfrac{t^\rho\ -a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}+\mathscr{L}_\rho{}^{-1}\ \left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)-U\frac{\partial U}{\partial x}-V\frac{\partial U}{\partial y}\right\}\right\}\right], \tag{42}$$

$$V(x,\ y,\ t)=-sin(x+y)-g_2\frac{\left(\dfrac{t^\rho\ -a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}+\mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)-U\frac{\partial V}{\partial x}-V\frac{\partial V}{\partial y}\right\}\right\}\right]. \tag{43}$$

Now, applying the homotopy perturbation method, we have:

$$\sum_{n=0}^{\infty} p^n U_n(x,\ y,\ t) = sin(x+y) + g_1 \frac{\left(\dfrac{t^\rho \ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)} + p\left[\mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{N\sum_{n=0}^{\infty} p^n H_n(U)\right\}\right\}\right]\right], \tag{44}$$

$$\sum_{n=0}^{\infty} p^n V_n(x,\ y,\ t) = -sin(x+y) - g_2 \frac{\left(\dfrac{t^\rho \ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)} + p\left[\mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{N\sum_{n=0}^{\infty} p^n H_n(V)\right\}\right\}\right]\right]. \tag{45}$$

where $H_n(U)$ and $H_n(V)$ are He's polynomials which signifies the nonlinear terms.

$$\begin{cases} H_n(U) = H_n(U_0,\ U_1,\ U_2,\ \ldots\ U_n) = \dfrac{1}{n!}\dfrac{\partial}{\partial p^n}\left[N\sum_{n=0}^{\infty} p^n U_n\right], \\ \\ H_n(V) = H_n(V_0,\ V_1,\ V_2,\ \ldots\ V_n) = \dfrac{1}{n!}\dfrac{\partial}{\partial p^n}\left[N\sum_{n=0}^{\infty} p^n V_n\right]. \end{cases} \quad n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{46}$$

where:

$$\begin{cases} U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \ldots \\ V = V_0 + pV_1 + p^2V_2 + p^3V_3 + \ldots \end{cases} \tag{47}$$

The first few components of He's polynomials are given as:

$$\begin{cases} H_0(U) = \rho_0\left(\dfrac{\partial^2 U_0}{\partial x^2} + \dfrac{\partial^2 U_0}{\partial y^2}\right) - U_0\dfrac{\partial U_0}{\partial x} - V_0\dfrac{\partial U_0}{\partial y}, \\ \\ H_0(V) = \rho_0\left(\dfrac{\partial^2 V_0}{\partial x^2} + \dfrac{\partial^2 V_0}{\partial y^2}\right) - U_0\dfrac{\partial V_0}{\partial x} - V_0\dfrac{\partial V_0}{\partial y}, \end{cases} \tag{48}$$

$$\begin{cases} H_1(U) = \rho_0\left(\dfrac{\partial^2 U_1}{\partial x^2} + \dfrac{\partial^2 U_1}{\partial y^2}\right) - U_0\dfrac{\partial U_1}{\partial x} - V_0\dfrac{\partial U_1}{\partial y} - U_1\dfrac{\partial U_0}{\partial x} - V_1\dfrac{\partial U_0}{\partial y}, \\ \\ H_1(V) = \rho_0\left(\dfrac{\partial^2 V_1}{\partial x^2} + \dfrac{\partial^2 V_1}{\partial y^2}\right) - U_0\dfrac{\partial V_1}{\partial x} - V_0\dfrac{\partial V_1}{\partial y} - U_1\dfrac{\partial V_0}{\partial x} - V_1\dfrac{\partial V_0}{\partial y}, \end{cases} \tag{49}$$

$$\begin{cases} H_2(U) = \rho_0\left(\dfrac{\partial^2 U_2}{\partial x^2} + \dfrac{\partial^2 U_2}{\partial y^2}\right) - \ldots, \\ \\ H_2(V) = \rho_0\left(\dfrac{\partial^2 V_2}{\partial x^2} + \dfrac{\partial^2 V_2}{\partial y^2}\right) - \ldots. \end{cases} \tag{50}$$

Equating the coefficients of like powers of p in Equations (44) and (45), we get the following results:

$$p^0 : \begin{cases} U_0(x,\ y,\ t) = sin(x+y) + g_1 \dfrac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}, \\ V_0(x,\ y,\ z,\ t) = -sin(x+y) - g_2 \dfrac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}, \end{cases} \tag{51}$$

$$p^1 : \begin{cases} U_1(x,\ y,\ z,\ t) = \mathscr{L}_\rho{}^{-1}\left[\dfrac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{H_0(U)\right\}\right\}\right] = -2\rho_0 \sin(x+y) \dfrac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}, \\ V_1(x,\ y,\ t) = \mathscr{L}_\rho{}^{-1}\left[\dfrac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{H_0(V)\right\}\right\}\right] = 2\rho_0 \sin(x+y) \dfrac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}, \end{cases} \tag{52}$$

$$p^2 : \begin{cases} U_2(x,\ y,\ t) = \mathscr{L}_\rho{}^{-1}\left[\dfrac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{H_1(U)\right\}\right\}\right] = 4{\rho_0}^2 \sin(x+y) \dfrac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^{2\alpha}}{\Gamma(2\alpha+1)}, \\ V_2(x,\ y,\ t) = \mathscr{L}_\rho{}^{-1}\left[\dfrac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{H_1(V)\right\}\right\}\right] = -4{\rho_0}^2 \sin(x+y) \dfrac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases} \tag{53}$$

$$p^3 : \begin{cases} U_3(x,\ y,\ t) = \mathscr{L}_\rho{}^{-1}\left[\dfrac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{H_2(U)\right\}\right\}\right] = -8{\rho_0}^3 \sin(x+y) \dfrac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^{3\alpha}}{\Gamma(3\alpha+1)}, \\ V_3(x,\ y,\ t) = \mathscr{L}_\rho{}^{-1}\left[\dfrac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{H_2(V)\right\}\right\}\right] = 8{\rho_0}^3 \sin(x+y) \dfrac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^{3\alpha}}{\Gamma(3\alpha+1)}, \end{cases} \tag{54}$$

$$\vdots$$

Thus, the solutions $U(x,\ y,\ t)$ and $V(x,\ y,\ t)$ are written in the form of:

$$\begin{cases} U(x,\ y,\ t) = U_0(x,\ y,\ t) + U_1(x,\ y,\ t) + U_2(x,\ y,\ t) + U_3(x,\ y,\ t) + \ldots \\ \qquad\qquad\quad = sin(x+y)e^{-2\rho_0\left(\frac{t^\rho\ -a^\rho}{\rho}\right)^\alpha} + g_1\frac{\left(\dfrac{t^\rho\ -a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}, \\ V(x,\ y,\ t) = V_0(x,\ y,\ t) + V_1(x,\ y,\ t) + V_2(x,\ y,\ t) + V_3(x,\ y,\ t) + \ldots \\ \qquad\qquad\quad = -sin(x+y)e^{-2\rho_0\left(\frac{t^\rho\ -a^\rho}{\rho}\right)^\alpha} - g_2\frac{\left(\dfrac{t^\rho\ -a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}. \end{cases} \tag{55}$$

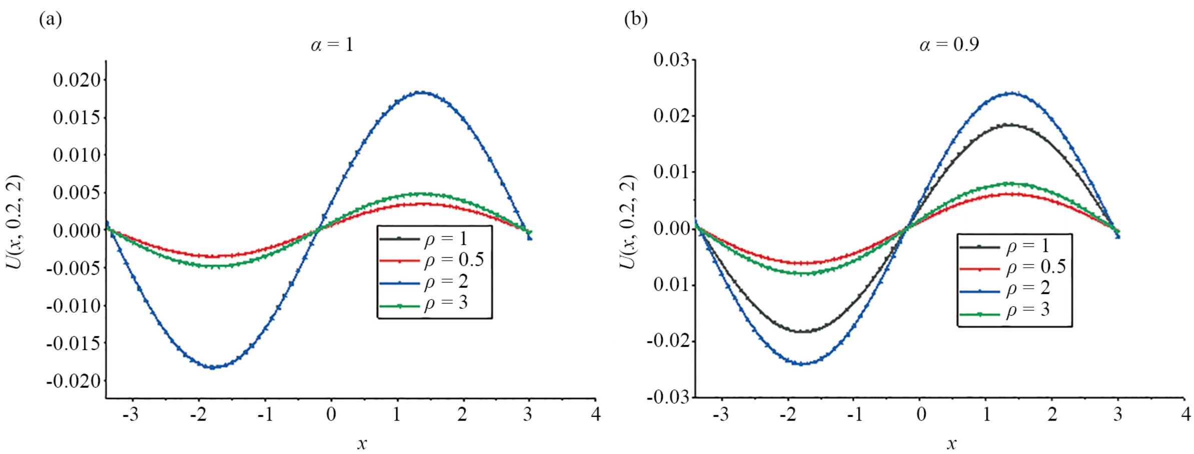
Finally, we work on writing infinite sums in terms of the Mittag-Leffler function, the Equation (9) is:

$$\begin{cases} U(x,\ y,\ t) = sin(x+y)\mathbb{E}_{\alpha,\ 1}\left(-2\rho_0\left(\dfrac{t^\rho\ -a^\rho}{\rho}\right)^\alpha\right) + g_1\dfrac{\left(\dfrac{t^\rho\ -a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}, \\ V(x,\ y,\ t) = -sin(x+y)\mathbb{E}_{\alpha,\ 1}\left(-2\rho_0\left(\dfrac{t^\rho\ -a^\rho}{\rho}\right)^\alpha\right) - g_2\dfrac{\left(\dfrac{t^\rho\ -a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}. \end{cases} \tag{56}$$

These solutions are in agreement with the solutions found by [35], using (HPETM) and operator The Riemann-Liouville fractional integral, which are also are in agreement with the solutions found by by [36], using (FRDTM). The plots of Equation (56) are depicted in Figures 1-5, for different values of $\rho = 1,\ 0.5,\ 2,\ 3,\ \alpha =\ 1,\ 0.9,\ 0.6,\ 0.3,\ \rho_0 = 1,\ a = 0$, and $g_1 = g_2 = 0$.

And $g_1 = g_2 = 0$, $\rho = \alpha = 1,\ \ a = 0$, Equation (56) the exact solution of classical NS equation for the velocity.

$$\begin{cases} U(x,\ y,\ t) = sin(x+y)e^{-2\rho_0 t}, \\ V(x,\ y,\ t) = -sin(x+y)e^{-2\rho_0 t}. \end{cases} \tag{57}$$



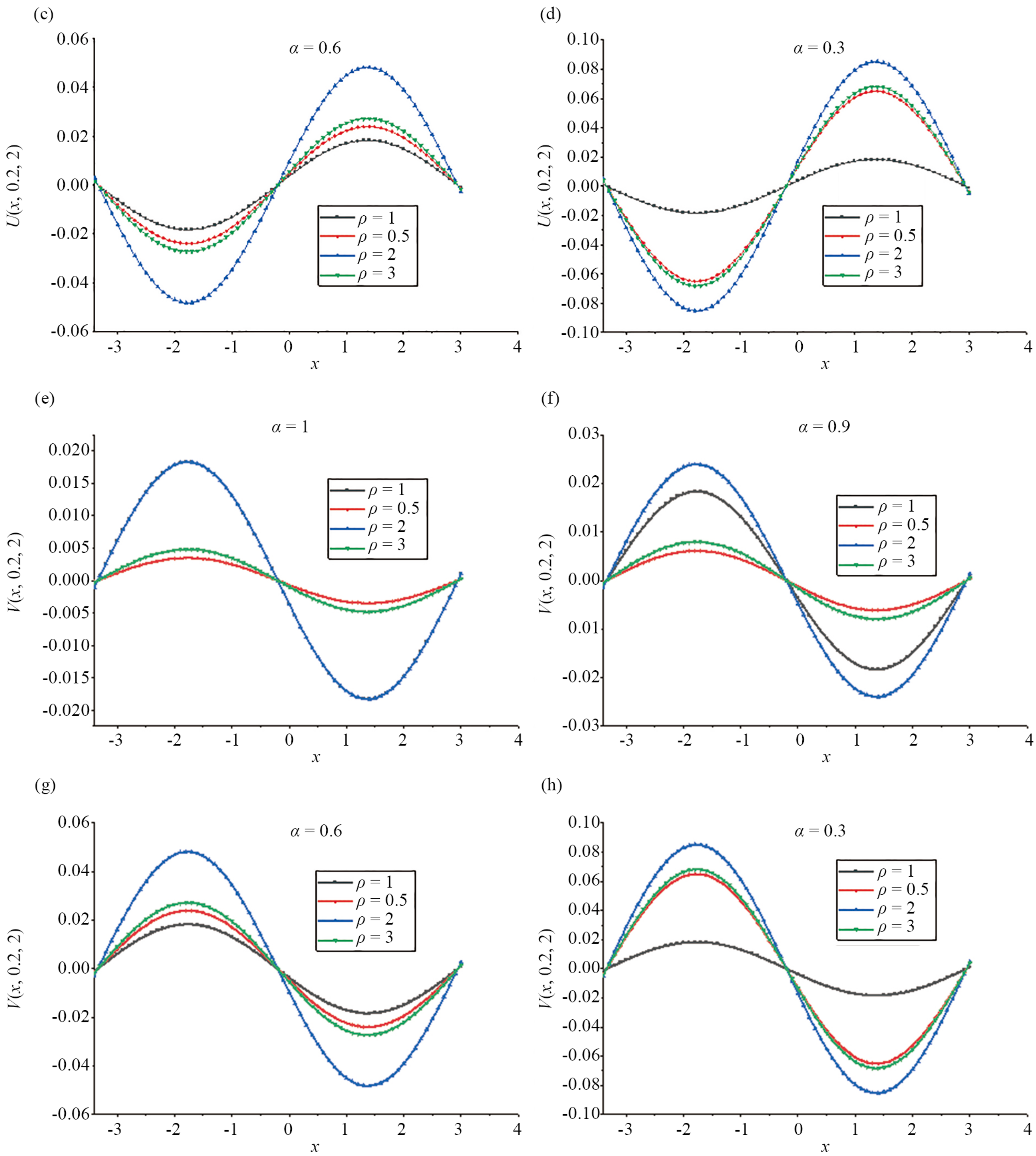


Figure 1. The graphs of Equation (56) different value of parameter α with, $\rho = 1,\ \rho = 0.5,\ \rho = 2,\ \rho = 3,\ \rho_0 = 1,\ g_1 = g_2 = 0,\ a = 0,\ y = 0.2,\ t = 2$

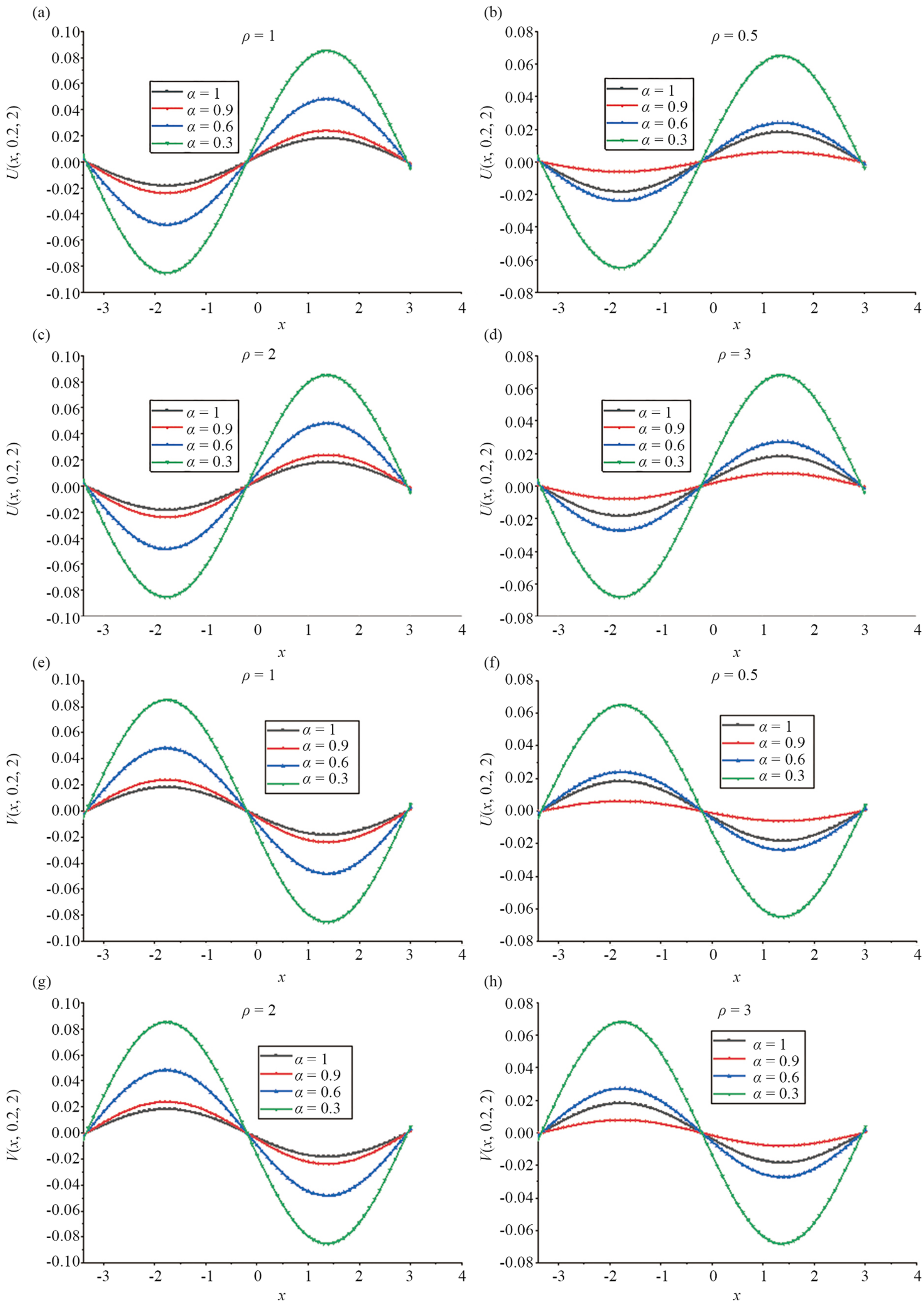


Figure 2. The graphs of Equation (56) different value of parameter ρ with, $\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.6$, $\alpha = 0.3$, $\rho_0 = 1$, $g_1 = g_2 = 0$, $a = 0$, $y = 0.2$, $t = 2$

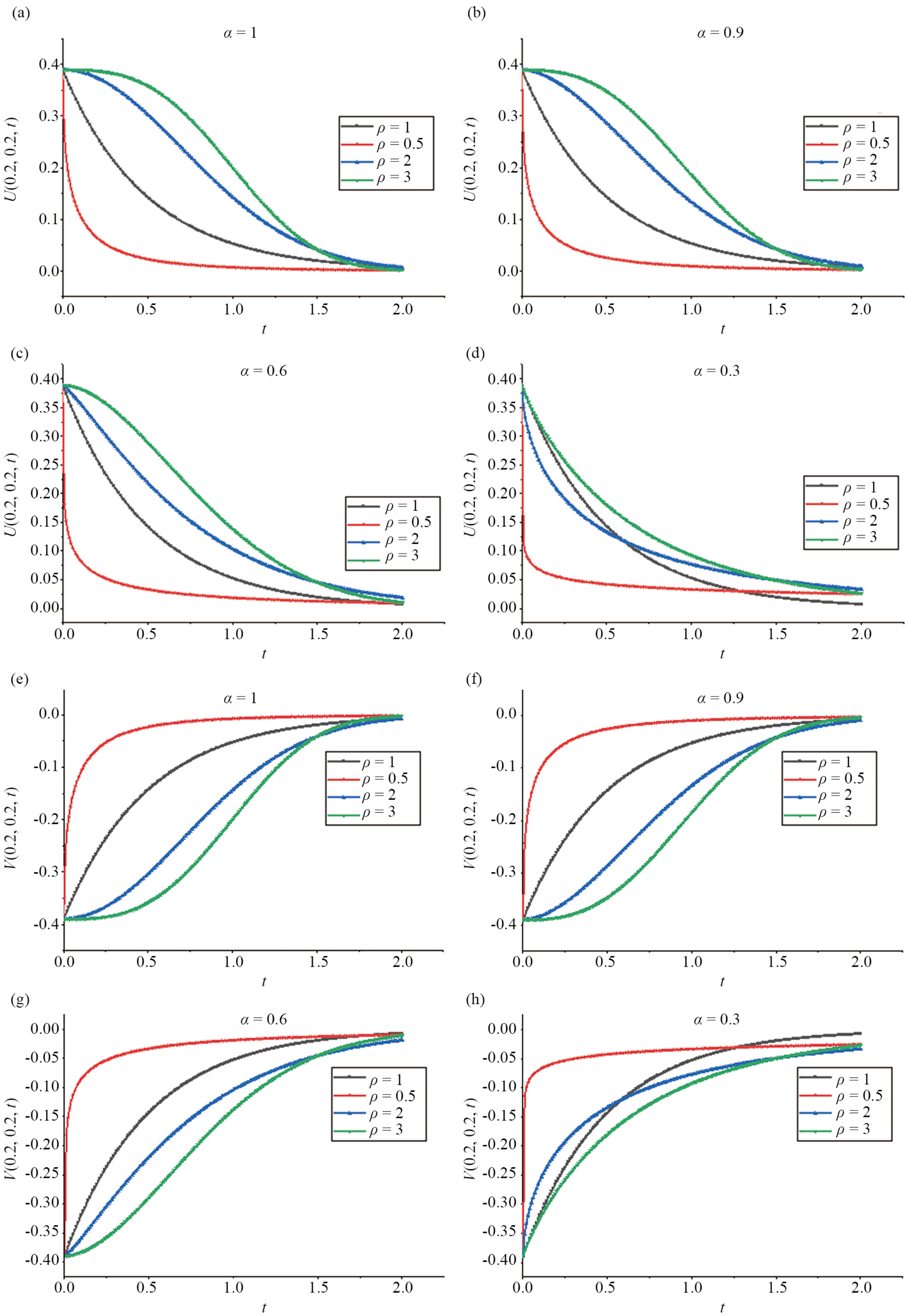


Figure 3. The graphs of Equation (56) different value of parameter α with, $\rho = 1,\ \rho = 0.5,\ \rho = 2,\ \rho = 3, \rho_0 = 1, g_1 = g_2 = 0,\ a = 0,\ x = y = 0.2$

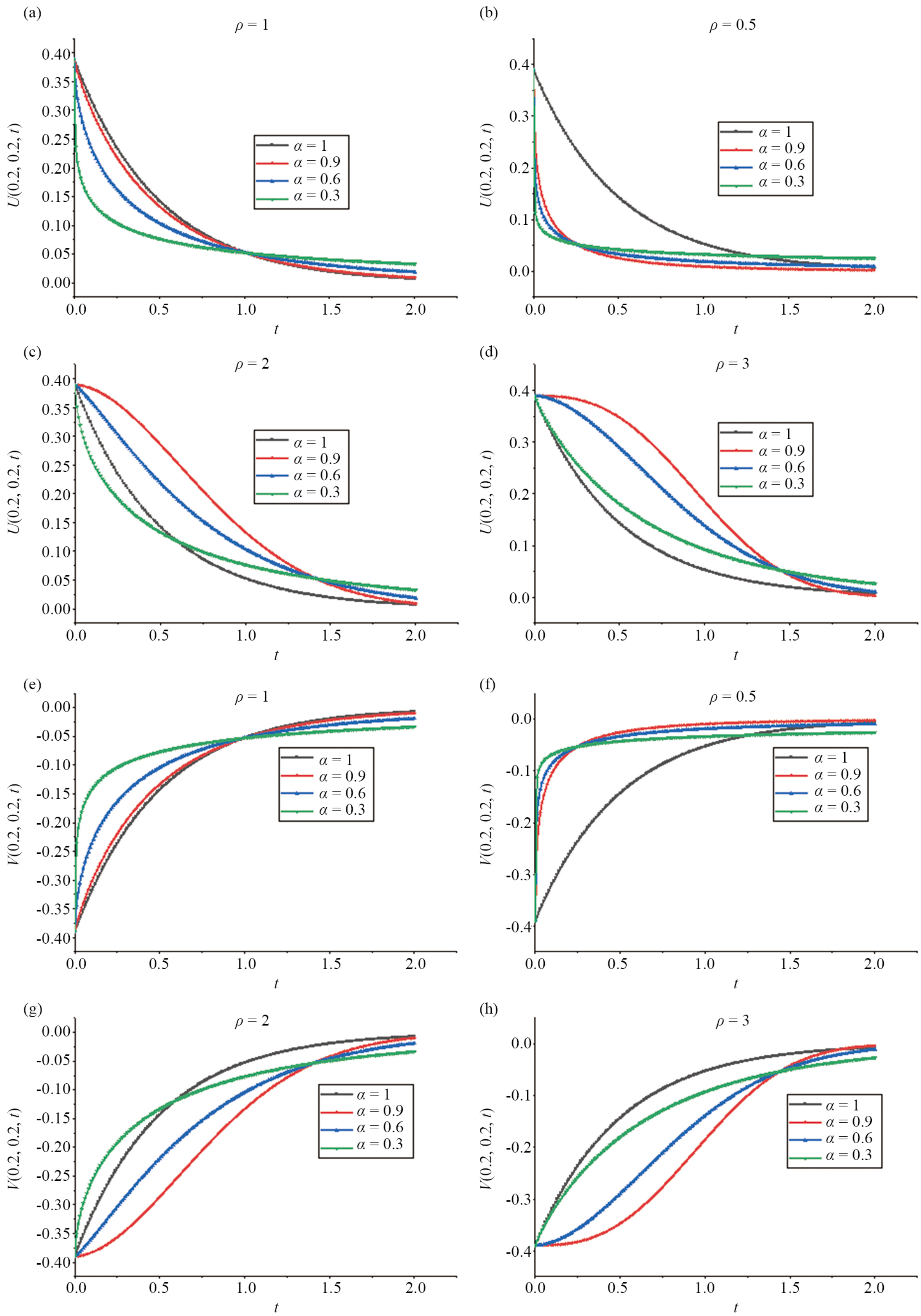


Figure 4. The graphs of Equation (56) different value of parameter ρ with, $\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.6$, $\alpha = 0.3$, $\rho_0 = 1$, $g_1 = g_2 = 0$, $a = 0$, $x = y = 0.2$

Tables 1 and 2 show numerical values of the solution obtained by HPLTM and the exact solution for Application among different values of t and α, ρ when $g_1 = g_2 = 0$.

Table 1. Numerical values of the approximate solutions to Applications 1 for different values of α and ρ with $x = y = 0.2$

t	$\alpha = 1,\ \rho = 1$	$\alpha = 1,\ \rho = 0.5$	$\alpha = 1,\ \rho = 2$	$\alpha = 1,\ \rho = 3$
	$u_{\text{exa}} = u_{\text{HPLTM}}$	u_{HPLTM}	u_{HPLTM}	u_{HPLTM}
0	0.38941834	0.38941834	0.38941834	0.38941834
0.1	0.31882877	0.10991893	0.38554357	0.38915882
0.2	0.26103492	0.06509188	0.37414903	0.38734697
0.3	0.21371732	0.04354365	0.35590157	0.38247152
0.4	0.17497694	0.0310262	0.33184042	0.37315263
0.5	0.143259	0.02301686	0.30327931	0.35828217
0.6	0.11729055	0.01757131	0.27168796	0.33719257
0.7	0.09602938	0.01370847	0.23856795	0.30981877
0.8	0.07862221	0.01088021	0.20533734	0.2768071
0.9	0.06437042	0.00875759	0.17323589	0.23952414
1	0.05270204	0.00713245	0.143259	0.19993404
1.1	0.04314878	0.00586744	0.11612349	0.1603439
1.2	0.03532723	0.00486893	0.09226402	0.1230578
1.3	0.02892349	0.00407126	0.07185529	0.09001601
1.4	0.02368055	0.00342736	0.05485285	0.06250964
1.5	0.019388	0.0029028	0.04104439	0.04104439
1.6	0.01587355	0.00247196	0.03010388	0.02538066
1.7	0.01299616	0.0021155	0.0216424	0.01472167
1.8	0.01064036	0.00181864	0.01525114	0.00797773
1.9	0.00871159	0.00156994	0.01053449	0.00402284
2	0.00713245	0.00136043	0.00713245	0.00188009

Table 2. Numerical values of the approximate solutions to Applications 1 for different values of α and ρ with $x = y = 0.2$

t	$\rho = 1,\ \alpha = 1$	$\rho = 1,\ \alpha = 0.9$	$\rho = 1,\ \alpha = 0.6$	$\rho = 1,\ \alpha = 0.3$
	$u_{\text{exa}} = u_{\text{HPLTM}}$	u_{HPLTM}	u_{HPLTM}	u_{HPLTM}
0	0.38941834	0.38941834	0.38941834	0.38941834
0.1	0.31882877	0.30273841	0.23563333	0.14291924
0.2	0.26103492	0.24342445	0.1818519	0.11336203
0.3	0.21371732	0.19792482	0.1474468	0.09663718
0.4	0.17497694	0.16204719	0.1227923	0.08522873
0.5	0.143259	0.13333713	0.10407859	0.076719
0.6	0.11729055	0.11014112	0.08935439	0.07002272
0.7	0.09602938	0.09126967	0.07747579	0.06456079
0.8	0.07862221	0.07583424	0.06771278	0.05998877
0.9	0.06437042	0.06315481	0.05957127	0.05608576
1	0.05270204	0.05270204	0.05270204	0.05270204
1.1	0.04314878	0.0440587	0.04684988	0.04973171
1.2	0.03532723	0.03689274	0.04182327	0.04709723
1.3	0.02892349	0.0309379	0.03747535	0.0447403
1.4	0.02368055	0.02597927	0.03369144	0.04261601
1.5	0.019388	0.02184257	0.03038063	0.0406891
1.6	0.01587355	0.01838574	0.0274699	0.03893145
1.7	0.01299616	0.01549263	0.02489994	0.03732026
1.8	0.01064036	0.0130679	0.02262206	0.03583685
1.9	0.00871159	0.01103306	0.02059597	0.03446571
2	0.00713245	0.00932335	0.01878805	0.03319387

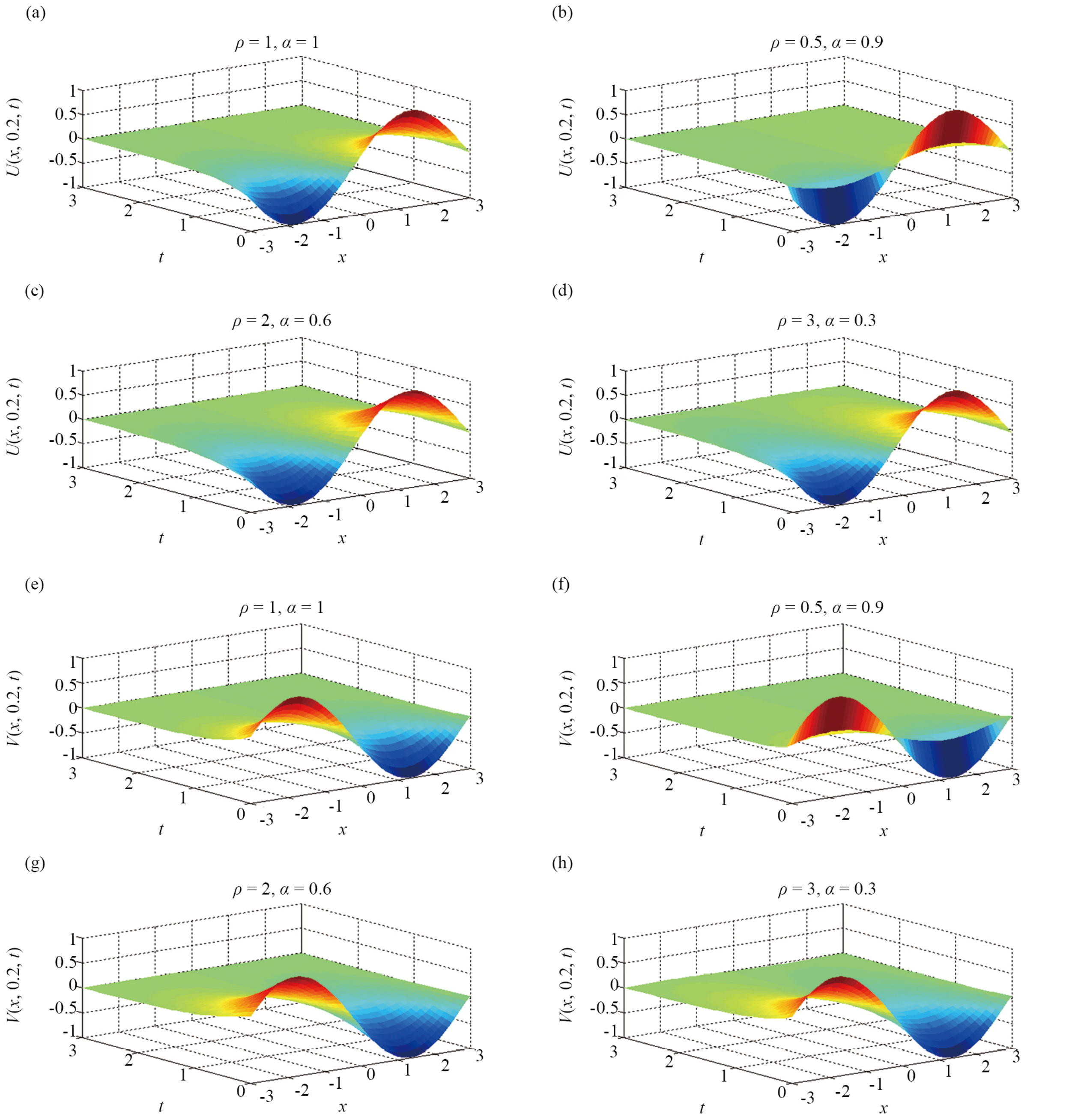


Figure 5. Plots solution Application 1 using Equation (56) with $a = 0$, $\rho_0 = 1$, $(\rho = 1,\ \alpha = 1)$, $(\rho = 0.5,\ \alpha = 0.9)$, $(\rho = 2,\ \alpha = 0.6)$, $(\rho = 3,\ \alpha = 0.3)$, $g_1 = g_2 = 0$, $y = 0.2$

Application 2 From Equation (36), 2-dimensional NS equation of fractional order with $g_1 = -g_2$ and with IC: $U(x,\ y,\ 0) = e^{x+y}$, $V(x,\ y,\ 0) = -e^{x+y}$.

Applying LT on both sides of Equation (32), we get:

$$\mathscr{L}_\rho\left\{U(x,\ y,\ t)\right\} = \frac{1}{s}e^{x+y} + \ \frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + g_1 - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right\}\right\} \tag{58}$$

$$\mathscr{L}_\rho\left\{V(x,\ y,\ t)\right\} = -\frac{1}{s}e^{x+y} + \ \frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) - g_2 - U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y}\right\}\right\} \tag{59}$$

The inverse ρ-Laplace transform of Equations (58) and (59) implies that:

$$U(x,\ y,\ t) = e^{x+y} + \mathscr{L}_\rho{}^{-1}\ \left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + g_1 - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right\}\right\}\right], \tag{60}$$

$$V(x,\ y,\ t) = -e^{x+y} + \mathscr{L}_\rho{}^{-1}\ \left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) - g_2 - U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y}\right\}\right\}\right]. \tag{61}$$

Simplifying Equations (60) and (61), we get:

$$U(x,\ y,\ z,\ t) = e^{x+y} + g_1\frac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)} + \mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right\}\right\}\right], \tag{62}$$

$$V(x,\ y,\ t) = -e^{x+y} - g_2\frac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)} + \mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{\rho_0\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) - U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y}\right\}\right\}\right]. \tag{63}$$

Now, applying the homotopy perturbation method, we have:

$$\sum_{n=0}^{\infty} p^n U_n(x,\ y,\ t) = e^{x+y} + g_1\frac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)} + p\left(\mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{N\sum_{n=0}^{\infty} p^n H_n(U)\right\}\right\}\right]\right), \tag{64}$$

$$\sum_{n=0}^{\infty} p^n V_n(x,\ y,\ t) = -e^{x+y} - g_2\frac{\left(\dfrac{t^\rho\ - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)} + p\left(\mathscr{L}_\rho{}^{-1}\left[\frac{1}{s^\alpha}\left\{\mathscr{L}_\rho\left\{N\sum_{n=0}^{\infty} p^n H_n(V)\right\}\right\}\right]\right). \tag{65}$$

where $H_n(U)$ and $H_n(V)$ are He's polynomials which signifies the nonlinear terms.

$$\begin{cases} H_n(U) = H_n(U_0,\ U_1,\ U_2,\ \ldots U_n) = \dfrac{1}{n!}\dfrac{\partial}{\partial p^n}\left[N\sum_{n=0}^{\infty} p^n U_n\right], \\ \\ H_n(V) = H_n(V_0,\ V_1,\ V_2,\ \ldots V_n) = \dfrac{1}{n!}\dfrac{\partial}{\partial p^n}\left[N\sum_{n=0}^{\infty} p^n V_n\right], \end{cases} \quad n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{66}$$

where:

$$\begin{cases} U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \ldots \\ V = V_0 + pV_1 + p^2V_2 + p^3V_3 + \ldots \end{cases} \tag{67}$$

The first few components of He's polynomials are given as:

$$\begin{cases} H_0(U) = \rho_0\left(\dfrac{\partial^2 U_0}{\partial x^2} + \dfrac{\partial^2 U_0}{\partial y^2}\right) - U_0\dfrac{\partial U_0}{\partial x} - V_0\dfrac{\partial U_0}{\partial y}, \\ \\ H_0(V) = \rho_0\left(\dfrac{\partial^2 V_0}{\partial x^2} + \dfrac{\partial^2 V_0}{\partial y^2}\right) - U_0\dfrac{\partial V_0}{\partial x} - V_0\dfrac{\partial V_0}{\partial y}, \end{cases} \tag{68}$$

$$\begin{cases} H_1(U) = \rho_0\left(\dfrac{\partial^2 U_1}{\partial x^2} + \dfrac{\partial^2 U_1}{\partial y^2}\right) - U_0\dfrac{\partial U_1}{\partial x} - V_0\dfrac{\partial U_1}{\partial y} - U_1\dfrac{\partial U_0}{\partial x} - V_1\dfrac{\partial U_0}{\partial y}, \\ \\ H_1(V) = \rho_0\left(\dfrac{\partial^2 V_1}{\partial x^2} + \dfrac{\partial^2 V_1}{\partial y^2}\right) - U_0\dfrac{\partial V_1}{\partial x} - V_0\dfrac{\partial V_1}{\partial y} - U_1\dfrac{\partial V_0}{\partial x} - V_1\dfrac{\partial V_0}{\partial y}, \end{cases} \tag{69}$$

$$\begin{cases} H_2(U) = \rho_0\left(\dfrac{\partial^2 U_2}{\partial x^2} + \dfrac{\partial^2 U_2}{\partial y^2}\right) - U_0\dfrac{\partial U_1}{\partial x} - V_0\dfrac{\partial U_1}{\partial y} - U_1\dfrac{\partial U_0}{\partial x} - V_1\dfrac{\partial U_0}{\partial y}, \\ \\ H_2(V) = \rho_0\left(\dfrac{\partial^2 V_2}{\partial x^2} + \dfrac{\partial^2 V_2}{\partial y^2}\right) - U_0\dfrac{\partial V_1}{\partial x} - V_0\dfrac{\partial V_1}{\partial y} - U_1\dfrac{\partial V_0}{\partial x} - V_1\dfrac{\partial V_0}{\partial y}, \end{cases} \tag{70}$$

$$\vdots$$

Equating the coefficients of like powers of p in Equations (64) and (65), we get the following results:

$$p^0:\begin{cases} U_0(x,\ y,\ t)=e^{x+y}+g_1\dfrac{\left(\dfrac{t^{\rho}\ -a^{\rho}}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}, \\ V_0(x,\ y,\ t)=-e^{x+y}-g_2\dfrac{\left(\dfrac{t^{\rho}\ -a^{\rho}}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}, \end{cases} \tag{71}$$

$$p^1:\begin{cases} U_1(x,\ y,\ t)=\mathscr{L}_{\rho}{}^{-1}\left[\dfrac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{H_0(U)\right\}\right\}\right]=-2\rho_0e^{x+y}\dfrac{\left(\dfrac{t^{\rho}\ -a^{\rho}}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}, \\ V_1(x,\ y,\ t)=\mathscr{L}_{\rho}{}^{-1}\left[\dfrac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{H_0(V)\right\}\right\}\right]=2\rho_0e^{x+y}\dfrac{\left(\dfrac{t^{\rho}\ -a^{\rho}}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}, \end{cases} \tag{72}$$

$$p^2:\begin{cases} U_2(x,\ y,\ t)=\mathscr{L}_{\rho}{}^{-1}\left[\dfrac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{H_1(U)\right\}\right\}\right]=4{\rho_0}^2e^{x+y}\dfrac{\left(\dfrac{t^{\rho}\ -a^{\rho}}{\rho}\right)^{2\alpha}}{\Gamma(2\alpha+1)}, \\ V_2(x,\ y,\ t)=\mathscr{L}_{\rho}{}^{-1}\left[\dfrac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{H_1(V)\right\}\right\}\right]=-4{\rho_0}^2e^{x+y}\dfrac{\left(\dfrac{t^{\rho}\ -a^{\rho}}{\rho}\right)^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases} \tag{73}$$

$$p^3:\begin{cases} U_3(x,\ y,\ t)=\mathscr{L}_{\rho}{}^{-1}\left[\dfrac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{H_2(U)\right\}\right\}\right]=-8{\rho_0}^3e^{x+y}\dfrac{\left(\dfrac{t^{\rho}\ -a^{\rho}}{\rho}\right)^{3\alpha}}{\Gamma(3\alpha+1)}, \\ V_3(x,\ y,\ t)=\mathscr{L}_{\rho}{}^{-1}\left[\dfrac{1}{s^{\alpha}}\left\{\mathscr{L}_{\rho}\left\{H_2(V)\right\}\right\}\right]=8{\rho_0}^3e^{x+y}\dfrac{\left(\dfrac{t^{\rho}\ -a^{\rho}}{\rho}\right)^{3\alpha}}{\Gamma(3\alpha+1)}, \end{cases} \tag{74}$$

$$\vdots$$

Thus, the solutions $U(x,\ y,\ t)$ and $V(x,\ y,\ t)$ are written in the form of:

$$\begin{cases} U(x,\, y,\, t) = U_0(x,\, y,\, z,\, t) + U_1(x,\, y,\, z,\, t) + U_2(x,\, y,\, z,\, t) + U_3(x,\, y,\, z,\, t) + \ldots \\ \qquad = e^{x+y} e^{-2\rho_0 \left(\frac{t^\rho - a^\rho}{\rho} \right)^\alpha} + g_1 \dfrac{\left(\dfrac{t^\rho - a^\rho}{\rho} \right)^\alpha}{\Gamma(\alpha+1)}, \\ V(x,\, y,\, t) = V_0(x,\, y,\, z,\, t) + V_1(x,\, y,\, z,\, t) + V_2(x,\, y,\, z,\, t) + V_3(x,\, y,\, z,\, t) + \ldots \\ \qquad = -e^{x+y} e^{-2\rho_0 \left(\frac{t^\rho - a^\rho}{\rho} \right)^\alpha} - g_2 \dfrac{\left(\dfrac{t^\rho - a^\rho}{\rho} \right)^\alpha}{\Gamma(\alpha+1)}. \end{cases} \tag{75}$$

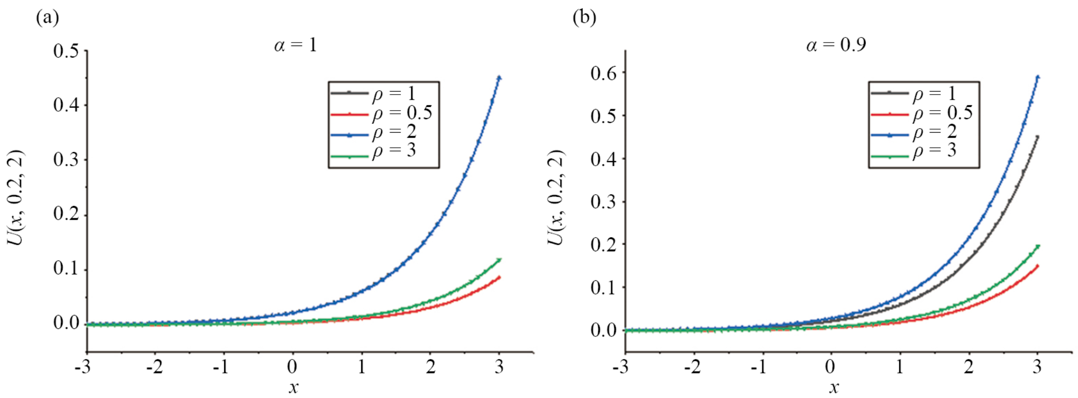
Finally, we work on writing infinite sums in terms of the Mittag-Leffler function, the Equation (76) [9] is:

$$\begin{cases} U(x,\, y,\, t) = e^{x+y} \mathbb{E}_{\alpha,\, 1} \left(-2\rho_0 \left(\dfrac{t^\rho - a^\rho}{\rho} \right)^\alpha \right) + g_1 \dfrac{\left(\dfrac{t^\rho - a^\rho}{\rho} \right)^\alpha}{\Gamma(\alpha+1)}, \\ V(x,\, y,\, t) = -e^{x+y} \mathbb{E}_{\alpha,\, 1} \left(-2\rho_0 \left(\dfrac{t^\rho - a^\rho}{\rho} \right)^\alpha \right) - g_2 \dfrac{\left(\dfrac{t^\rho - a^\rho}{\rho} \right)^\alpha}{\Gamma(\alpha+1)}. \end{cases} \tag{76}$$

These solutions are in agreement with the solutions found by [35] using (HPETM) and operator The Riemann-Liouville fractional integral, which are alike are in agreement with the solutions found by [36] applying the (FRDTM). Besides, the plots of Equation (76) are depicted in Figures 5-10, for different values of $\rho = 1,\ 0.5,\ 2,\ 3$ and $\alpha = 1,\ 0.9,\ 0.6,\ 0.3,\ \rho_0 = 1,\ a = 0$ and $g_1 = g_2 = 0$.

And $g_1 = g_2 = 0$, $\rho = \alpha = 1$ and $a = 0$ Equation (76) the exact solution of classical NS equation for the velocity.

$$\begin{cases} U(x,\, y,\, t) = e^{x+y} e^{-2\rho_0(t)}, \\ V(x,\, y,\, t) = -e^{x+y} e^{-2\rho_0(t)}. \end{cases} \tag{77}$$



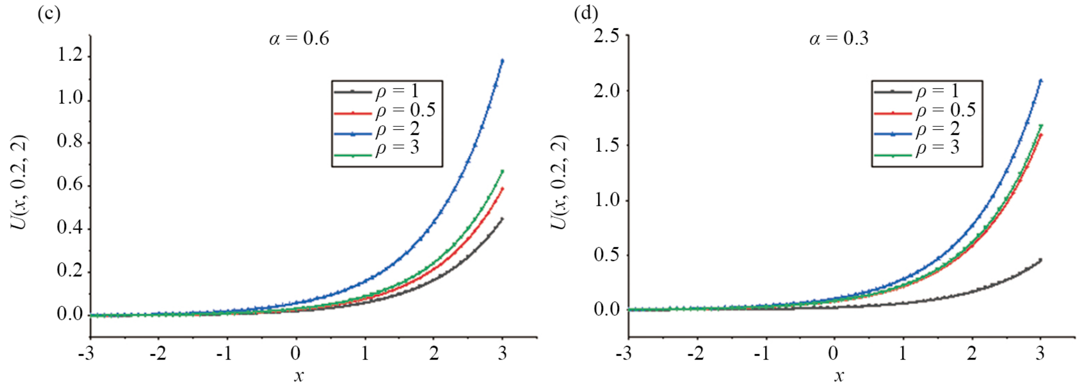


Figure 6. The graphs of Equation (76) different value of parameter α with, $\rho = 1$, $\rho = 0.5$, $\rho = 2$, $\rho = 3$, $\rho_0 = 1$, $g_1 = g_2 = 0$, $a = 0$, $y = 0.2$, $t = 2$

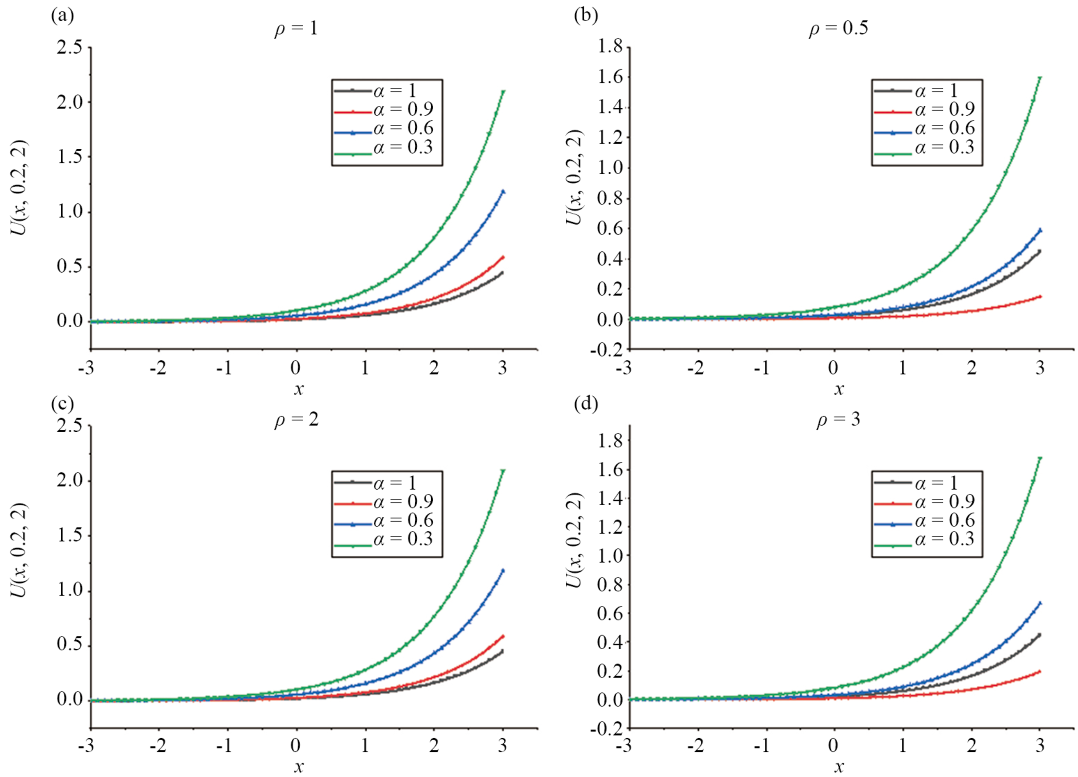


Figure 7. The graphs of Equation (76) different value of parameter ρ with, $\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.6$, $\alpha = 0.3$, $\rho_0 = 1$, $g_1 = g_2 = 0$, $a = 0$, $y = 0.2$, $t = 2$

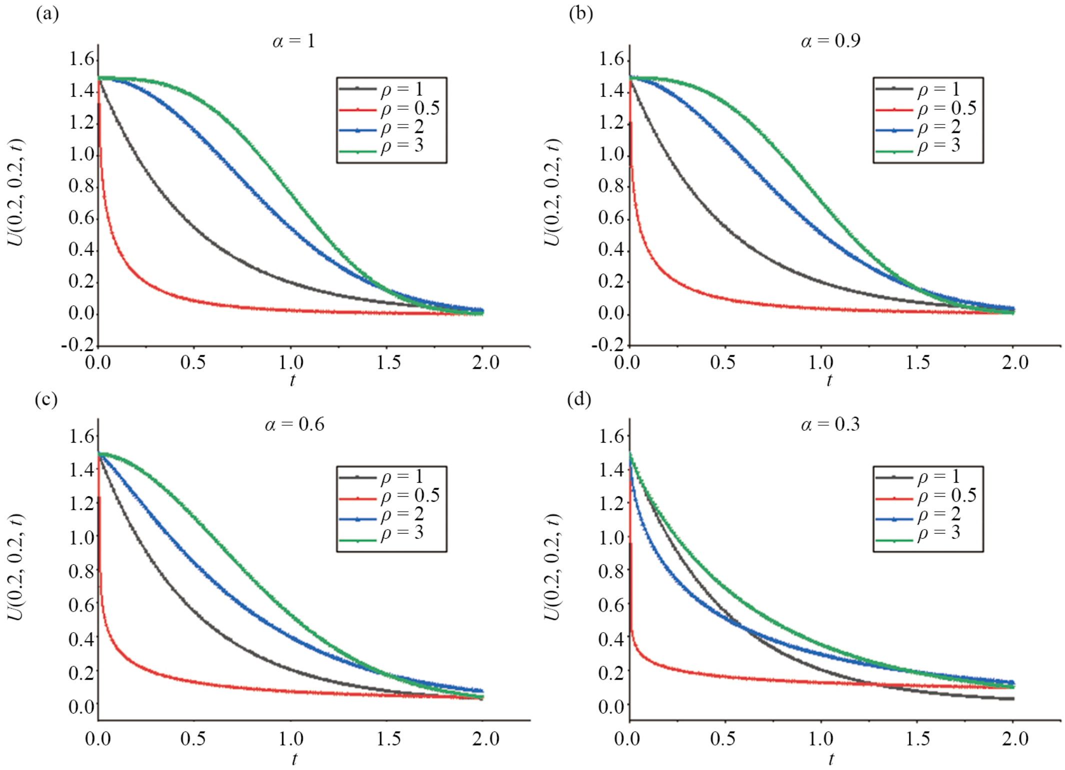


Figure 8. The graphs of Equation (76) different value of parameter ρ with, $\alpha=1$, $\alpha=0.9$, $\alpha=0.6$, $\alpha=0.3$, $\rho_0=1$, $g_1=g_2=0$, $a=0$, $x=y=0.2$

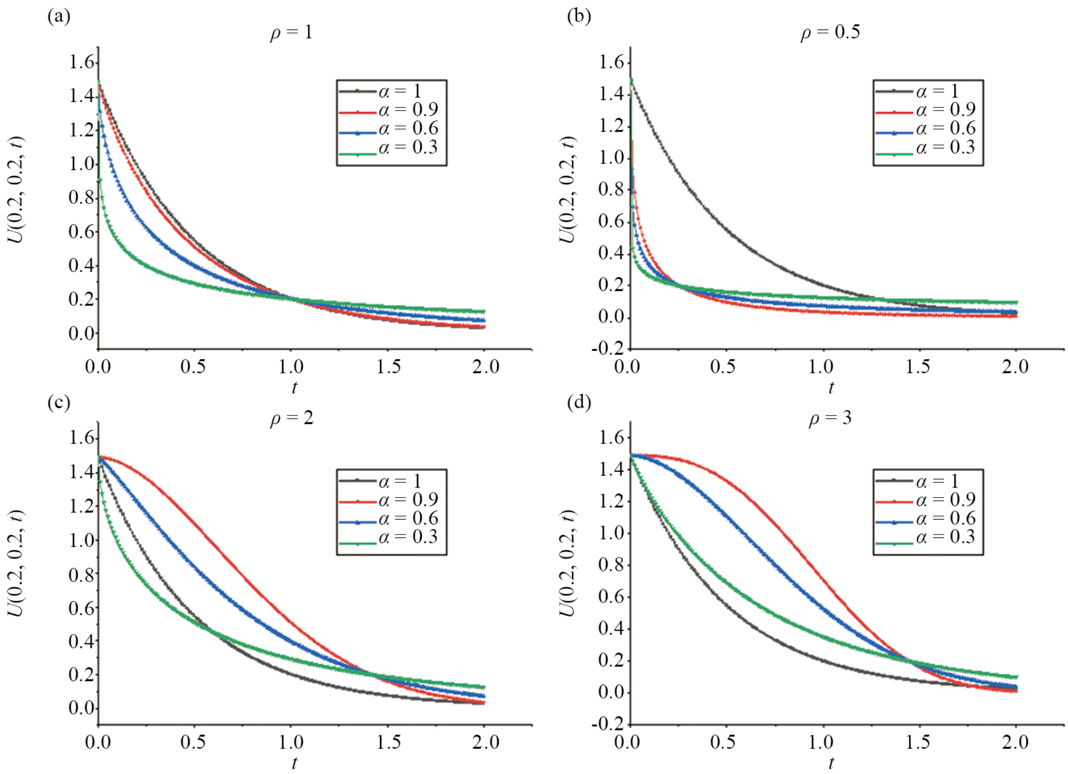


Figure 9. The graphs of Equation (76) different value of parameter ρ with, $\alpha=1$, $\alpha=0.9$, $\alpha=0.6$, $\alpha=0.3$, $\rho_0=1$, $g_1=g_2=0$, $a=0$, $x=y=0.2$

Tables 3 and 4 show numerical values of the solution obtained by HPLTM solution for Application among different values of t and α, ρ when $g_1 = g_2 = 0$.

Table 3. Numerical values of the approximate solutions to Applications 2 for different values of α and ρ with $x = y = 0.2$

t	$\alpha = 1,\ \rho = 1$	$\alpha = 1,\ \rho = 0.5$	$\alpha = 1,\ \rho = 2$	$\alpha = 1,\ \rho = 3$
	$u_{\text{exa}} = u_{\text{HPLTM}}$	u_{HPLTM}	u_{HPLTM}	u_{HPLTM}
0	14, 918.247	14, 918.247	14, 918.247	14, 918.247
0.1	122, 140.276	0.421089	147, 698.079	149, 083.048
0.2	1	0.24936081	143, 332.941	148, 388.948
0.3	0.81873075	0.1668116	136, 342.511	146, 521.209
0.4	0.67032005	0.11885843	127, 124.915	142, 951.229
0.5	0.54881164	0.08817541	116, 183.424	137, 254.498
0.6	0.44932896	0.06731402	104, 081.077	129, 175.273
0.7	0.36787944	0.05251586	0.91393119	118, 688.631
0.8	0.30119421	0.04168105	0.78662786	106, 042.171
0.9	0.24659696	0.0335495	0.66365025	0.91759423
1	0.20189652	0.02732372	0.54881164	0.76592834
1.1	0.16529889	0.02247761	0.44485807	0.61426224
1.2	0.13533528	0.0186524	0.35345468	0.47142276
1.3	0.11080316	0.01559661	0.27527078	0.34484278
1.4	0.09071795	0.01312989	0.21013607	0.23946851
1.5	0.07427358	0.01112034	0.15723717	0.15723717
1.6	0.06081006	0.00946983	0.11532512	0.0972309
1.7	0.04978707	0.00810429	0.08290997	0.05639734
1.8	0.0407622	0.00696705	0.05842567	0.03056193
1.9	0.03337327	0.00601427	0.04035661	0.01541111
2	0.02732372	0.00521167	0.02732372	0.00720246

Table 4. Numerical values of the approximate solutions to Applications 2 for different values of α and ρ with $x = y = 0.2$

t	$\rho = 1,\ \alpha = 1$	$\rho = 1,\ \alpha = 0.9$	$\rho = 1,\ \alpha = 0.6$	$\rho = 1,\ \alpha = 0.3$
	$u_{\text{exa}} = u_{\text{HPLTM}}$	u_{HPLTM}	u_{HPLTM}	u_{HPLTM}
0	0.38941834	14, 918.247	14, 918.247	14, 918.247
0.1	0.31882877	0.421089	147, 698.079	149, 083.048
0.2	0.26103492	0.24936081	143, 332.941	148, 388.948
0.3	0.21371732	0.1668116	136, 342.511	146, 521.209
0.4	0.17497694	0.11885843	127, 124.915	142, 951.229
0.5	0.143259	0.08817541	116, 183.424	137, 254.498
0.6	0.11729055	0.06731402	104, 081.077	129, 175.273
0.7	0.09602938	0.05251586	0.91393119	118, 688.631
0.8	0.07862221	0.04168105	0.78662786	106, 042.171
0.9	0.06437042	0.0335495	0.66365025	0.91759423
1	0.05270204	0.02732372	0.54881164	0.76592834
1.1	0.04314878	0.02247761	0.44485807	0.61426224
1.2	0.03532723	0.0186524	0.35345468	0.47142276
1.3	0.02892349	0.01559661	0.27527078	0.34484278
1.4	0.02368055	0.01312989	0.21013607	0.23946851
1.5	0.019388	0.01112034	0.15723717	0.15723717
1.6	0.01587355	0.00946983	0.11532512	0.0972309
1.7	0.01299616	0.00810429	0.08290997	0.05639734
1.8	0.01064036	0.00696705	0.05842567	0.03056193
1.9	0.00871159	0.00601427	0.04035661	0.01541111
2	0.00713245	0.00521167	0.02732372	0.00720246

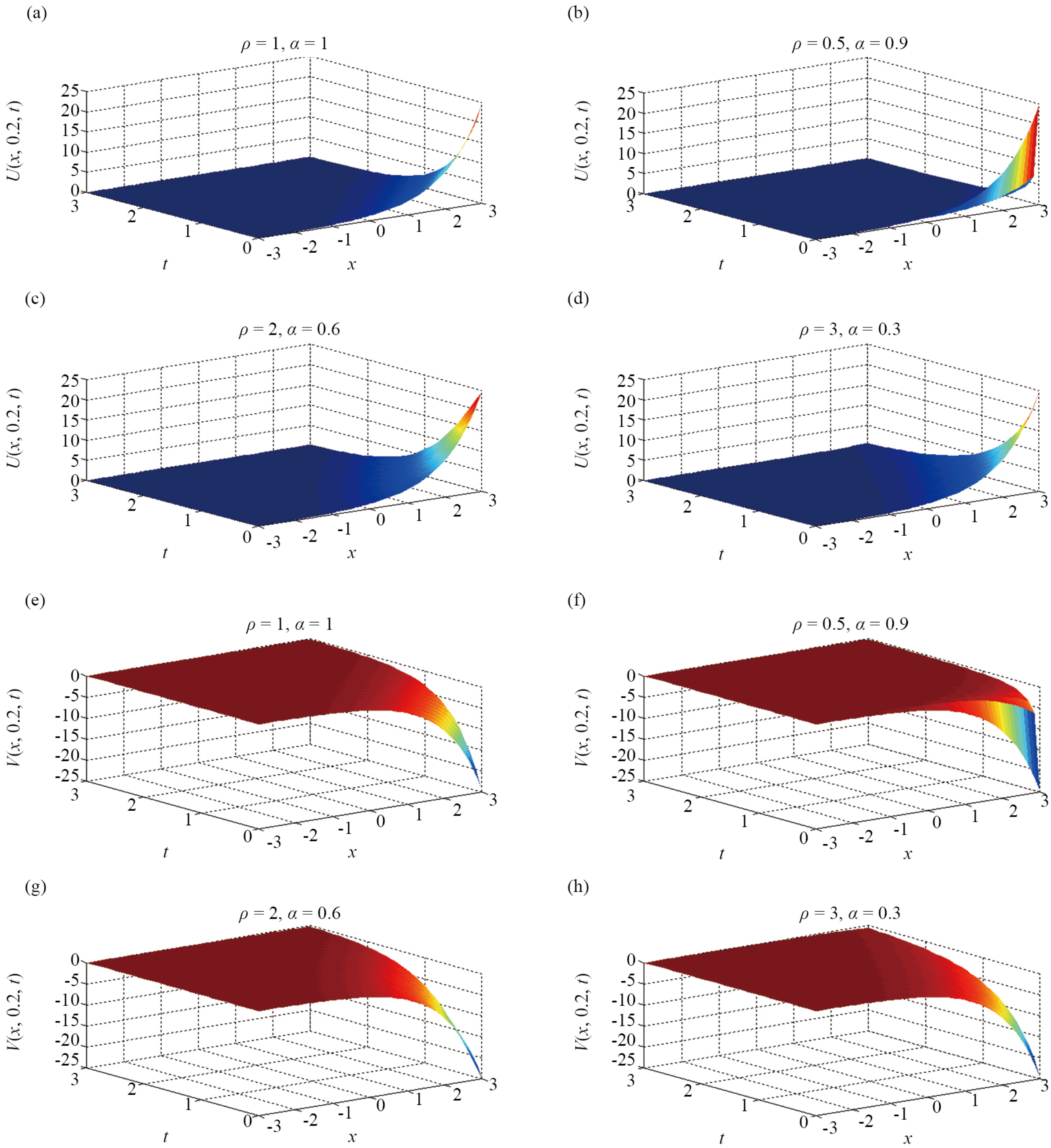


Figure 10. Plots solution Application 2 using Equation (76) with $a = 0$, $\rho_0 = 1$, $(\rho = 1,\ \alpha = 1)$, $(\rho = 0.5,\ \alpha = 0.9)$, $(\rho = 2,\ \alpha = 0.6)$, $(\rho = 3,\ \alpha = 0.3)$, $g_1 = g_2 = 0$

5. Conclusions

In light of the facts set out in this paper, HPLTM is used for computing the time-fractional model of Navier-Stokes equations with initial variables, whereat we adopted the definition of Caputo-Katugampola fractional derivative of time. In addition, analytical results are presented in the form of a power series. To validate and illustrate the method's efficacy, two test problems are done. However, the provided solutions have great agreement with HPETM and FRDTM [35, 36] are approximated without resorting to transformation, perturbation, discretization, or imposing restrictive conditions.

However, the performed calculations show that the described method needs very small size of computation in comparison with HPM and ADM [41, 43]. and we anticipate that this work is a step towards extending applications of the HPLT method to solve fractional problems with boundary conditions at infinity which I believe that this method is applicable for, which will be discussed in detail in further works Especially in the field of fluid physics.

Conflict of interest

The authors declare no competing financial interest.

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